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Group extensions and cohomology

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1. Introduction.

Let G be a group (not necessary finite), and M a left $\mathbb{Z}G$-module. For $n \geq 0$, the n-th cohomology group $\mathrm{H}^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$ of M is defined as the n-th homology of $\mathrm{Hom}_{\mathbb{Z}G}(P_*, M)$, where P_* is a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. As well-known, there are some interpretations for low dimensional cohomology groups. By taking the Bar resolution as a projective resolution of $\mathbb{Z}$, $\mathrm{H}^1(G, M)$ is isomorphic to the group of derivations from G to M modulo principal derivations. However a derivation defines a splitting monomorphism from G into the fixed semidirect product of M by G. Hence $\mathrm{H}^1(G, M)$ is also bijective to the set of G-conjugacy classes of semidirect products of M by G. By the same way, $\mathrm{H}^2(G, M)$ is isomorphic to the group of factor sets modulo principal factor sets. It is also bijective to the set of equivalent classes of extensions of M by G in which the conjugate action of G on M is the given one. The latter becomes an abelian group by a certain sum, called Baer sum. So this bijection is an isomorphism.

In this report, an exact sequence

$$0 \to A \to B_{n-1} \to B_{n-2} \to \cdots \to B_1 \to B_0 \to \mathrm{C} \to 0$$

is said to *start* at A, *end* at C, and have *length* n. Since an extension of M by G is a short exact sequence $0 \to M \to E \to G \to 1$, elements of $\mathrm{H}^2(G,M)$ have length 1. A semidirect product may be regarded as an exact sequence of length 0. Although third cohomology has an interpretation in terms of obstructions to the construction of extensions by G of a non-abelian kernel, it can not be straightforward regarded as such exact sequences.

On the other hand, there is Yoneda's nice interpretation for cohomology. Elements of $\mathrm{H}^n(G,M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$ correspond to equivalent classes of exact sequences of left $\mathbb{Z}G$-modules, which have length n, start at M and end at $\mathbb{Z}$. Their sum is Baer sum. Moreover cup product corresponds to connecting two sequences. So the connecting homomorphisms which appear in the cohomology long exact sequence for a short exact sequence ζ of $\mathbb{Z}G$-modules, correspond just to connecting ζ. Since the existence of cohomology long exact sequences is a basic tool for methods of dimension shifting, it is sensitive to give such a clear image.

The interpretation as semidirect products or group extensions is similar to Yoneda's one, although G itself appears, and lengths decrease in the former. Then, can it be generalized for $n \geq 3$? The answer is yes. It was discovered by several people simultaneously, but independently. In the report we study this interpretation, i.e. $\mathrm{H}^n(G,M)$ is isomorphic to some equivalent classes of exact sequences of groups (say, crossed extensions) which have length $n-1$, start at M and end at G. However the result seems not to be the best style, because crossed extensions are not enough general forms. They still have many functorial properties like Yoneda's interpretation. For example, suppose $H \leq G$, and $\imath : H \to G$ be the inclusion map. Let $\imath^\sharp : \mathrm{H}^n(G, M) \to \mathrm{H}^n(H, M)$ be the natural map constructed by taking the pullback of a crossed extension with $\imath$ as in Yoneda's interpretation. Then $\imath^\sharp$ coincides with the restriction map. At this point of view, it is very interesting to make something like the cohomology long exact sequences for extensions of groups. If we could make them, we might use the methods of dimension shifting rather than spectral sequences. In section 4, Ratcliffe's result is introduced, which is related to $n = 3$ terms of such cohomology long exact sequences. Unfortunately, the author does not generalize his results yet. The fundamental concepts of them seem to lie in Rinehart's abstruct argument [**8**].

Historical note and references can be found in [**6**].

Notations. In this paper we treat exact sequences in several categories, i.e. in groups, in $\mathbb{Z}G$-modules, etc. The following notations are commonly used in them.

For morphisms $A \to B$ and $C \to B$, $A \times_B C$ denotes the pullback of them. Similarly for $B \to A$ and $B \to C$, $A \amalg_B C$ denotes the pushout of them.

Let $\alpha : 1 \to X \to A_{r-1} \to A_{r-2} \to \cdots \to A_0 \to Y \to 1$, $\beta : 1 \to X \to B_{r-1} \to B_{r-2} \to \cdots \to B_0 \to Y \to 1$ be exact sequences. We write $\alpha \sim \beta$ if there is a chain map (i.e. a family of maps which makes the below diagram commutative)

$$\begin{array}{ccccccccccccccc} 1 & \longrightarrow & X & \longrightarrow & A_{r-1} & \longrightarrow & A_{r-2} & \longrightarrow & \cdots & \longrightarrow & A_0 & \longrightarrow & Y & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\ 1 & \longrightarrow & X & \longrightarrow & B_{r-1} & \longrightarrow & B_{r-2} & \longrightarrow & \cdots & \longrightarrow & B_0 & \longrightarrow & Y & \longrightarrow & 1 \end{array}$$

which is the identity on X and Y. Then $\sim$ generates an equivalent relation in those exact sequences. We will adopt this equivalent relation in the report unless stated.

2. Yoneda's interpretation.

In this section we recall Yoneda's interpretation, since it is the basic model in the report.

Let L, M be left $\mathbb{Z}G$-modules. The definition of $\mathrm{Ext}^n_{\mathbb{Z}G}(L, M)$ is the value at L of the left derived functor of the additive left exact functor $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ from the category of left $\mathbb{Z}G$-modules to the category of abelian groups. Namely, let

$$(2.1) \qquad \cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} L \longrightarrow 0$$

be a $\mathbb{Z}G$-projective resolution of L. For $n \geq 1$, $\mathrm{Ext}^n_{\mathbb{Z}G}(L, M)$ is defined as the n-th homology group of $\mathrm{Hom}_{\mathbb{Z}G}(P_*, M)$, i.e. $\mathrm{Ker}\ \partial^\sharp_{n+1}/\mathrm{Im}\ \partial^\sharp_n$, where $\partial^\sharp_n$ is the natural map $\mathrm{Hom}_{\mathbb{Z}G}(P_{n-1}, M) \to \mathrm{Hom}_{\mathbb{Z}G}(P_n, M)$. For $n = 0$, $\mathrm{Ext}^0_{\mathbb{Z}G}(L, M)$ is defined as $\mathrm{Hom}_{\mathbb{Z}G}(L, M)$. We remark that a projective resolution (2.1) has two important properties,

(1) for any exact sequences $\cdots \to B_1 \to B_0 \to N \to 0$ of $\mathbb{Z}G$-modules, and for any $\mathbb{Z}G$-homomorphism $f : L \to N$, there is a chain map $P_* \to B_*$ whose last term is f,

(2) such chain maps are homotopic.

These properties imply that the cohomology groups are independent of the choice of projective resolutions.

Yoneda's interpretation is as follows. Consider an exact sequence

$$0 \to M \to B_{n-1} \to B_{n-2} \to \cdots \to B_1 \to B_0 \to L \to 0$$

of $\mathbb{Z}G$-modules of length n. By the above, there is a chain map

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, "\mu"'] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & B_{n-1} \arrow[r] & B_{n-2} \arrow[r] & \cdots \arrow[r] & B_0 \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

Then μ is coycle (i.e. $\mu \in \mathrm{Ker}\ \partial^\sharp_{n+1}$). Conversely let $\mu \in \mathrm{Ker}\ \partial^\sharp_{n+1}$. Then μ is regarded as a map to M from $\Omega^n = \mathrm{Ker}\ \partial_{n-1} \simeq P_n/\mathrm{Im}\ \partial_{n+1}$. So we can construct an exact sequence as

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n \arrow[r] \arrow[d, "\mu"] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & L \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] & M \amalg_{\Omega^n} P_{n-1} \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

This correspondence implies that $\mathrm{Ext}^n_{\mathbb{Z}G}(L, M)$ is bijective to the set of equivalent classes of exact sequences which have length n, start at M and end at L.

The bijection induces sum and product between exact sequences. For $\alpha : 0 \to M \to A_{r-1} \to \cdots \to A_0 \to L \to 0$ and $\beta : 0 \to M \to B_{r-1} \to \cdots \to B_0 \to L \to 0$, their sum $\alpha + \beta$ is

$$0 \to M \to A_{r-1} \amalg_M B_{r-1} \to A_{r-2} \oplus B_{r-2} \to \cdots \to A_1 \oplus B_1 \to A_0 \times_L B_0 \to L \to 0.$$

This is called Baer sum. For $\gamma : 0 \to N \to C_{s-1} \to \cdots \to C_0 \to M \to 0$ and $\beta : 0 \to M \to B_{r-1} \to \cdots \to B_0 \to L \to 0$, their composition (or inner cup) product $\gamma\beta$ is

$$0 \to N \to C_{s-1} \to \cdots \to C_0 \to B_{r-1} \to \cdots \to B_0 \to L \to 0,$$

i.e. connecting them. It is called Yoneda splice.

Functorial properties of Ext are interpreted as follows. Let N be a $\mathbb{Z}G$-module having a projective resolution $\cdots \to Q_1 \to Q_0 \to N \to 0$. Suppose $\mathbb{Z}G$-homomorphism $f : N \to L$ is given. Then f induces a chain map $f^* : Q_* \to P_*$, and $f^{*\sharp} : \mathrm{Hom}_{\mathbb{Z}G}(P_*, M) \to \mathrm{Hom}_{\mathbb{Z}G}(Q_*, M)$. Hence f induces a homomorphism $f^\sharp : \mathrm{Ext}^n_{\mathbb{Z}G}(L, M) \to \mathrm{Ext}^n_{\mathbb{Z}G}(N, M)$. In terms of exact sequences, only using a pull-back, it is

$$\begin{array}{ccccccccccccccc}
f^\sharp(\beta): & 0 & \longrightarrow & M & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0 \times_L N & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & & & & & & & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\beta\ : & 0 & \longrightarrow & M & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & L & \longrightarrow & 0.
\end{array}$$

Similarly, for $f: M \to N$, the induced map $f_\sharp : \mathrm{Ext}^n_{\mathbb{Z}G}(L, M) \to \mathrm{Ext}^n_{\mathbb{Z}G}(L, N)$ is

$$\begin{array}{lcccccccccccccc}
\beta & : 0 & \longrightarrow & M & \longrightarrow & B_{n-1} & \longrightarrow & B_{n-2} & \longrightarrow & \cdots & \longrightarrow & B_0 & \longrightarrow & L \longrightarrow & 0 \\
 & & & {\scriptstyle f}\downarrow & & \downarrow & & & & & & & & & \\
f_\sharp(\beta) & : 0 & \longrightarrow & N & \longrightarrow & N \amalg_M B_{n-1} & \longrightarrow & B_{n-2} & \longrightarrow & \cdots & \longrightarrow & B_0 & \longrightarrow & L \longrightarrow & 0.
\end{array}$$

Let $\zeta : 0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a short exact sequence. In the cohomology long exact sequence

$$\cdots \xrightarrow{\delta} \mathrm{Ext}^n_{\mathbb{Z}G}(L, M_1) \xrightarrow{f_\sharp} \mathrm{Ext}^n_{\mathbb{Z}G}(L, M_2) \xrightarrow{g_\sharp} \mathrm{Ext}^n_{\mathbb{Z}G}(L, M_3) \xrightarrow{\delta} \mathrm{Ext}^{n+1}_{\mathbb{Z}G}(L, M_1) \xrightarrow{f_\sharp} \cdots,$$

the connecting homomorphism δ is just the multiplication by ζ on the left. So cohomology long exact sequences can be naturally interpreted by this aspect.

3. Crossed extensions.

We fix a group G, and a left $\mathbb{Z}G$-module M. Next is the main theorem.

THEOREM. *$H^{n+1}(G, M) \simeq XExt^n(G, M)$ for $n \geq 1$.*

XExt is the group of crossed extensions (see below). It was discovered independently by Holt [**3**], Huebschmann [**4**], Hill [**1**](without proof), and for $n = 3$ case, by Ratcliffe [**7**], Leedham-Green and MacKay [**5**], Wu [**9**]. Our proof mainly follows Huebschmann's method, but it is reduced to somewhat simpler case with Holt's lemma to understand the decrease of length more naturally.

To state the definition of crossed extensions, we introduce Whitehead's crossed modules. A homomorphism $\delta : C_1 \to C_0$ of groups is a (C_0-)*crossed module* if

(1) C_1 is a C_0-group, i.e. C_0 acts on C_1 as group automorphisms,

(2) δ is a C_0-homomorphism, i.e. $\delta({}^a x) = a\delta(x)a^{-1}$ for $a \in C_0, x \in C_1$,

(3) $yxy^{-1} = {}^{\delta(y)}x$ for $x, y \in C_1$.

Note that Ker δ is a $\mathbb{Z}G$-module lying the center of C_1 (C_1 is not necessary abelian). For example, an inclusion map to a group from a normal subgroup is a crossed module.

An exact sequence of groups

$$0 \longrightarrow M \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} G \longrightarrow 1$$

is a *crossed extension* of length n if

(1) $\delta_1 : C_1 \to C_0$ is a crossed module,

(2) for $i \geq 2$, C_i are $\mathbb{Z}G$-modules, and δ_i are $\mathbb{Z}G$-homomorphisms.

Note that since Im $\delta_2 =$ Ker δ_1 is a $\mathbb{Z}G$-module, it makes sense to require δ_2 to be $\mathbb{Z}G$-linear. For example, $0 \to Z(G) \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$ is a crossed extension of length 2.

Suppose a crossed extension of G and a crossed extension of another group H are given. A *morphism* of them is a chain map of them which preserves all the structure of crossed extensions. Especially, among the crossed extensions starting at M and ending at G, morphisms whose heads and ends are identity maps generate an equivalent relation, as in section 1. $\mathrm{XExt}^n(G,M)$ denotes the set of equivalent classes of crossed extensions of length n. For example, $\mathrm{XExt}^1(G,M)$ is just the equivalent classes of group extensions of M by G. Hence the theorem is true for $n=1$, as well-known.

Holt shows [**3**, Proposition 2.7],

LEMMA 3.1. *Let α be a crossed extension of length 2. Then there is a crossed extension which is equivalent to α, and whose C_1-term is abelian (hence a $\mathbb{Z}G$-module).*

Outline of the proof is as follows. Take a free presentation $1 \to R \to F \to G \to 1$ of G. Then $0 \to M \to M \times R \to F \to G \to 1$ can have a crossed extension structure equivalent to α. Let X denote $M \times R$. Then $0 \to M \to X/[X,X] \to F/\delta[X,X] \to G \to 1$ is the crossed extension as in the lemma.

By the lemma, it is clear that each crossed extension is equivalent to a crossed extension whose C_1-term is abelian. So we consider only such crossed extensions. Then we can show that there is a projective object among them as follows.

Let $1 \to R \to F \to G \to 1$ be a free presentation of G, and $\overline{R} = R/[R,R]$, $\overline{F} = F/[F,F]$. Hence $0 \to \overline{R} \to \overline{F} \to G \to 1$ is a crossed extension of length 1. Next we take a $\mathbb{Z}G$-projective resolution $\cdots \to P_2 \to P_1 \to \overline{R} \to 0$ of $\overline{R}$. Combine them at $\overline{R}$ as

$$\pi : \quad \cdots \to P_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to P_0 \to G \to 1.$$

where $P_0 = \overline{F}$. Then

LEMMA 3.2. *For any crossed extension* $\alpha : 0 \to N \to D_{n-1} \to \cdots \to D_1 \to D_0 \to H \to 1$ *of a group* H *in which* D_1 *is abelian, and for any group homomorphism* $f : G \to H$, *there is a morphism from* π *to* α *whose last term is* f. *Moreover such morphisms are homotopic.*

This is easily shown like the module case (section 2), from the above natural construction.

We prove the theorem. By Lemma 3.1, $\mathrm{XExt}^n(G, M)$ is bijective to the set of equivalent classes of crossed extensions of length n, whose C_1-term is abelian. However, we have Lemma 3.2. So it is bijective to the n-th homology group $\mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}G}(P_*, M))$ by the same argument as Yoneda's interpretation. Hence it is sufficient to show $\mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}G}(P_*, M)) = \mathrm{H}^{n+1}(G, M)$. But there is the *Gruenburg resolution*

$$0 \to \overline{R} \to \mathbb{Z}G \otimes_{\mathbb{Z}F} I_F \to \mathbb{Z}G \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}G \otimes_{\mathbb{Z}F} I_F$ is $\mathbb{Z}G$-projective (I_F is the augumentaion ideal of $\mathbb{Z}F$). Therefore, we can construct a $\mathbb{Z}G$-projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to \mathbb{Z}G \otimes_{\mathbb{Z}F} I_F \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$. This proves the bijection of the theorem.

By Lemma 3.2, we can treat crossed extensions like exact sequences of modules. Let $f : H \to G$ be a group homomorphism. Then $\mathbb{Z}G$-modules are $\mathbb{Z}H$-modules via f.

The induced map is

$$\begin{array}{llllllllllllllll}
f^\sharp(\beta): & 0 & \longrightarrow & M & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & C_0 \times_G H & \longrightarrow & H & \longrightarrow & 0 \\
 & & & & & & & & & & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\beta \quad : & 0 & \longrightarrow & M & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & G & \longrightarrow & 0,
\end{array}$$

in terms of crossed extensions. If f is monic, then it is just the restriction map. If f is epic, then it is just the inflation map.

Like modules, we can define the sum of two crossed extensions as Baer sum. It makes the bijection of the theorem an isomorphism. However, it may not be easy to define their products. It is also difficult to construct something like the cohomology long exact sequence, for a *group extension*, especially difficult near the connecting homomorphisms. It is still possible for $n = 3$ term which is discussed in the next section.

We note Holt's method. He showed

(1) Baer sum in $\mathrm{XExt}^n(G, M)$ is well-defined.

(2) XExt^n is a bi-functor to abelian groups.

(3) $\mathrm{XExt}^*(G, -)$ has cohomology long exact sequences for short exact sequences of modules.

(4) $\mathrm{XExt}^n(G, I) = 0$ for any injective module I.

Hence we can use the dimension shifting. Since $\mathrm{XExt}^1(G, M) \simeq \mathrm{H}^2(G, M)$, we get result.

4. Cohomology long exact sequences.

Ratcliffe [7] shows an interesting result for lower degrees. We fix a group extension $1 \to N \to G \to Q \to 1$ and a left $\mathbb{Z}Q$-module M. Then M is a $\mathbb{Z}G$-module on which N acts trivially. An exact sequence $0 \to M \to C \to N \to 1$ of groups is called a *G-crossed extension* if the induced map $C \to G$ is a crossed module. Morphisms preserving the structure generate an equivalent relation among G-crossed extensions. $\mathrm{H}^2_G(N, M)$ denotes the set of equivalent classes of G-crossed extensions. Note that the natural map $\imath : \mathrm{H}^2_G(N, M) \to \mathrm{H}^2(N, M)$ may not be monic. Ker $\imath$ consists of the classes of extensions which split as groups, but not necessary split as G-groups.

PROPOSITION. *There is an exact sequence*

$$H^2(Q, M) \xrightarrow{\inf} H^2(G, M) \xrightarrow{\rho} H^2_G(N, M) \xrightarrow{\delta} H^3(Q, M) \xrightarrow{\inf} H^3(G, M).$$

Here inf are the inflation maps, and $\imath\rho$ coincides with the restriction map. Moreover δ is just Yoneda splice, i.e. for $\alpha : 0 \to M \to C \to N \to 1$, $\delta(\alpha)$ is $0 \to M \to C \to G \to Q \to 1$. If we could generalize it for higher degrees, we might more roughly and easily replace groups — the first variant of cohomology — than the use of the spectral sequences. However, the author can not do it yet. Ratcliffe's proof uses a generalized concept of factor sets, say *factor systems.* Its calculating method may not be suitable for higher degrees.

We introduce the Rinehart's foresighted theory [8], without proofs, to conclude the report. Although he discussed on very general categories, we concentrate only our situation.

Let G be a group, and $\mathfrak{C}$ =(Groups,G) the category of groups over G, i.e. the category of group homomorphisms into G. $|\mathfrak{C}|$ denotes the class of its objects. We write only $A \in |\mathfrak{C}|$ for $(A \to G) \in |\mathfrak{C}|$ if no confusion. $\mathfrak{C}(A, B)$ denotes the morphisms from A to B. $P \in |\mathfrak{C}|$ is called projective in case $\mathfrak{C}(P, f)$ is surjective for every epimorphism in $\mathfrak{C}$, namely P is a free group. $\mathcal{P}$ denotes the class of projectives.

Let $\mathfrak{A}$ be the dual of the category of abelian groups. $Z \in |\mathfrak{C}|$ is called an abelian group in $\mathfrak{C}$ if $\mathfrak{C}(-, Z)$ factorize through $\mathfrak{A}$, that is, Z is a semidirect product of a $\mathbb{Z}G$-module M by G. So we write it only M. In this case, $\mathfrak{C}(-, Z)$ is the derivations $\mathrm{Der}(-, M)$. $\mathcal{R}$ denotes the full subcategory of the functor category $\mathfrak{A}^{\mathfrak{C}}$ whose objects are right exact functors. It can be shown that $\mathcal{R}$ is an abelian category and enough projectives. $\mathfrak{C}(-, Z) = \mathrm{Der}(-, M)$ is in $\mathcal{R}$.

For $n \geq 0$, let $\mathrm{S}_n : \mathcal{R} \to \mathfrak{A}^{\mathfrak{C}}$ be the n-th left derived functor of the inclusion functor $\mathcal{R} \to \mathfrak{A}^{\mathfrak{C}}$. Define $\mathrm{Ext}^n_{\mathfrak{C}}(B, M) = (\mathrm{S}_n\mathrm{Der}(-, M))(B)$ for $B \in |\mathfrak{C}|$.

Theorem 4.1. *$Ext^n_{\mathfrak{C}}(B, M) = H^{n+1}(B, M)$, the usual cohomology group.*

Its form is similar to our main theorem of section 3.

Let $\mathfrak{C}'$ be the full subcategory of the morphism category of $\mathfrak{C}$ whose objects are epimorphisms. For $i = 0, 1$, $\Gamma_i : \mathfrak{C}' \to \mathfrak{C}$ is the functor defined by $\Gamma_i(A_0 \to A_1) = A_i$. $\mathcal{R}' = \mathcal{R}(\mathfrak{C}', \mathfrak{A})$ denotes the category of right exact functors from $\mathfrak{C}'$ to $\mathfrak{A}$. Let $\Delta : \mathcal{R} \to \mathcal{R}'$ be the functor defined by $\Delta F = \mathrm{Ker}_{\mathcal{R}'}(F\Gamma_0 \to F\Gamma_1)$. So we can define $\mathrm{S}'^n \Delta F$

where S'^n is the n-th left derived functor of the inclusion functor $\mathcal{R}' \to \mathfrak{A}^{\mathfrak{C}'}$.

THEOREM 4.2. *For every $F \in \mathcal{R}$, and for every epimorphism $A \to B$ in $\mathfrak{C}$, there is an exact sequence*

$$\cdots \to S_{n+1}F(B) \to S_n\Delta F(A \to B) \to S_nF(A) \to S_nF(B) \to \cdots .$$

Combining Theorem 4.1, this is just the cohomology long exact sequence for a group extension. We hope that it will be understood in terms of Yoneda's interpretation.

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