

ON SYMMETRIC ALGEBRAS

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Throughout this note, all algebras and modules are finite dimensional over an algebraically closed field  $K$ .

An algebra  $\Lambda$  is said to be symmetric if the regular module  $\Lambda$  is isomorphic to its dual  $D(\Lambda)$  as a bi- $\Lambda$ -module, where  $D = \text{Hom}(-, K)$ . It is well known that any block ideal of  $KG$ , the group algebra of a finite group  $G$ , is symmetric. Therefore, it seems interesting to know the way of constructing symmetric algebras from ring theoretical view point.

The trivial extension algebra  $A \ltimes D(A)$ , for any algebra  $A$ , is symmetric. This is also well known and the trivial extension algebras  $A \ltimes D(A)$  have been used for studying representation-finite self-injective algebras by many authors. Our construction is given by generalizing trivial extension algebras.

We start by giving the definition of nilpotent Morita context.

We call a linear map  $\phi: A^{\text{N} \otimes A^{\text{N}}} \rightarrow A^{\text{N}}_A$  a (generalized) Morita

context if  $\phi$  is associative, i.e.,  $\phi(\phi(n \otimes n') \otimes n'') = \phi(n \otimes \phi(n' \otimes n''))$ .

It is easy to see that  $B = A \oplus N$  has an algebra-structure by  $(a, n) \cdot (a', n') = (a \cdot a', a \cdot n' + n \cdot a' + \phi(n \otimes n'))$ . If  $\phi = 0$  then

$B$  is the same with  $A \otimes N$ . In the algebra  $B$ ,  $N$  is an ideal.

We call  $\phi$  nilpotent if the ideal  $N$  is nilpotent. In this

case we have  $J(B) = J(A) \oplus N$ .

In the case  ${}_A N_A = {}_A M_A \oplus {}_A S_A$  and  $\phi$  is given by

$\phi((m, s) \otimes (m', s')) = (\varphi(m \otimes m'), \psi(m \otimes m'))$ , where  $\varphi: {}_A M \otimes {}_A M_A \rightarrow {}_A M_A$  and

$\psi: {}_A M \otimes {}_A M_A \rightarrow {}_A S_A$ , we denote  $\phi = (\varphi, \psi)$ . Then we know that  $(\varphi, \psi)$

is a (nilpotent) Morita context if and only if 1)  $\varphi$  is a

(nilpotent) Morita context and 2)  $\psi(\varphi(m \otimes m') \otimes m'') = \psi(m \otimes \varphi(m' \otimes m''))$ .

In the case  $(\varphi, \psi)$  is a nilpotent Morita context, we denote the algebra  $A \oplus M \oplus S$  with the multiplication  $(a, m, s) \cdot (a', m', s')$

$= (a \cdot a', a \cdot m' + m \cdot a' + \varphi(m \otimes m'), a \cdot s' + s \cdot a' + \psi(m \otimes m'))$  by  $\Lambda(\varphi, \psi)$ .

Now, by using nilpotent Morita contexts, we define QF-systems.

We call  $(\varphi, \gamma, \theta, f)$  a QF-system if 1)  $A$  is an algebra and  ${}_A M_A$

is a bimodule, 2)  $\varphi: {}_A M \otimes {}_A M_A \rightarrow {}_A M_A$  is a nilpotent Morita context,

3)  $\gamma: {}_M A \rightarrow D(M)_A$  is an isomorphism with the property  $\gamma(m)(\varphi(m' \otimes m''))$

$= \gamma(\varphi(m \otimes m'))(m'')$  and 4)  $\theta$  is an algebra automorphism of  $A$  and

$f \in D(\text{Im } \varphi)$  such that  $(\gamma(am) - \theta(a)\gamma(m))(m') = (fa - \theta(a)f)(\varphi(m \otimes m'))$ .

Let us denote by  $\theta D(A)_A$  the bimodule  $D(A)$  defined as the

following manner:  $(a \cdot q)(a') = q(a' \cdot \theta(a))$ ,  $(q \cdot a)(a') = q(a \cdot a')$  for

$a, a' \in A$  and  $q \in D(A)$ . Then, by defining  $\psi(m \otimes m')(a) =$

$\chi(m)(m' \cdot a) - f(\varphi(m \otimes m' \cdot a))$ , we get a nilpotent Morita context

$(\varphi, \psi)$  on the bimodule  ${}_A M_A \oplus \theta^{D(A)}_A$ . We will denote the algebra

$\Lambda(\varphi, \psi)$  by  $\Lambda(\varphi, \chi, \theta, f)$ .

Theorem 1. For any QF-system  $(\varphi, \chi, \theta, f)$ , the algebra  $\Lambda(\varphi, \chi, \theta, f)$  is Frobenius, i.e.,  $\Lambda(\varphi, \chi, \theta, f)$  is isomorphic to its dual as a one-sided module.

Theorem 2. Assume  $\Lambda$  is basic, indecomposable and self-injective. If  $\Lambda$  is not isomorphic to  $K$ , there exists a QF-system  $(\varphi, \chi, \theta, f)$  and  $\Lambda \cong \Lambda(\varphi, \chi, \theta, f)$ .

We call a QF-system  $(\varphi, \chi, \text{id}_A, 0)$  a symmetric QF-system (or SQF-system for short) if  $\chi$  is symmetric, i.e.,  $\chi(m)(m') = \chi(m')(m)$ . Precisely describing,  $(\varphi, \chi)$  is an SQF-system if

- 1)  $A$  is an algebra and  ${}_A M_A$  is a bimodule,
- 2)  $\varphi$  is a nilpotent Morita context defined on  ${}_A M_A$  and
- 3)  $\chi: {}_A M_A \rightarrow {}_A^{D(M)}_A$  is an isomorphism with the properties  $\chi(m)(m') = \chi(m')(m)$  and  $\chi(\varphi(m \otimes m'))(m'') = \chi(m)(\varphi(m' \otimes m''))$ .

Corresponding to the above results, we have

Theorem 3. For any SQF-system  $(\varphi, \chi)$ , the algebra  $\Lambda(\varphi, \chi)$  is symmetric.

Theorem 4. Assume  $\Lambda$  is basic, indecomposable and symmetric. If  $\Lambda$  is not isomorphic to  $K$ , there exists an SQF-system  $(\varphi, \chi)$  and  $\Lambda \cong \Lambda(\varphi, \chi)$ .

Now, let  $(\varphi, \chi)$  be an SQF-system and  $P_A$  a progenerator with  $B = \text{End}(P_A)$ . Then, it is easy to see that, on the modules  ${}_B M^*_B = {}_B P \otimes_A M \otimes_A P^*_B$  and  ${}_B D(B)_B = {}_B P \otimes_A D(A) \otimes_A P^*_B$ , where  ${}_A P^*_B = \text{Hom}(P_A, A_A)$ , we have an SQF-system  $(\varphi^*, \chi^*)$  defined by

$$\varphi^*(p \otimes m \otimes h \otimes p' \otimes m' \otimes h') = p \otimes \varphi(m \otimes h(p') \cdot m') \otimes h'$$

$$\chi^*(p \otimes m \otimes h)(p' \otimes m' \otimes h') = \chi(m)(h(p') \cdot m' \cdot h'(p))$$

for  $m, m' \in M$ ,  $p, p' \in P$  and  $h, h' \in P^*$ . Further, it is checked that  $\Lambda(\varphi^*, \chi^*) \cong \text{End}({}_B P \otimes_A \Lambda(\varphi, \chi) \otimes_A P^*_B)$ . Therefore, we have

Corollary 5. For any symmetric algebra  $\Lambda$ , there exists an SQF-system  $(\varphi, \chi)$  and  $\Lambda \cong \Lambda(\varphi, \chi) \times S$ , where  $S$  is a product of full matrix algebras (= the semi-simple part of  $\Lambda$ ).

By the above corollary, we know that we have to study the way of constructing SQF-systems, in order to get symmetric algebras.

Here, we list some constructions of SQF-systems:

(Construction I) Let  $(\varphi_i, \chi_i)$  be SQF-systems. Then, the direct sum  $\bigoplus_i (\varphi_i, \chi_i)$  is again an SQF-system.

(Construction II) Let  $\phi: {}_A I \otimes_A I_A \rightarrow {}_A I_A$  be a nilpotent Morita context. Then, by putting  ${}_A M_A = {}_A I_A \oplus {}_A D(I)_A$  and

$$\varphi((x, q) \otimes (x', q')) = (\phi(x \otimes x'), q'(\phi(- \otimes x)) + q(\phi(x' \otimes -)))$$

$$\chi((x, q) \otimes (x', q')) = q'(x) + q(x') \quad \text{for } x, x' \in I \text{ and } q, q' \in D(I)$$

we have an SQF-system  $(\varphi, \chi)$ . We call this system the trivial extension of  $\phi$  and denote it by  $\phi \ltimes_D(\phi)$ . If  ${}_A I_A$  is a nilpotent ideal of  $A$  and  $\phi$  is given by  $\phi(x \otimes y) = x \cdot y$ , the multiplication in  $A$ , we denote more simply by  $I \ltimes_D(I)$ .

(Construction III) Let  $(\varphi_0, \chi_0)$  be an SQF-system defined on a bimodule  ${}_A X_A$  and  $G$  a finite group. We put  ${}_A M_A = \bigoplus_{g \in G} X^{(g)}$ ,  $\varphi = \varphi_0 : {}_A X^{(g)} \otimes_A X^{(h)} \rightarrow {}_A X^{(g \cdot h)}$  and  $\chi = \chi_0 : {}_A X^{(g)} \rightarrow {}_A D(X^{(g^{-1})})_A$ . It is easy to see that  $(\varphi, \chi)$  is an SQF-system defined on  ${}_A M_A$ .

(Construction IV) Let  $E$  be a symmetric algebra with an isomorphism  $d: {}_E E_E \rightarrow {}_E D(E)_E$ . Assume there is an algebra map  $\xi: A \rightarrow E$ . Then, putting  ${}_A M_A = \bigoplus_{i=1}^{k-1} E^{(i)}$ ,  $\varphi: {}_A E^{(i)} \otimes_A E^{(j)} \rightarrow {}_A E^{(i+j)}$  for  $i+j \leq k-1$  and  $\chi = d: {}_A E^{(i)} \rightarrow {}_A D(E^{(k-i)})_A$ , we have an SQF-system  $(\varphi, \chi)$ . We denote the

algebra  $\Lambda(\varphi, \chi)$  by  $\Lambda_k(A, \xi)$ . For any module  $V_A$ , we may take  $\text{End}({}_K V)$  (= a full matrix algebra, and, therefore, symmetric) as  $E$  and  $\xi_V =$  the representation of  $V_A$  as  $\xi$ . In this case, the corresponding algebra will be denoted by  $\Lambda_k(A, V_A)$ .

(Construction V) Let  $(\varphi_X, \chi_X)$  be an SQF-system defined on

a bimodule  ${}_A X_A$ . Let  $B = A \oplus X$  be the algebra defined by the multiplication  $(a, x) \cdot (a', x') = (a \cdot a', a \cdot x' + x \cdot a' + \varphi_X(x \otimes x'))$ .

Assume  $(\varphi_Y, \chi_Y)$  is an SQF-system defined on a bimodule  ${}_B Y_B$ .

Then, on the bimodule  ${}_A X_A \oplus {}_A Y_A$ , we can define an SQF-system

$(\varphi, \chi)$  as follows:

$$\varphi((x, y) \otimes (x', y')) = (\varphi_X(x \otimes x') + \chi_X^{-1}(\chi_Y(y)(y')), x \cdot y' + y \cdot x' + \varphi_Y(y \otimes y')),$$

$$\chi((x, y)((x', y'))) = \chi_X(x)(x') + \chi_Y(y)(y').$$