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# On Nilpotent Blocks of Finite Groups

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Let G be a finite group, F be an algebraically closed field of prime characteristic p and B be a p-block of G. If Q is a p-subgroup of G and b is a p-block of $C_G(Q)$ associated with B, we call (Q, b) a B-Brauer pair. When Q = $\langle\pi\rangle$, we call ($\pi$, b) a B-Brauer element. In Alperin-Broué [1], the method of the p-local theory for finite groups is generalized to a local theory for p-blocks. In the local theory for p-blocks the Brauer pairs correspond to the p-subgroups, and the B-Brauer elements correspond to the p-elements. Between Brauer pairs, "inclusion" is defined, and if for B-Brauer pair (Q, $b_Q$) and B-Brauer element ($\pi$, $b_\pi$), ($\langle\pi\rangle$, $b_\pi$) $\subset$ (Q, $b_Q$) then we write ($\pi$, $b_\pi$) $\in$ (Q, $b_Q$). G naturally acts on the B-Brauer pairs and on the B-Brauer elements, respectively. Maximal B-Brauer pairs are mutually conjugate and they behave like Sylow p-subgroups of G.

On the other hand, Puig introduced so called pointed group theory in [6]. For any subgroup Q of G, put $(FG)^Q = \{a \in FG \mid ax = xa$ for all $x \in Q\}$. A pointed group over FG is any pair $Q_\delta$ where Q is a subgroup of G and $\delta$ is a equivalence class of primitive idempotents of $(FG)^Q$, and we say that $\delta$ is a point of Q over FG. If Q = $\langle u\rangle$, we say that the pair $u_\delta$ is a pointed element. For pointed groups $Q_\delta$ and $P_\varepsilon$ , we write $Q_\delta \subset P_\varepsilon$ and say that $Q_\delta$ is contained in $P_\varepsilon$ if Q $\subset$ P and for any $i \in \varepsilon$ there exists $j \in \delta$ such that $ij = j = ji$. If $Q = \langle u \rangle$, then we write $u_\delta \in P_\varepsilon$ whenever $Q_\delta \subset P_\varepsilon$. We say that a pointed group $Q_\delta$ is local or that $\delta$ is a local point of Q over FG, if Q is a p-subgroup and for $i \in \delta$ , the image of i by the Brauer homomorphism from $(FG)^Q$ to $FC_G(Q)$ does not vanish. Thus there is a bijection between the set of local points of Q over FG and the set of isomorphism classes of irreducible $FC_G(Q)$-modules. G acts on the pointed groups over FG and the maximal local pointed groups contained in $G_{\{B\}}$ are mutually conjugate. (We must add that pointed groups are defined in more general situation).

Now nilpotent blocks were introduced in Broue-Puig [2] : We say that the block B is nilpotent if for any B-Brauer pair $(Q, b_Q)$, $N_G(Q, b_Q)/C_G(Q)$ is a p-group. The structure of nilpotent blocks is described in [2] and Puig [7], and in Külshammer [4] which is recently published. Let (D, b) be a maximal B-Brauer pair and $D_\gamma$ be a maximal local pointed group over FG contained in $G_{\{B\}}$. If B is nilpotent, then FGB is isomorphic to a full matrix algebra over FD from the main theorem in [7]. Moreover, if B is nilpotent, then we have the following.

(I) The number of conjugacy classes of B-Brauer elements is equal to the number of conjugacy classes of D, that is, for B-Brauer elements $(\pi, b_\pi)$, $(\sigma, b_\sigma) \in (D, b)$, if $(\pi, b_\pi)$ and $(\sigma, b_\sigma)$ are conjugate, we have $(\pi, b_\pi) = (\sigma, b_\sigma)^d$ where $d \in D$.

(II) For any B-Brauer pair $(Q, b_Q)$, $l(b_Q) = 1$.

(III) For any local pointed element $u_\varepsilon \in D_\gamma$ and any $x \in G$ such that $(u_\varepsilon)^x \in D_\gamma$ we have $x = zd$ where $z \in C_G(u)$ and $d \in D$.

Here $l(B)$ denote the number of isomorphism classes of irreducible

FG-modules in B. In this report we consider the converse of (I), (II) and (III), respectively.

# I

The converse of (I) is true :

**Theorem 1**. Let B be a p-block of G and (D, b) be a maximal B-Brauer pair. Suppose that for any B-Brauer elements $(\pi, b_\pi)$ and $(\sigma, b_\sigma) \in (D, b)$, if $(\pi, b_\pi)$ and $(\sigma, b_\sigma)$ are conjugate, then we have $(\pi, b_\pi) = (\sigma, b_\sigma)^d$ where $d \in D$. Then B is nilpotent.

Let $B_0$ be the principal p-block of G. By the Frobenius criterion for p-nilpotent groups and the third main theorem on p-blocks, $B_0$ is nilpotent if and only if G is p-nilpotent. Let P be a Sylow p-subgroup of G and suppose that for any $\pi, \sigma \in P$, if $\pi$ and $\sigma$ are conjugate, then we have $\pi = \sigma^d$ where $d \in P$. Then G is p-nilpotent. So Theorem 1 is a kind of generalization of this well known result to p-blocks.

The following is used to prove Theorem 1.

**Lemma 2**. Let Z be a central p-subgroup of G and $\bar{B}$ be the p-block of $\bar{G}$ corresponding to B, where $\bar{G} = G/Z$. Then B is nilpotent if and only if $\bar{B}$ is nilpotent.

*Sketch of proof of Theorem 1*. We prove the theorem by induction on the order $|G|$ of G. Put $Z = Z(D)$ and $b_Z = b^{C_G(Z)}$. We can show by routine argument that $b_Z$ satisfies the assumption of the theorem. Therefore if $C_G(Z) \neq G$, then $b_Z$ is nilpotent.

In the case $C_G(Z) \neq G$. Let $(R, b_R)$ be a B-Brauer pair such that $(R, b_R) \subset (D, b)$ and $(R, b_R)$ is extremal in $(D, b)$ and that R is a defect group of $b_R$ as a p-block of $RC_G(R)$. Then by [1, Proposition 4.4], $R \supset Z$ and $b_R^{C_G(Z)} = b_Z$. So $(R, b_R)$ can be seen as a $b_Z$-Brauer pair. Furthermore for $\sigma \in Z$ and $t \in N_G(R, b_R)$, $(\sigma, b_\sigma) \in (R, b_R)$ and $(\sigma, b_\sigma)^t \in (R, b_R) \subset (D, b)$. So by the assumption $N_G(R, b_R) \subset C_G(Z)$, and hence $N_G(R, b_R)/C_G(R)$ is a p-group, because $b_Z$ is nilpotent. Next let $(Q, b_Q)$ be an arbitrary B-Brauer pair and S a defect group of $b_Q$ regarded as a p-block of $QC_G(Q)$ and $(S, b_S)$ be a maximal $b_Q$-Brauer pair. By replacing $(Q, b_Q)$ by its conjugate if necessary, we may assume $(S, b_S) \subset (D, b)$ and $(S, b_S)$ is extremal in $(D, b)$. As is shown in just above, $N_G(S, b_S)/C_G(S)$ is a p-group. Since $N_G(Q, b_Q) \subset N_G(S, b_S)C_G(Q)$, $N_G(Q, b_Q)/C_G(Q)$ is a p-group.

In the case $C_G(Z) = G$. Put $\bar{G} = G/Z$ and $\bar{B}$ be the p-block of $\bar{G}$ corresponding to B. We can show that $\bar{B}$ satisfies the assumption of the theorem. Since we may assume that $D \neq 1$, $\bar{B}$ is nilpotent by the induction hypothesis. Therefore by Lemma 2, B is nilpotent. Thus the proof is complete.

## II

In Puig [7, 1.9] he says that the converse of (II) is probably true. A partial answer for his question is given.

**Proposition 3.** Let G be a p-solvable group and B be a p-block of G. If $l(b_Q) = 1$ for any B-Brauer pair $(Q, b_Q)$, then B is nilpotent.

We can show that if Alperin's weight conjecture is true, then the converse of (II) is true. For a p-block B and a p-subgroup Q of G, denote by $Bl(N_G(Q), B)$ the set of p-blocks of $N_G(Q)$ associated with B and denote by $l_B(Q)$ the number of isomorphism classes of irreducible FG-modules in B with vertex Q. By Okuyama [5, Theorem], Alperin's conjecture is true for p-solvable groups as follows.

**Lemma 4.** Let G be a p-solvable group and B be a p-block of G. Then the following hold.

(i) We have $l(B) = \sum_Q \sum_{b \in Bl(N_G(Q),B)} l_b(Q)$, where Q ranges over a complete set of representaives for the conjugacy classes of p-subgroups of G.

(ii) Let P be a normal p-subgroup of G. then we have $l(B) = \sum_Q \sum_{b \in Bl(N_G(Q),B)} l_b(Q)$, where Q ranges over a complete set of representatives for the conjugacy classes of p-subgroups of G containing P.

## III

The converse of (III) is true :

**Theorem 5.** Let B be a p-block of G and $D_\gamma$ be a maximal local pointed group over FG contained in $G_{(B)}$. Suppose that for any local pointed element $u_\varepsilon \in D_\gamma$ and any element $x \in G$ such that $(u_\varepsilon)^x \in D_\gamma$ we have $x = zd$ where $z \in C_G(u)$ and $d \in D$ . Then B is nilpotent.

The above result is due to L. Puig. He proves it by the same argument as in the proof of Theorem 1. But in this time , $\varepsilon$ is no longer uniquely determined by $\gamma$ and u, and hence its proof is more complicated than Brauer element case. In particular it is not easy to show that $\bar{B}$ satisfies the assumption. Section 3 in Kulshammer-Puig [3] satisfactorily meets the requirement.

The detail version of this report will probably be published elsewhere.

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