数理解析研究所講究録
第 794 巻 1992 年 61-72

A Mixed Knuth Correspondence for (A, B)-Partially Strict Tableaux

Masao Ishikawa

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Bunkyo-ku Tokyo, Japan 113

Fix finite totally ordered sets $\mathcal{A}$, $\mathcal{A}'$throughout this article. A pair (U, C) of subsets of $\mathcal{A}$ is called a *division* of $\mathcal{A}$ if it satisfies

$$U \uplus C = \mathcal{A}. \qquad \text{(disjoint union)}$$

Henceforth, we fix a division (U, C) of $\mathcal{A}$, and we call elements of U *uncircled letters* and elements of C *circled letters*. Fix another division (A, B) of $\mathcal{A}$. Set $k = |A|$ and $l = |B|$ so that we have $|\mathcal{A}| = k + l$. We have two pairs (A, B) and (U, C) which are divisions of $\mathcal{A}$. We write

$$A_u = A \cap U, \qquad A_c = A \cap C$$
$$B_u = B \cap U, \qquad B_c = B \cap C$$

Example 1.1
Set $A = \{1, 3^\circ, 5, 7^\circ\}$, $B = \{\mathbf{2}, \mathbf{4}^\circ, \mathbf{6}, \mathbf{8}^\circ\}$, $U = \{1, \mathbf{2}, 5, \mathbf{6}\}$ and $C = \{3^\circ, \mathbf{4}^\circ, 7^\circ, \mathbf{8}^\circ\}$. Then (A, B) and (U, C) are divisions of $[8]$ and we have $A_u = \{1, 5\}$, $A_c = \{3^\circ, 7^\circ\}$, $B_u = \{\mathbf{2}, \mathbf{6}\}$, and $B_c = \{\mathbf{4}^\circ, \mathbf{8}^\circ\}$. As in this example we write elements of A in lightface and elements of B in boldface.

We take the word "(A, B)-partially strict" from [Ok], but the original definition is due to [St]. For the definition of (k, l)-semistandard tableaux see [BR] or [Re]. A *reverse plane partition* π is a filling of a Young diagram with letters of $\mathcal{A}$ wherein in each row from left to right and in each column from top to bottom the letters are arranged in weakly increasing order.

Definition 1.1
Let π be a reverse plane partition. π is said to be (A, B)-partially strict if it satisfies the conditions:

(i) For any $m \in A$, m appears at most once in each column.
(ii) For any $m \in B$, m appears at most once in each row.

We call a (A, B)-partially strict reverse plane partition a (A, B)-partially strict tableau. A $(\mathbf{P}, \emptyset)$-partially strict skew tableau is usually called a column-strict skew tableau and a $(\emptyset, \mathbf{P})$-partially strict skew tableau, a row-strict skew tableau. If $A = \{1, 2, \dots, k\}$ and $B = \{\mathbf{1}', \mathbf{2}', \dots, \mathbf{l}'\}$, where $1 < 2 < \dots < k < \mathbf{1}' < \mathbf{2}' < \dots < \mathbf{l}'$, then a (A, B)-partially strict tableau is called a (k, l)-semistandard tableau.

Example 1.2
Set (A, B) to be the division given in Example 4.1. Then

$\pi =$

1	1	1	$\mathbf{2}$	$^\circ\mathbf{4}$	5	5	$^\circ 7$
$^\circ 3$	$^\circ 3$	$^\circ 3$	$^\circ\mathbf{4}$	$\mathbf{6}$	$^\circ 7$	$^\circ 7$	
$^\circ\mathbf{4}$	5	5	$\mathbf{6}$				
$^\circ\mathbf{4}$	$^\circ 7$	$^\circ 7$	$^\circ\mathbf{8}$				
5	$^\circ\mathbf{8}$						

is a (A, B)-partially strict tableau.

$\pi' =$

1	1	3	**2'**	**3'**
4	4	**1'**	**2'**	
1'	**3'**			
1'				

is an exmple of $(4, 3)$-semistandard tableau, where $1 < 2 < 3 < 4 < \mathbf{1'} < \mathbf{2'} < \mathbf{3'}$.

DEFINITION 1.2
Let λ/μ be a skew diagram. Let $\mathcal{T}_{(A,B)}(\lambda/\mu)$ denote the set of all (A, B)-partially strict skew tableaux of shape λ/μ. For $\pi \in \mathcal{T}_{(A,B)}(\lambda/\mu)$ set the weight $wt(\pi)$ of π to be $\prod_{a \in \mathcal{A}} x_a^{m_a}$ where

$$m_a = \text{the number of times } a \text{ occurs in } \pi$$

and x_a's are indeterminates. Set

$$HS_{\lambda/\mu}^{(A,B)}(x) = \sum_{\pi \in \mathcal{T}_{(A,B)}(\lambda/\mu)} wt(\pi).$$

It is clear from the defintion that

$$HS_{\lambda/\mu}^{(A,B)}(x) = HS_{\lambda'/\mu'}^{(B,A)}(x).$$

In particular if $A = \{1, 2, \ldots, k\}$ and $B = \{\mathbf{1'}, \mathbf{2'}, \ldots, \mathbf{l'}\}$, where $1 < 2 < \ldots < k < \mathbf{1'} < \mathbf{2'} < \ldots < \mathbf{l'}$, we write $HS_{\lambda/\mu}^{(A,B)}(x)$ as $HS_{\lambda/\mu}^{(k,l)}(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l)$.

PROPOSITION 1.1
Set $A = \{a_1, a_2, \ldots, a_k\}$ *and* $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_l\}$. *Then*

$$HS_{\lambda/\mu}^{(A,B)}(x) = HS_{\lambda/\mu}^{(k,l)}(x_{a_1}, x_{a_2}, \ldots, x_{a_k}, x_{\mathbf{b}_1}, x_{\mathbf{b}_2}, \ldots, x_{\mathbf{b}_l})$$

Proof.
We can easily construct a bijection beteween $\mathcal{T}_{(A,B)}(\lambda/\mu)$ and $\mathcal{T}_{(\{1,2,\ldots,k\},\{\mathbf{1'},\mathbf{2'},\ldots,\mathbf{l'}\})}(\lambda/\mu)$ using the jeu de taquin method in [Re], Section 3, pp.266. For details see [Re]. ∎

Now we define a mixed Knuth insertion.

DEFINITION 1.3
Let π be a (A, B)-partially strict tableau and let $x \in \mathcal{A}$. We define $\mathrm{INSERT}_{(A,B;U,C)}(x)$ as follow.

If $x \in U$, insert x into the first row of π; if $x \in C$, insert x into the first column of π. If the bumped elemet y is uncircled, then we insert y into the row immediately below or if the bumped element y is circled, then we insert y into the column immediately to its right by the following rules.

y replace the least element which is $> y$ if $y \in A_u \cup B_c$: or y replace the least element which is $\geq y$ if $y \in B_u \cup A_c$.

Continue until an insertion takes place at the end of a row or column, bumping no new element. This procedure terminates in a finite number of steps. Then set (s, t) to be the cell which is added to π.

Similarly we define $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(x)$ by swapping U and C in the foregoing definition. If $x \in U$, insert x into the first column of π; if $x \in C$, insert x into the first row of π. The uncircled letters which are bumped are inserted into the column immediately to its right and circled letters are inserted into the row immediately below by the following rule.

y replace the least element which is $> y$ if $y \in A_c \cup B_u$: or y replace the least element which is $\geq y$ if $y \in B_c \cup A_u$.

It is easy to see that the resulting tableau is also (A, B)-partially strict. Let $\pi \leftarrow^m x$ (resp. $x \rightarrow^m \pi$) denote the tableau which is obtained after we applied $\text{INSERT}_{(A,B;U,C)}(x)$ (resp. $\overline{\text{INSERT}}_{(A,B;U,C)}(x)$) to π.

EXAMPLE 1.3
Let π be the (A, B)-partially strict tableau in EXAMPLE 4.2.

$\pi \leftarrow^m 4 =$

1	1	1	1	°**4**	5	5	°7
2	°3	°3	°3	°**4**	°7	°7	
°**4**	5	5	**6**	°**8**			
°**4**	**6**	°7	°7				
5	°**8**						

And we have $(s, t) = (4, 4)$.

$2 \rightarrow^m \pi =$

1	1	1	**2**	°**4**	5	5	°7
2	°3	°3	°**4**	6	°7	°7	
°3	5	5	5	**6**			
°**4**	°7	°7	°**8**				
°**4**	°**8**						

And we have $(s, t) = (6, 1)$.

REMARK 1.1
In [Re] two insertion procedures are defined for (k, l)-semistandard tableaux. Set $A = \{1, 2, \ldots, k\}$ and $B = \{\mathbf{1}', \mathbf{2}', \ldots, \mathbf{l}'\}$, where $1 < 2 < \ldots < k < \mathbf{1}' < \mathbf{2}' < \ldots < \mathbf{l}'$. If $U = A$ and $C = B$, then the insertion algorithm in Definition 4.2 is called RS1 insertion in [Re]. If $U = \mathcal{A}$ and $C = \emptyset$, then the insertion algorithm is called RS2 insertion.

DEFINITION 1.4
Let π be a (A, B)-partially strict tableau. Set m_x to be the number of times x occurs in π for each $x \in \mathcal{A}$. Let $m = \sum_{x \in \mathcal{A}} m_x$ We make a partial tableau $pt(\pi)$ with letters in $[m]$ from π as follows. If $x \in A$, then replace m_x x's in π to $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, ..., $\sum_{y \leq x} m_x$ from left to right. If $x \in B$, then replace m_x x's in π to $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, ..., $\sum_{y \leq x} m_x$ from top to bottom. If $x \in U$ then $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, ..., $\sum_{y \leq x} m_x$ are in U, and vice versa.

EXAMPLE 1.4
If π is as in Example 4.2, then $pt(\pi)$ is as in Example 2.2.

DEFINITION 1.5
A *word with repetition* is a sequence $w = w_1 w_2 \ldots w_m$ of letters in $\mathcal{A}$ wherein each $a \in \mathcal{A}$ can appear more than once. Given a word with repetition $w = w_1 w_2 \ldots w_m$, we make the insertion tableau $\pi = \emptyset \leftarrow^m w$ for w as follows. For $i = 1, 2, \ldots m$ we define inductively $\pi_0 = \emptyset$ and $\pi_i = \pi_{i-1} \leftarrow^m w_i$. Let $\pi = \pi_m$.

EXAMPLE 1.5

$$w = {}^\circ\mathbf{2}\quad {}^\circ\mathbf{2}\quad 1\quad {}^\circ 3\quad \mathbf{4}\quad {}^\circ 3\quad {}^\circ 3\quad 1\quad \mathbf{4}\quad {}^\circ\mathbf{2}\quad \mathbf{4}$$

is a word with repetition and the insertion tableau for w is as follows.

$\emptyset \leftarrow^m w =$

1	1	$^\circ\mathbf{2}$	$^\circ 3$	$\mathbf{4}$
$^\circ\mathbf{2}$	$^\circ 3$	$^\circ 3$	$\mathbf{4}$	
$^\circ\mathbf{2}$	$\mathbf{4}$			

DEFINITION 1.6
For a given word with repetition $w = w_1 w_2 \ldots w_m$ we make a permutation $p(w)$ of $[m]$ as follows. For each $x \in \mathcal{A}$ let m_x denote the number of times x appears in w. For each $x \in \mathcal{A}$, if $x \in A_u \cup B_c$ then replace all x in w by $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, …, $\sum_{y\le x} m_x$ in increasing order. For each $x \in \mathcal{A}$, if $x \in A_c \cup B_u$ then replace all x in w by $\sum_{y\le x} m_x$, $\sum_{y\le x} m_x - 1$, …, $\sum_{y<x} m_x + 1$ in decreasing order. If $x \in U$ then $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, …, $\sum_{y\le x} m_x$ are in U, and vice versa.

EXAMPLE 1.6
Let w be as in Example 4.5.

$$p(w) = {}^\circ 3\quad {}^\circ 4\quad 1\quad {}^\circ 8\quad 9\quad {}^\circ 7\quad {}^\circ 6\quad 2\quad 10\quad {}^\circ 5\quad 11$$

$\emptyset \leftarrow^m p(w) =$

1	2	$^\circ 8$	$^\circ 3$	9
$^\circ 4$	$^\circ 6$	$^\circ 7$	10	
$^\circ 5$	11			

The following proposition is easy to see from definitions.

PROPOSITION 1.2
Let w be a word with repetition. Let π be the insertion tableau of w Then the following diagram commutes.

$$\begin{array}{ccc} w & \longleftrightarrow & \pi \\ \downarrow p & & \downarrow pt \\ p(w) & \longleftrightarrow & pt(\pi) \end{array}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively. ∎

LEMMA 1.1
Let π be a (A,B)-partially strict tableau and $x, x' \in \mathcal{A}$. If $\text{INSERT}_{(A,B;U,C)}(x)$, *determining s and t, is immediately followed by* $\text{INSERT}_{(A,B;U,C)}(x')$, *determining (s', t'), then*
(Case 1) $x, x' \in U$
(a) If $x < x'$ or $x = x' \in A$ then we have $s \ge s'$ and $t < t'$.
(b) If $x > x'$ or $x = x' \in B$ then we have $s < s'$ and $t \ge t'$.
(Case 2) $x, x' \in C$
(a) If $x > x'$ or $x = x' \in A$ then $s \ge s'$ and $t < t'$.
(b) If $x < x'$ or $x = x' \in B$ then $s < s'$ and $t \ge t'$.

(Case 3) $x \in U$ *and* $x' \in C$, *we always have* $s < s'$ *and* $t \geq t'$.
(Case 4) $x \in C$ *and* $x' \in U$, *we always have* $s \geq s'$ *and* $t < t'$.

Proof.
Choose arbitrary word with repetition w such that $\pi = \emptyset \leftarrow^m w$. Let $w' = wxx'$. Then it is easy to verify the lemma by using Proposition 4.2, Corollary 2.2, and Lemma 1.1. For example, we verify Case 3. Assume that $x \in U$ and $x' \in C$. Then $x' \in C$ is changed into some negative letter $-x'$ which is less than x so that we obtain $s < s'$ and $t \geq t'$ immediately by Lemma 1.1. ∎

REMARK 1.2
In the foregoing lemma by changing $\mathrm{INSERT}_{(A,B;U,C)}(x)$ *and* $\mathrm{INSERT}_{(A,B;U,C)}(x')$ *into* $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(x)$ *and* $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(x')$, *respectively and swapping* U *and* C, *we obtain a similar result on* $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(\cdot)$.

Fix another finite totally orderd set $\mathcal{A}'$ and its divisions (A', B') and (U', C') such that $|A'| = k'$ and $|B'| = l'$ We write

$$A'_u = A' \cap U', \qquad A'_c = A' \cap C'$$
$$B'_u = B' \cap U', \qquad B'_c = B' \cap C'$$

DEFINITION 1.7
Let a be a $(k' + l') \times (k + l)$ matrix of nonnegative integers

$$a = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

whose rows are labeled by elements of $\mathcal{A}'$ and columns are labeled by elements of $\mathcal{A}$. a is said to be *addmissible* if it satisfies:
(1) If $(i,j) \in A' \times A \cup B' \times B$, $\qquad a_{i,j} \in \mathbf{N}$.
(2) If $(i,j) \in A' \times B \cup B' \times A$, $\qquad a_{i,j} \in \{0,1\}$.
Let $\mathcal{M}(A', B', A, B)$ denote the set of all admissible $(k' + l') \times (k + l)$ matrices.

EXAMPLE 1.7
Let $A' = \{2, {}^\circ 4\}$, $B' = \{\mathbf{1}, {}^\circ\mathbf{3}\}$, $A = \{{}^\circ 3, 4\}$, and $B = \{\mathbf{1}, {}^\circ\mathbf{2}\}$. Then

$$a = \begin{pmatrix} \mathit{2} & \mathit{0} & 0 & 0 \\ 0 & 1 & \mathit{1} & \mathit{2} \\ \mathit{0} & \mathit{3} & 1 & 0 \\ 1 & 0 & \mathit{1} & \mathit{2} \end{pmatrix}$$

is a admissibe 4×4 matrix. As in this example we write $a_{i,j}$ such that $(i,j) \in A' \times A \cup B' \times B$ in italic.

DEFINITION 1.8
Let $a \in \mathcal{M}(A', B', A, B)$. From a we make a two-line array

$$l(a) = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

as follows. We arrange $a_{u,v}$ pairs of row and column labels $\begin{pmatrix} u \\ v \end{pmatrix}$ by the following rule.

First we assume that

$$u_1 \le u_2 \le \cdots \le u_m.$$

(1) For each $u \in A'_u \cup B'_c$ we arrenge all labels $\binom{u_i}{v_i}$ such that $u_i = u$ as follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \dots, \dots, v_{p_r}}_{\substack{\text{elements of } C \\ \text{in decreasing order}}}, \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \dots, \dots, v_{p_{r+s}}}_{\substack{\text{elements of } U \\ \text{in increasing order}}}$$

(2) For each $u \in A'_c \cup B'_u$ we arrenge all labels $\binom{u_i}{v_i}$ such that $u_i = u$ as follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \dots, \dots, v_{p_r}}_{\substack{\text{elements of } U \\ \text{in decreasing order}}}, \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \dots, \dots, v_{p_{r+s}}}_{\substack{\text{elements of } C \\ \text{in increasing order}}}$$

It is easy to see this gives an one to one correspondence between admissible matrices and two line arrays satisfying the above conditions. We call this two line array the *matrix word* of a and denote by $l(a)$. The top (resp. bottom) line of $l(a)$ is denoted by $\hat{l}(a) = u_1, u_2, \cdots, u_m$ (resp. $\check{l}(a) = v_1, v_2, \cdots, v_m$).

EXAMPLE 1.8
The two line array which correspond to the matrix a in EXAMPLE 4.5 is

$$l(a) = \begin{pmatrix} \mathbf{1} & \mathbf{1} & 2 & 2 & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ 4 & {}^\circ 4 & {}^\circ 4 & {}^\circ 4 \\ \mathbf{1} & \mathbf{1} & {}^\circ 3 & {}^\circ\mathbf{2} & 4 & 4 & {}^\circ 3 & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & 4 & 4 & \mathbf{1} & {}^\circ 3 \end{pmatrix}.$$

DEFINITION 1.9
Let $a \in \mathcal{M}(A', B', A, B)$. From a we make a two-line array $l(a)$ in Definition 4.8.

$$l = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

We construct a sequence of tableaux pairs:

$$(\emptyset, \emptyset) = (\pi_0, \sigma_0), (\pi_1, \sigma_1), \dots, (\pi_m, \sigma_m) = (\pi, \sigma)$$

inductively as follows. For each $i = 1, 2, \dots, m$ form π_i from π_{i-1} by performing $\mathrm{INSERT}_{(A,B;U,C)}(v_i)$ on π_{i-1} if u_i is a uncircled letter, or performing $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(v_i)$ on π_{i-1} if u_i is a circled letter. Form σ_i from σ_{i-1} by placing u_i on σ_{i-1} in the cell added to π_i. By Lemma 4.1 σ is a (A', B')-partially strict tableau and π and σ have the same shape.

EXAMPLE 1.9
Let a be as in Example 4.7. Then

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & 4 & 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{2} & 4 \\ \cline{1-3} \mathbf{1} & {}^\circ 3 \\ \cline{1-2} {}^\circ\mathbf{2} & 4 \\ \cline{1-2} {}^\circ\mathbf{2} \\ \cline{1-1} \end{array} \qquad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & 2 & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-3} 2 & {}^\circ\mathbf{3} \\ \cline{1-2} {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-2} {}^\circ 4 \\ \cline{1-1} \end{array}$$

DEFINITION 1.10
Let $a \in \mathcal{M}(A', B', A, B)$. Let

$$l(a) = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

be the two line array which correspond to a. We construct a biword w from l as follow. For each $x \in \mathcal{A}$ (resp. $x \in \mathcal{A}'$) let m_x (resp. m'_x) denote the numeber of times x occurs in the bottom (resp. top) line of l. Replace m'_x x's in the top line of l by $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, ..., $\sum_{y\le x} m_x$ from left to right. The circles are transfered unchanged in this replacement. For each $x \in \mathcal{A}$ let r_x (resp. s_x) be the number of pairs $\binom{u_i}{v_j}$ such that $v_j = x$ and $u_i \in U$ (resp. $u_i \in C$). So we have $r_x + s_x = m_x$. For each $x \in \mathcal{A}$ we replace v_j's such that $v_j = x$ by the following rules. The circles are transfered unchanged in this replacement.

(Case 1) : $x \in A_u \cup B_c$
Replace the v_j's of pairs $\binom{u_i}{v_j}$ such that $v_j = x$ and $u_i \in C$ by $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, ..., $\sum_{y<x} m_x + s_x$ from right to left. Then replace the v_j's of pairs $\binom{u_i}{v_j}$ such that $v_j = x$ and $u_i \in U$ by $\sum_{y<x} m_x + s_x + 1$, $\sum_{y<x} m_x + s_x + 2$, ..., $\sum_{y\le x} m_x$ from left to right.

(Case 2) : $x \in A_c \cup B_u$
Replace the v_j's of pairs $\binom{u_i}{v_j}$ such that $v_j = x$ and $u_i \in U$ by $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2$, ..., $\sum_{y<x} m_x + r_x$ from right to left. Then replace the v_j's of pairs $\binom{u_i}{v_j}$ such that $v_j = x$ and $u_i \in C$ by $\sum_{y<x} m_x + r_x + 1$, $\sum_{y<x} m_x + r_x + 2$, ..., $\sum_{y\le x} m_x$ from left to right.

Let $p(a)$ denote the resulting biword.

EXAMPLE 1.10
Let a be as Examle 4.7 and $l(a)$ as Example 4.8. Then we have

$$p(a) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & {}^\circ 7 & {}^\circ 8 & {}^\circ 9 & {}^\circ 10 & {}^\circ 11 & {}^\circ 12 & {}^\circ 13 & {}^\circ 14 \\ 2 & 1 & {}^\circ 8 & {}^\circ 7 & 13 & 14 & {}^\circ 9 & {}^\circ 6 & {}^\circ 5 & {}^\circ 4 & 12 & 11 & 3 & {}^\circ 10 \end{pmatrix}.$$

The following proposition is easy to see from definitions.

PROPOSITION 1.3
Let $a \in \mathcal{M}(A', B', A, B)$. Then the following diagram commutes.

$$\begin{array}{ccc} a & \longleftrightarrow & (\pi, \sigma) \\ \downarrow p & & \downarrow \\ p(a) & \longleftrightarrow & (pt(\pi), pt(\sigma)) \end{array}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively.
∎

EXAMPLE 1.11
Let $p(a)$ be as in Example 4.9. Then the insertion pair of $p(a)$ is as follows.

$\pi =$

1	°4	°9	°10	13	14
2	°5	12			
3	°8				
°6	11				
°7					

$\sigma =$

1	4	5	6	°7	°14
2	°8	°13			
3	°9				
°10	°12				
°11					

From Proposition 4.3 we obtain the following theorem.

THEOREM 1.1
Fix $\mathcal{A}$ and its divisions (U, C) and (A, B). Fix another $\mathcal{A}'$ and its divisions (U', C') and (A', B'). The map in Definition 4.9 from admissible matrices $a \in \mathcal{M}(A', B', A, B)$ to pairs (π, σ), where π is (A, B)-partially strict tableau, σ is (A', B')-partially strict tableau and π and σ have the same shape, is a bijction.

The following proposition is also easy to see from definitions.

PROPOSITION 1.4
Let $a \in \mathcal{M}(A', B', A, B)$. If $p(a)$ correspond to a by the map in Definition 4.10, then the inverse biword $p(a)^{-1}$ correspond to a^t. Here a^t denote the conjugate matix of a.

From Proposition 4.4 we obtain the following theorem.

THEOREM 1.2
Fix $\mathcal{A}$ and its divisions (U, C) and (A, B). Assume that (π, σ) correspond to a by the bijection in Definition 4.9, where $a \in \mathcal{M}(A, B, A, B)$, and π and σ are (A, B)-partially strict tableau having the same shape. Then (σ, π) correspond to a^t by the same bijection.

EXAMPLE 1.12
Let a be as Examle 4.7. Then

$$a^t = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

and

$$l(a) = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & 4 & 4 & 4 & 4 \\ \mathbf{1} & \mathbf{1} & {}^\circ 4 & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ 4 & {}^\circ 4 & {}^\circ 4 & 2 & 2 \end{pmatrix}.$$

It's easy to make sure that

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & 4 & 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{2} & 4 \\ \cline{1-3} \mathbf{1} & {}^\circ 3 \\ \cline{1-2} {}^\circ\mathbf{2} & 4 \\ \cline{1-2} {}^\circ\mathbf{2} \\ \cline{1-1} \end{array} \qquad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & 2 & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-3} 2 & {}^\circ\mathbf{3} \\ \cline{1-2} {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-2} {}^\circ 4 \\ \cline{1-1} \end{array}$$

DEFINITION 1.11
Fix $\mathcal{A}$ and its divisions (U, C) and (A, B). Let $a = (a_{ij})_{i,j \in \mathcal{A}} \in \mathcal{M}(A, B, A, B)$ be an admissible symmetric matrix. We define $\mathrm{tr}_{(A,B)}\, a$ by

$$\mathrm{tr}_{(A,B)}\, a = \sum_{i \in A} a_{ii} + \sum_{i \in B} \mathrm{odd}\{a_{ii}\}$$

where $\mathrm{odd}\{x\} = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}$.

Corollary 1.1

Fix $\mathcal{A}$ and its divisions (U, C) and (A, B). The map in Definition 4.9 gives a bijection from admissible symmetric matrices $a \in \mathcal{M}(A, B, A, B)$ onto (A, B)-partially strict tableaux π. In this bijection we have

$$\mathrm{tr}_{(A,B)}\, a = \mathrm{odd}(\lambda)$$

where λ is the shape of π and $\mathrm{odd}(\lambda)$ stands for the number of odd length columns in λ.

Example 1.13

Let $A = \{1, {}^\circ 3\}$ and $B = \{{}^\circ\mathbf{2}, \mathbf{4}\}$. Let a be an admissible symmetric matrix given by

$$a = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & 4 & 4 \\ {}^\circ 3 & {}^\circ 3 & 1 & {}^\circ 3 & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & \mathbf{4} & 1 & 1 & {}^\circ\mathbf{2} & 4 & {}^\circ 3 \end{pmatrix}.$$

and

$\pi =$

1	1	1	$^\circ\mathbf{2}$	4
$^\circ\mathbf{2}$	$^\circ 3$	$^\circ 3$	$^\circ 3$	
$^\circ\mathbf{2}$	4			
$^\circ 3$				

Corollary 1.2

Fix $\mathcal{A}$ and its division (A, B).

$$\sum_{\lambda} HS_\lambda^{(A,B)}(x)\, t^{\mathrm{odd}(\lambda)} = \prod_{\substack{(i,j)\in A\times A \cup B\times B \\ i<j}} \frac{1}{1 - x_i x_j} \prod_{(i,j)\in A\times B} (1 + x_i x_j) \prod_{i\in A} \frac{1}{1 - t x_i} \prod_{i \in B} \frac{1 + t x_i}{1 - x_i^2}$$

In particular,

$$\sum_{\lambda' \ \mathit{even}} HS_\lambda^{(A,B)}(x) = \prod_{\substack{(i,j)\in A\times A \cup B\times B \\ i<j}} \frac{1}{1 - x_i x_j} \prod_{(i,j)\in A\times B} (1 + x_i x_j) \prod_{i\in B} \frac{1}{1 - x_i^2}$$

Now we investigate skew case. Let $\mathrm{PST}_{(A,B)}(\lambda/\mu)$ denote the set of (A, B)-partially strict skew tableaux which have skew shape λ/μ.

Theorem 1.3

Fix $\mathcal{A}$ and its divisions (U, C) and (A, B). Fix another $\mathcal{A}'$ and its divisions (U', C') and (A', B'). Let α and β be fixed partitions. Then the map

$$(a, \tau, \kappa) \leftrightarrow (\pi, \sigma)$$

defined below is a bijection between admissible matrices $a \in \mathcal{M}(A, B, A', B')$ with $\tau \in \mathrm{PST}_{(A,B)}(\alpha/\mu)$ and $\kappa \in \mathrm{PST}_{(A',B')}(\beta/\mu)$, on the one hand, and $\pi \in \mathrm{PST}_{(A,B)}(\lambda/\beta)$ and $\sigma \in \mathrm{PST}_{(A',B')}(\lambda/\alpha)$, on the other, such that $\tau \cup \check{l}(a) = \pi$ and $\kappa \cup \hat{l}(a) = \sigma$.

Proof.
Let the largest letter of $\kappa \cup \widehat{l}(a)$ be n. We construct (π_r, σ_r) for $r = 0, 1, \ldots, n$ as follows. Start with $(\pi_0, \sigma_0) = (\tau, \emptyset_\alpha)$. Form π_r from π_{r-1} as follows.

Case 1 : $r \in A'_u \cup B'_c$

At first we insert all the circled letters of $\check{l}(a)$ paired with r's in $\widehat{l}(a)$, where these circled letters are arranged in decreasing order. Next we internally insert all the letters of π_{r-1} corresponding to r's in σ_{r-1}. If $r \in A'_u$, the insertion proceed left to right, and if $r \in B'_c$, the insertion proceed top to bottom. Finally we insert all the uncircled letters of $\check{l}(a)$ paired with r's in $\widehat{l}(a)$, where these uncircled letters are arranged in increasing order.

Case 2 : $r \in A'_c \cup B'_u$

At first we insert all the uncircled letters of $\check{l}(a)$ paired with r's in $\widehat{l}(a)$, where these uncircled letters are arranged in decreasing order. Next we internally insert all the letters of π_{r-1} corresponding to r's in σ_{r-1}. If $r \in A'_c$, the insertion proceed left to right, and if $r \in B'_u$, the insertion proceed top to bottom. Finally we insert all the circled letters of $\check{l}(a)$ paired with r's in $\widehat{l}(a)$, where these uncircled letters are arranged in increasing order.

In either case placing r's in the appropriate cells of σ_{r-1} result in σ_r. It is not hard to see that the cells where r's are placed are horizontal or vertical strip in σ_r. At last we put $(\pi_n, \sigma_n) = (\pi, \sigma)$. ∎

EXAMPLE 1.14

Let $A = \{1, {}^\circ 2\}$, $B = \{\mathbf{{}^\circ 3}, \mathbf{4}\}$, $A' = \{1, {}^\circ 3\}$, $B = \{\mathbf{2}, \mathbf{{}^\circ 4}\}$, $\alpha = (221)$ and $\beta = (43)$. Let $a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

so that the matrix word of a is $l(a) = \begin{pmatrix} 1 & \mathbf{2} & \mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & \mathbf{{}^\circ 4} \\ {}^\circ 2 & \mathbf{4} & {}^\circ 2 & 1 & 1 & {}^\circ 2 & \mathbf{{}^\circ 3} \end{pmatrix}$. Let

$\tau =$

	1
1	**4**
°3	

$\kappa =$

	1	**2**	**°4**
1	°3	**°4**	

Then we have

$\pi =$

				1	1	**°3**	**°4**
			1	°2			
1	°2	**4**					
°2	**°3**						

$\sigma =$

		1	1	**2**	°3	°3	**°4**
		2	°3	°3			
	1	**°4**					
2	**°4**						

COROLLARY 1.3
Fix $\mathcal{A}$ and its division (A, B). Fix another $\mathcal{A}'$ and its division (A', B'). Let α and β be fixed partitions. Then

$$\sum_\lambda HS^{(A,B)}_{\lambda/\beta}(x)\, HS^{(A',B')}_{\lambda/\alpha}(y)$$

$$= \sum_\mu HS^{(A,B)}_{\alpha/\mu}(x)\, HS^{(A',B')}_{\beta/\mu}(y) \prod_{(i,j)\in A\times A' \cup B\times B'} \frac{1}{1 - x_i y_j} \prod_{(i,j)\in A\times B' \cup B\times A'} (1 + x_i y_j)$$

THEOREM 1.4
Let $\mathcal{A} = \mathcal{A}'$, $(A,B) = (A',B')$, $(U,C) = (U',C')$ and $\alpha = \beta$ in Theorem 4.3. If (a,τ,κ) correspond to (π,σ) by the bijection in Theorem 4.3 then (a^t,κ,τ) correspond to (σ,π) by the same bijection.

THEOREM 1.5
Fix $\mathcal{A}$ and its divisions (U,C) and (A,B). Let α be a fixed partition. Then the mapping in Theorem 4.3 restricts to a bijection

$$(a,\tau) \leftrightarrow \pi$$

where $a \in \mathcal{M}(A,B,A,B)$ is a symmetric matrix, $\tau \in \mathrm{PST}_{(A,B)}(\alpha/\mu)$, $\pi \in \mathrm{PST}_{(A,B)}(\lambda/\mu)$, and $\check{l}(a) \cup \tau = \pi$. In this bijection we always have

$$\mathrm{tr}_{(A,B)}\, a + \mathrm{odd}(\mu) = \mathrm{odd}(\lambda)$$

EXAMPLE 1.15

Let $A = \{1, {}^\circ 3\}$, $B = \{{}^\circ\mathbf{2}, \mathbf{4}\}$ and $\alpha = (221)$. Let $a = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ so that the matrix word of a is

$l(a) = \begin{pmatrix} 1 & 1 & 1 & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ 3 & \mathbf{4} \\ {}^\circ 3 & {}^\circ\mathbf{2} & \mathbf{4} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & 1 & 1 & 1 \end{pmatrix}$. Let

$$\tau = \begin{array}{|c|c|} \hline & \\ \hline & \mathbf{4} \\ \hline {}^\circ 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & {}^\circ\mathbf{2} & {}^\circ 3 \\ \hline & 1 & {}^\circ\mathbf{2} & \mathbf{4} \\ \hline {}^\circ\mathbf{2} & {}^\circ 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

COROLLARY 1.4
Fix $\mathcal{A}$ and its division (A,B). Let α be a fixed partition.

$$\sum_{\lambda} HS^{(A,B)}_{\lambda/\alpha}(x)\ t^{\mathrm{odd}(\lambda)}$$
$$= \sum_{\mu} HS^{(A,B)}_{\alpha/\mu}(x)\ t^{\mathrm{odd}(\mu)} \cdot \prod_{\substack{(i,j)\in A\times A \cup B\times B \\ i<j}} \frac{1}{1-x_i x_j} \prod_{(i,j)\in A\times B} (1+x_i x_j) \prod_{i\in A} \frac{1}{1-t x_i} \prod_{i\in B} \frac{1+t x_i}{1-x_i^2}$$

In particular,

$$\sum_{\lambda'\ even} HS^{(A,B)}_{\lambda/\alpha}(x) = \sum_{\mu'\ even} HS^{(A,B)}_{\alpha/\mu}(x) \prod_{\substack{(i,j)\in A\times A \cup B\times B \\ i<j}} \frac{1}{1-x_i x_j} \prod_{(i,j)\in A\times B} (1+x_i x_j) \prod_{i\in B} \frac{1}{1-x_i^2}$$

Reference

[BR] A. Berele and A. Regev, Hook Young Diagrams with Applications to Combinatorics and to Representations of Lie Superalgebras, *Adv. Math.* **64**, (1987) 118–175

[Fo1] Sergey V. Fomin, Finite partially ordered sets and Young tableaux, *Soviet Math. Dokl.* **19**, (1978) 1510–1514

[Fo2] Sergey V. Fomin, Generalized Robinson-Sdhensted-Knuth Correspondence, *preprint*

[Gr] Curtis Greene, An Extension of Schensted's Theorem, *Adv. in Math.* **14**, (1974) 254–265

[GK] Curtis Greene and Daniel J. Kleitman, The Structure of Sperner k-Families, *J.Combi. Theory Ser. A* **20** (1976), 41–68

[Ha] Mark D. Haiman, On Mixed Insertion, Symmetry, and Shifted Young Tableux, *J.Combi. Theory Ser. A* **50** (1989), 196–225

[Mc] I. G. Macdonald, *Symmetric Functions and Hall Polynomials Clarendon Press*, Oxford 1979

[Ok] Soichi Okada, Partially Strict Shifted Plane Partitions, *J.Combi. Theory Ser. A* **53** (1990), 143–156

[Re] Jeffrey B. Remmel, The Combinatorics of (k,l)-Hook Schur Functions *Contemporary Math. Vol.* **34**, (1984), 253–287

[Ro] Thomas W. Roby, Applications and Extensions of Fomin's Generalization of the Robinson-Shensted Correspondence to Differential posets, *Ph.D. thesis, MIT*, 1991

[Sa] Bruce E. Sagan, *The Symmetric Group Representations, Combinatorial Algorithms, and Symmetric Functions*, Wadsworth & Brooks/Cole

[SS] Bruce E. Sagan and Richard P. Stanley, Robinson-Schensted Algorithm for Skew Tableax, *J.Combi. Theory Ser. A* **55** (1990), 161–193

[Sc] M. P. Schützenberger, La correspondence de Robinson, in *Combinatoire et Représentation du Groupe Symetrique,* D. Foata ed., Lecture Notes in Math., Vol. **579**, Springer-Verlag, New York, NY/ (1977), 59–135

[St] Richard. P. Stanley, Unimodality and Lie Superalgebra, *Studies in Applied Math.* **72** (1985), 263–281

[To] G. P. Thomas, On A Construction of Schüzenberger, *Descrete Math.* **17** (1977), 107–118