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Modular representations and quantum algebras
—— after G.Lusztig, and H.H.Andersen, P.Polo and Wen K.

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"Usually, the representation theory of [finite Chevalley groups] is approached via that of the ambient algebraic groups. Many results are of the form: If we know some data for the algebraic group, then we know (in principle) some data for the finite group," J.C.Jantzen [JA], p.127.

For example, if $\mathfrak{G}_{\mathbb{Z}}$ is a split semisimple simply connected $\mathbb{Z}$-group (scheme) and q a power of a prime p, then $\mathfrak{G}_{\mathbb{Z}}(\mathbb{F}_q)$ is a universal Chevalley group and all the irreducible $\mathfrak{G}_{\mathbb{Z}}(\mathbb{F}_q)$-modules over $\mathbb{F}_q$ are obtained as the restriction of certain $\mathfrak{G}$-simples, where we put $\mathfrak{G} = \mathfrak{G}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p$. We do not know, however, even the dimension of simple $\mathfrak{G}$-modules. Recently, G.Lusztig has proposed a novel program to attack the problem using the quantized enveloping algebras, or quantum algebras for short, discovered independently by V.G.Drinfeld and Jimbo M.

The purpose of this note is to convey some of Lusztig's ideas, and subsequent developments by H.H.Andersen, P.Polo and Wen K. There is an excellent introduction to the subject by Lusztig himself [LI], which has helped us organize the present survey.

1° Definition of the quantum algebras

(1.1) Let $[\![a_{ij}]\!]_{1\le i,j\le n}$ be a Cartan matrix: $a_{ii} = 2\ \forall i$, $-a_{ij} \in \mathbb{N}$ $\forall i \ne j$, and $\exists d_i \in \mathbb{Z}^+$: $[\![d_i a_{ij}]\!]$ is symmetric and positive definite. We will take the d_i so that $d_1 + \ldots + d_n$ is as small as possible, hence $d_i \in \{1, 2, 3\}$.

Let $X = \coprod_{i=1}^{n} \mathbb{Z}\omega_i$, $Y = \coprod_{i=1}^{n} \mathbb{Z}\alpha_i^\vee$, and define a bilinear pairing $<\ ,\ > : X \times Y \longrightarrow \mathbb{Z}$ by $<\omega_i, \alpha_j^\vee> = \delta_{ij}\ \forall i, j$. Let $\alpha_i \in X$ such that $<\alpha_i, \alpha_j^\vee> = a_{ji}\ \forall j$, and define $s_i \in GL(X)$ by $x \longmapsto x - <x, \alpha_i^\vee>\alpha_i$. We call $W := <s_1, \ldots, s_n> \le GL(X)$ the Weyl group, $R := W\{\alpha_1, \ldots, \alpha_n\} \subseteq X$ the set of roots, the pair (R, W) the root system associated to the Cartan matrix $[\![a_{ij}]\!]$, and $R^+ := R \cap \sum_{i=1}^{n} \mathbb{N}\alpha_i$ a positive system of R.

(1.2) To each root system R there is associated a unique semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{Q}$. If $\overline{U}$ is the universal enveloping algebra of $\mathfrak{g}$, then $\overline{U}$ has a presentation as $\mathbb{Q}$-algebra by generators e_i, f_i, h_i, $1 \le i \le n$, and relations

$$h_i h_j = h_j h_i,$$

$$h_i e_j - e_j h_i = a_{ij} e_j, \qquad h_i f_j - f_j h_i = -a_{ij} f_j,$$

$$e_i f_j - f_j e_i = \delta_{ij} h_i,$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} e_i^r e_j e_i^s = 0 \qquad \text{if } i \ne j$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} f_i^r f_j f_i^s = 0 \qquad \text{if } i \ne j.$$

Further, $\overline{U}$ carries a structure of Hopf algebra with comultiplication $\overline{\Delta} \in \mathbb{Q}\underline{\mathrm{Alg}}(\overline{U}, \overline{U} \otimes_{\mathbb{Q}} \overline{U})$, counit $\overline{\varepsilon} \in \mathbb{Q}\underline{\mathrm{Alg}}(\overline{U}, \mathbb{Q})$, and antipode $\overline{S} \in \mathbb{Q}\underline{\mathrm{Alg}}(\overline{U}, \overline{U}^{\mathrm{op}})$ given by $\forall x \in \mathfrak{g}$,

$$\overline{\Delta} : x \longmapsto 1 \otimes x + x \otimes 1,$$

$$\overline{\varepsilon} : x \longmapsto 0,$$

$$\overline{S} : x \longmapsto -x.$$

Let $\overline{U}_{\mathbb{Z}}$ be the subalgebra of $\overline{U}$ generated by

$$e_i^{(r)} := \frac{e_i^{\,r}}{r!}, \quad f_i^{(r)} := \frac{f_i^{\,r}}{r!}, \quad 1 \le i \le n,\ r \in \mathbb{N}.$$

Then $\overline{U}_{\mathbb{Z}}$ inherits from $\overline{U}$ the structure of Hopf algebra such that $\overline{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \overline{U}$. If Γ is a commutative ring, we will put $\overline{U}_{\Gamma} = \overline{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Gamma$.

(1.3) Let $\mathfrak{G}_{\mathbb{Z}}$ be the split semisimple simply connected $\mathbb{Z}$-group with the root system R. If Γ is a commutative ring, we set $\mathfrak{G}_{\Gamma} = \mathfrak{G}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Gamma$. In case $\Gamma = \mathbb{F}_p$ we will abbreviate $\mathfrak{G}_{\mathbb{F}_p}$ as $\mathfrak{G}$.

If $\mathrm{Dist}(\mathfrak{G}_{\mathbb{Z}})$ is the algebra of distributions of $\mathfrak{G}_{\mathbb{Z}}$, then

$$\overline{U}_{\mathbb{Z}} \simeq \mathrm{Dist}(\mathfrak{G}_{\mathbb{Z}}) \quad \text{as Hopf algebras.}$$

Hence for any commutative ring Γ

$$\overline{U}_{\Gamma} = \overline{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Gamma \simeq \mathrm{Dist}(\mathfrak{G}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \Gamma \simeq \mathrm{Dist}(\mathfrak{G}_{\Gamma}).$$

We recall that in case Γ is a field, the category $\mathfrak{G}_\Gamma\underline{\text{Mod}}$ of $\mathfrak{G}_\Gamma$-modules is equivalent to the category of locally finite Dist($\mathfrak{G}_\Gamma$)-modules [J], (II.1.20), hence of locally finite $\overline{U}_\Gamma$-modules.

(1.4) Let v be an indeterminate. Following Gauss we set $\forall m \in \mathbb{Z}$, $r \in \mathbb{N}$, and $d \in \mathbb{Z}^+$,

$$\begin{bmatrix} m \\ r \end{bmatrix}_d = \prod_{s=1}^{r} \frac{v^{d(m-s+1)} - v^{-d(m-s+1)}}{v^{ds} - v^{-ds}} \quad \in \quad \mathbb{Q}(v),$$

$$[r]!_d = \prod_{s=1}^{r} \frac{v^{ds} - v^{-ds}}{v^{d} - v^{-d}} \quad \in \quad \mathbb{Q}(v).$$

Both $\begin{bmatrix} m \\ r \end{bmatrix}_d$ and $[r]!_d$ actually belong to $\mathbb{Z}[v,v^{-1}]$, and under the specialization $v \longmapsto 1$ we get

$$\begin{bmatrix} m \\ r \end{bmatrix}_d \longmapsto \begin{pmatrix} m \\ r \end{pmatrix} \quad \text{and} \quad [r]!_d \longmapsto r!.$$

Now the quantized enveloping algebra, or quantum algebra for short, associated to the Cartan matrix $[\![a_{ij}]\!]$ is the $\mathbb{Q}(v)$-algebra U' with generators E_i, F_i, $K_i^{\pm 1}$, $1 \le i \le n$, and relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1 = K_i^{-1}K_i,$$

$$K_iE_j = v^{d_ia_{ij}}E_jK_i, \quad K_iF_j = v^{-d_ia_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i{}^r E_j E_i{}^s = 0 \qquad \text{if } i \neq j$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i{}^r F_j F_i{}^s = 0 \qquad \text{if } i \neq j.$$

The quantum algebra U' is equipped with a structure of Hopf algebra with comultiplication Δ, counit ε, and antipode S defined by

$$\Delta : E_i \longmapsto E_i \otimes 1 + K_i \otimes E_i , \quad F_i \longmapsto F_i \otimes K_i^{-1} + 1 \otimes F_i ,$$

$$K_i \longmapsto K_i \otimes K_i ,$$

$$\varepsilon : E_i \longmapsto 0, \quad F_i \longmapsto 0, \quad K_i \longmapsto 1,$$

$$S \text{ antihom} : E_i \longmapsto -K_i^{-1} E_i , \quad F_i \longmapsto -F_i K_i , \quad K_i \longmapsto K_i^{-1} .$$

(1.5) In order to define a quantum analogue of $\overline{U}_{\mathbb{Z}}$, we let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and put $\forall r \in \mathbb{N}$,

$$E_i^{(r)} := \frac{E_i{}^r}{[r]!_{d_i}}, \qquad F_i^{(r)} := \frac{F_i{}^r}{[r]!_{d_i}},$$

$$\begin{bmatrix} K_i \\ r \end{bmatrix} := \prod_{s=1}^{r} \frac{K_i v^{d_i(-s+1)} - K_i^{-1} v^{-d_i(-s+1)}}{v^{d_i s} - v^{-d_i s}}.$$

We define $U_{\mathcal{A}}$ to be the $\mathcal{A}$-subalgebra of U' generated by $E_i^{(r)}$, $F_i^{(r)}$, $K_i^{\pm 1}$, $1 \leq i \leq n$, $r \in \mathbb{N}$. Then $U_{\mathcal{A}}$ is a Hopf $\mathcal{A}$-subalgebra of U' and contains all $\begin{bmatrix} K_i \\ r \end{bmatrix}$.

Let $U_{\mathscr{A}}^+$ (resp. $U_{\mathscr{A}}^-$) be the $\mathscr{A}$-subalgebra of $U_{\mathscr{A}}$ generated by $E_i^{(r)}$ (resp. $F_i^{(r)}$), $1 \le i \le n$, $r \in \mathbb{N}$, and let $U_{\mathscr{A}}^0$ the $\mathscr{A}$-subalgebra of $U_{\mathscr{A}}$ generated by $K_i^{\pm 1}$, $\begin{bmatrix} K_i \\ r \end{bmatrix}$, $1 \le i \le n$, $r \in \mathbb{N}$. Then

$$U_{\mathscr{A}}^- \otimes_{\mathscr{A}} U_{\mathscr{A}}^0 \otimes_{\mathscr{A}} U_{\mathscr{A}}^+ \xrightarrow[\sim]{\text{mult}} U_{\mathscr{A}} \quad \text{in } \underline{\mathscr{A}\text{Mod}},$$

and $U_{\mathscr{A}} \otimes_{\mathscr{A}} \mathbb{Q}(v) \simeq U'$ as Hopf algebras. In fact, each of $U_{\mathscr{A}}^{\pm}$ and $U_{\mathscr{A}}^0$ has an $\mathscr{A}$-free basis.

If Γ is a commutative $\mathscr{A}$-algebra, we will put $U_\Gamma = U_{\mathscr{A}} \otimes_{\mathscr{A}} \Gamma$, $U_\Gamma^{\pm} = U_{\mathscr{A}}^{\pm} \otimes_{\mathscr{A}} \Gamma$, and $U_\Gamma^0 = U_{\mathscr{A}}^0 \otimes_{\mathscr{A}} \Gamma$.

(1.6) Fix $\ell \in \mathbb{Z}^+$ such that $(\ell, a_{ij}) = 1$ $\forall a_{ij} \neq 0$. In particular, ℓ is odd. Let Φ_ℓ be the ℓ-th cyclotomic polynomial and put $\mathscr{B} = \mathscr{A}/(\Phi_\ell)$ with $\mathscr{B}'$ the field of fractions of $\mathscr{B}$. Regarding $\mathscr{B}$ and $\mathscr{B}'$ as $\mathscr{A}$-algebras in a natural way, we set

$$\tilde{U}_{\mathscr{B}} = U_{\mathscr{B}}/(K_1^{\ell}-1, \ \ldots, \ K_n^{\ell}-1) \quad \text{and} \quad \tilde{U}_{\mathscr{B}'} = U_{\mathscr{B}'}/(K_1^{\ell}-1, \ \ldots, \ K_n^{\ell}-1).$$

$\tilde{U}_{\mathscr{B}}$ has a $\mathscr{B}$-basis that is also a $\mathscr{B}'$-basis of $\tilde{U}_{\mathscr{B}'}$. We note that in $U_{\mathscr{B}}$ $K_i^{2\ell} = 1$ and K_i^{ℓ} is central $\forall i$.

In case $\ell = 1$, $\mathscr{B} = \mathbb{Z}$, $\mathscr{B}' = \mathbb{Q}$, and we have an isomorphism of Hopf $\mathbb{Q}$-algebras

$$\bar{U} \xrightarrow{\sim} \tilde{U}_{\mathbb{Q}} \quad \text{such that} \quad e_i \longmapsto E_i \quad \text{and} \quad f_i \longmapsto F_i \ \forall i$$

that restricts to yield $\bar{U}_{\mathbb{Z}} \xrightarrow{\sim} \tilde{U}_{\mathbb{Z}}$ sending

$$\binom{h_i}{t} := \frac{h_i(h_i-1)\ldots(h_i-t+1)}{t!} \quad \text{onto} \quad \begin{bmatrix} K_i \\ t \end{bmatrix}, \quad 1 \le i \le n, \ t \in \mathbb{N}.$$

2° A Frobenius map in characteristic 0

(2.1) The Frobenius morphism $Frob$ on $\mathfrak{G}$ induces by taking the differential an endomorphism Fr of the Hopf $\mathbb{F}_p$-algebra $\overline{U}_{\mathbb{F}_p}$ such that

$$\begin{array}{ccc} \mathrm{Dist}(\mathfrak{G}) & \xrightarrow{d(Frob)} & \mathrm{Dist}(\mathfrak{G}) \\ \wr| & \circlearrowright & \wr| \\ \overline{U}_{\mathbb{F}_p} & \underset{Fr}{\dashrightarrow} & \overline{U}_{\mathbb{F}_p}. \end{array}$$

Let $\mathfrak{G}_1 = \ker(Frob)$. Under the isomorphism $\mathrm{Dist}(\mathfrak{G}) \simeq \overline{U}_{\mathbb{F}_p}$, $\mathrm{Dist}(\mathfrak{G}_1)$ is identified with the restricted enveloping algebra of $\mathrm{Lie}(\mathfrak{G}) = \mathrm{Dist}_1^+(\mathfrak{G})$:

$$\begin{array}{ccc} \mathrm{Dist}(\mathfrak{G}) & \xrightarrow{\sim} & \overline{U}_{\mathbb{F}_p} \\ \uparrow & \circlearrowright & \uparrow \\ \mathrm{Dist}(\mathfrak{G}_1) & \underset{\sim}{\dashrightarrow} & \langle e_i^{(r)}, f_i^{(r)} \rangle_{1 \le i \le n, 0 \le r \le p-1}. \end{array}$$

If ε_{11} is the counit of $\mathrm{Dist}(\mathfrak{G}_1)$, then $\ker(Fr) = (\ker(\varepsilon_{11}))$.

(2.2) Let ℓ be as in (1.6). Lusztig [LQG],(8.16) has found a remarkable map $\tilde{F}r \in \mathscr{B}\underline{\mathrm{Alg}}(U_{\mathscr{B}}, \overline{U}_{\mathscr{B}})$ such that

$$E_i^{(r)} \longmapsto \begin{cases} e_i^{(\frac{r}{\ell})} & \text{if } \ell \mid r \\ 0 & \text{otherwise,} \end{cases}$$

$$F_i^{(r)} \longmapsto \begin{cases} f_i^{(\frac{r}{\ell})} & \text{if } \ell \mid r \\ 0 & \text{otherwise,} \end{cases}$$

$$K_i \longmapsto 1.$$

Let $\mathrm{u}_{\mathscr{B}'}$ (resp. $\mathrm{u}_{\mathscr{B}}$) be the $\mathscr{B}'$ (resp. $\mathscr{B}$)-subalgebra of $U_{\mathscr{B}'}$ (resp. $U_{\mathscr{B}}$) generated by $E_i^{(r)}$, $F_i^{(r)}$, $K_i^{\pm 1}$, $1 \le i \le n$, $0 \le r \le \ell - 1$. Then $\mathrm{u}_{\mathscr{B}}$ is a Hopf subalgebra of $U_{\mathscr{B}}$, $\mathscr{B}$-free of rank $2^n \ell^{|R|+n}$, and

$$\ker(\tilde{F}r \otimes_{\mathscr{B}} \mathscr{B}') = (\ker(\varepsilon_{\mathrm{u}_{\mathscr{B}'}})) \qquad \text{[LQG],(8.16),}$$

where $\varepsilon_{\mathrm{u}_{\mathscr{B}'}}$ is the counit of $\mathrm{u}_{\mathscr{B}'}$.

Set $\tilde{\mathrm{u}}_{\mathscr{B}} = \mathrm{u}_{\mathscr{B}}/(K_i^{\ell}-1)_i$ and $\tilde{\mathrm{u}}_{\mathscr{B}'} = \mathrm{u}_{\mathscr{B}'}/(K_i^{\ell}-1)_i$. $\tilde{\mathrm{u}}_{\mathscr{B}}$ remains $\mathscr{B}$-free of rank $\ell^{|R|+n}$.

$\tilde{F}r$ factors through $\tilde{U}_{\mathscr{B}}$ to induce $\hat{F}r$:

$$\begin{array}{ccc} U_{\mathscr{B}} & \xrightarrow{\tilde{F}r} & \overline{U}_{\mathscr{B}} \\ {\scriptstyle \text{nat}}\downarrow & \circlearrowright & \uparrow {\scriptstyle \hat{F}r} \\ \tilde{U}_{\mathscr{B}} & = & U_{\mathscr{B}}/(K_i^{\ell}-1)_i. \end{array}$$

(2.3) Assume $\ell = p$ a prime and regard $\mathbb{F}_p$ as a $\mathscr{B}$-algebra under the specialization $v \longmapsto 1$. Then $\tilde{U}_{\mathscr{B}} \otimes_{\mathscr{B}} \mathbb{F}_p \simeq \overline{U}_{\mathbb{F}_p}$, and

$$\begin{array}{ccc} \tilde{U}_{\mathscr{B}} \otimes_{\mathscr{B}} \mathbb{F}_p & \xrightarrow{\hat{F}r \otimes_{\mathscr{B}} \mathbb{F}_p} & \overline{U}_{\mathscr{B}} \otimes_{\mathscr{B}} \mathbb{F}_p \\ \wr\mid & \circlearrowright & \wr\mid \\ \overline{U}_{\mathbb{F}_p} & \xrightarrow[Fr]{} & \overline{U}_{\mathbb{F}_p} \end{array}.$$

In this sense one can regard $\widetilde{Fr}$ as a quantization of Fr or as a lifting of Fr to characteristic 0.

Also $\tilde{\mathrm{u}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{F}_p \simeq \mathrm{Dist}(\mathfrak{G}_1)$, hence one may regard $\tilde{\mathrm{u}}_{\mathcal{B}}$, as a quantization of $\mathrm{Dist}(\mathfrak{G}_1)$.

3° Representations of the quantum algebras

(3.1) If $\lambda \in X$, we will write $\lambda_i = \langle\lambda, \alpha_i^\vee\rangle$. Set $X^+ = \{ \lambda \in X \mid \lambda_i \geq 0 \ \forall i \}$.

Any finite dimensional $\overline{U}$-module is semisimple and the set of isomorphism classes of finite dimensional simple $\overline{U}$-modules is classified by X^+:

$$\lambda \longmapsto \overline{L}(\lambda) \quad \text{simple module of highest weight } \lambda.$$

If we define

$$\overline{L}(\lambda)_\mu = \{ m \in \overline{L}(\lambda) \mid h_i m = \langle\mu, \alpha_i^\vee\rangle m \ \forall i \},$$

then $\mathrm{ch}\,\overline{L}(\lambda) := \sum_{\mu\in X} (\dim \overline{L}(\lambda)_\mu) e^\mu \in \mathbb{Z}[X]$ is given by Weyl's character formula, where $\mathbb{Z}[X]$ is the group ring of X with natural basis e^μ, $\mu \in X$.

(3.2) For each $\lambda \in X$, $\exists! \varepsilon_\lambda \in \mathcal{A}\underline{\mathrm{Alg}}(U_\mathcal{A}, \mathcal{A})$ such that $\forall i \in [1, n]$,

$$K_i \longmapsto v^{d_i\lambda_i}, \quad E_i \longmapsto 0, \quad F_i \longmapsto 0.$$

In particular, ε_0 is the counit of $U_{\mathcal{A}}$. Set $\chi_\lambda = \varepsilon_\lambda|_{U^0_{\mathcal{A}}}$.

If Γ is a commutative $\mathcal{A}$-algebra and $M \in U_\Gamma$Mod, we define the λ-weight space of M to be

$$M_\lambda = \{ m \in M \mid um = (\chi_\lambda \otimes_{\mathcal{A}} \Gamma)(u)m \ \forall u \in U^0_\Gamma \}.$$

In case M_λ is Γ-free of finite rank $\forall \lambda \in X$, we set

$$\Gamma\mathrm{ch}\, M = \sum_{\lambda \in X} \mathrm{rk}(M_\lambda) e^\lambda .$$

We say that M is a highest weight module if and only if M has a vector $m^+ \in M_\lambda \setminus 0$ for some $\lambda \in X$ such that $M = U_\Gamma m^+$ and that $E_i^{(r)} m^+ = 0 \ \forall i \in [1, n]$ and $r > 0$. In each case we consider below, λ (resp. m^+) is uniquely determined (resp. up to $\Gamma^\times$). We call m^+ (resp. λ) a highest weight vector (resp. the highest weight) of M.

Let $U^\#_\Gamma = U^0_\Gamma U^+_\Gamma$. If $\lambda \in X$, let Γ_λ be the $U^\#_\Gamma$-module Γ with $U^\#_\Gamma$ acting by $\varepsilon_\lambda \otimes_{\mathcal{A}} \Gamma$ and set $Y_\Gamma(\lambda) = U_\Gamma \otimes_{U^\#_\Gamma} \Gamma_\lambda$. Then each highest weight U_Γ-module of highest weight λ is a quotient of $Y_\Gamma(\lambda)$. We call $Y_\Gamma(\lambda)$ the Verma module of highest weight λ.

(3.3) Any finite dimensional U'-module is semisimple [R], and any finite dimensional simple U'-module is obtained as the head of a Verma module $Y_{\mathbb{Q}(v)}(\lambda)$, $\lambda \in X^+$. If $L'(\lambda)$ is such, then take a highest weight vector m^+ of $L'(\lambda)$. Then $L'(\lambda) \simeq U_{\mathcal{A}} m^+ \otimes_{\mathcal{A}} \mathbb{Q}(v)$ while $U_{\mathcal{A}} m^+ \otimes_{\mathcal{A}} \mathbb{Q}$ under the specialization $v \longmapsto 1$ is a simple $\tilde{U}_{\mathbb{Q}}$-module [LQD],(4.12). As $\tilde{U}_{\mathbb{Q}} \simeq \overline{U}$ (1.6), one gets

$$\mathbb{Q}(v)\mathrm{ch}\ L'(\lambda) = \mathscr{A}\mathrm{ch}(U_{\mathscr{A}}m^+) = \mathrm{ch}\ \overline{L}(\lambda).$$

(3.4) Throughout the rest of this section we will consider in the category $\mathscr{C}'_f$ of the finite dimensional $U_{\mathscr{B}'}$-modules on which the central element K_i^{ℓ} acts as 1 $\forall i$. If $M \in \mathscr{C}'_f$, then $\sum_{\lambda\in X} M_\lambda = \coprod_{\lambda\in X} M_\lambda$.

In case $\ell = 1$, $\mathscr{B}' = \mathbb{Q}$ and K_i acts as 1, hence any $U_{\mathscr{B}'}$-module is a $\widetilde{U}_{\mathscr{B}'}$-module. But $\widetilde{U}_{\mathscr{B}'} = U_{\mathscr{B}'}/(K_i-1)_i \simeq \overline{U}$, so nothing happens here.

If $\ell > 1$, however, not all objects of $\mathscr{C}'_f$ are semisimple. In what follows we will assume $\ell > 1$.

(3.5) Let $\lambda \in X^+$. If m^+ is a highest weight vector of $L'(\lambda)$, set $D'(\lambda) = U_{\mathscr{A}}m^+ \otimes_{\mathscr{A}} \mathscr{B}'$. Then $D'(\lambda)$ has a simple head, denoted $L_{\mathscr{B}'}(\lambda)$, and the simple objects of $\mathscr{C}'_f$ are exhausted in this way.

Let $X_\ell = \{\mu \in X^+ \mid \mu_i \leq \ell-1\ \forall i\}$ and write $\lambda = \lambda' + \ell\lambda''$ with $\lambda' \in X_\ell$, $\lambda'' \in X^+$. We have a quantum analogue, due to Lusztig [LMR], (7.4), of Steinberg's tensor product theorem:

$$L_{\mathscr{B}'}(\lambda) \simeq L_{\mathscr{B}'}(\lambda') \otimes_{\mathscr{B}'} L_{\mathscr{B}'}(\ell\lambda'') \quad \text{in} \quad U_{\mathscr{B}'}\underline{\mathrm{Mod}},$$

where $U_{\mathscr{B}'}$ acts on the right hand side via $\Delta \otimes_{\mathscr{A}} \mathscr{B}'$.

Moreover, $L_{\mathscr{B}'}(\ell\lambda'')$ is obtained by regarding $\overline{L}(\lambda'') \otimes_{\mathbb{Q}} \mathscr{B}'$ as $U_{\mathscr{B}'}$-module via $\widetilde{Fr} \otimes_{\mathscr{B}} \mathscr{B}' : U_{\mathscr{B}'} \longrightarrow \overline{U}_{\mathscr{B}'}$. In particular,

$$\mathscr{B}'\mathrm{ch}\ L_{\mathscr{B}'}(\ell\lambda'') = \mathrm{ch}\ \overline{L}(\lambda'').$$

On the other hand, $L_{\mathscr{B}'}(\lambda')$ remains simple upon restriction to $\mathfrak{u}_{\mathscr{B}'}$, and any simple $\mathfrak{u}_{\mathscr{B}'}$-module on which K_i^{ℓ} acts as 1 $\forall i$ arises in this way.

4° Lusztig's conjectures

(4.1) Assume in this section that R is irreducible and let α_0 be the highest short root of R^+, $\rho = \omega_1 + \ldots + \omega_n$, $h = \langle\rho, \alpha_0^\vee\rangle + 1$.

Define a permutation s_0 of X by $\mu - (\langle\mu, \alpha_0^\vee\rangle+\ell)\alpha_0$. If $S_a = \{s_0, s_1, \ldots, s_n\}$ and $W_\ell = \langle S_a\rangle \leq \mathfrak{S}_X$, (W_ℓ, S_a) forms a Coxeter system. Let $P_{x,y}$, $x,y \in W_\ell$, be the Kazhdan-Lusztig polynomials of (W_ℓ, S_a). Define a new action of W_ℓ on X by $w\cdot\mu = w(\mu + \rho) - \rho$, $w \in W_\ell$, $\mu \in X$. Let $A = \{ \mu \in X \mid \langle\mu + \rho, \alpha_i^\vee\rangle < 0 \ \forall i \in [1,n]$ and $\langle\mu + \rho, \alpha_0^\vee\rangle > -\ell \}$, and $\overline{A}$ its closure in $X \otimes_{\mathbb{Z}} \mathbb{R}$. If $\mu \in X$ and $y \in W_\ell$, let $\mu_y = yw_\mu^{-1}\cdot\mu$, where w_μ is the element of W_ℓ of minimal length such that $w_\mu^{-1}\cdot\mu \in \overline{A}$.

(4.2) If $\lambda \in X^+$, Lusztig conjectures [LMR],(8.2)

$$(1) \qquad \mathscr{B}'\mathrm{ch}\, L_{\mathscr{B}'}(\lambda) = \sum_{y\in W_\ell} (-1)^{\ell(yw_\lambda)} P_{y,w_\lambda}(1)\mathrm{ch}\, \overline{L}(\lambda_y).$$

Modulo L.Casian's [C] and S.Kumar's [K] proofs of Lusztig's similar conjecture on affine Kac-Moody Lie algebras [LQG],(2.5), D.Kazhdan and Lusztig [KL] has just announced the validity of (1).

(4.3) Assume $\ell = p$ a prime, and let $\lambda \in X_\ell$. Take a highest weight vector m^+ of $L_{\mathscr{B}'}(\lambda)$, and set $L_p(\lambda) = \mathrm{u}_{\mathscr{B}}m^+ \otimes_{\mathscr{B}} \mathbb{F}_p$ regarding $\mathbb{F}_p$ as a $\mathscr{B}$-algebra under $v \longmapsto 1$. Then $L_p(\lambda)$ comes equipped with a structure of Dist$(\mathfrak{G}_1)$-module. As such $L_p(\lambda)$ has a simple head, that is the simple $\mathfrak{G}$-module $\mathfrak{L}(\lambda)$ of highest weight λ.

Lusztig conjectures [LFD],(0.3) that for $p >> 0$

$$L_p(\lambda) = \mathfrak{L}(\lambda),$$

hence $\mathscr{B}'\mathrm{ch}\, L_{\mathscr{B}'}(\lambda) = \mathrm{ch}\, \mathfrak{L}(\lambda)$. Indeed, Lusztig conjectured in 1978 [LS] that if $\mu \in X^+$ with $w_\mu^{-1}\cdot\mu \in A$ and if $<\mu + \rho, \alpha_0^\vee> < p(p-h+2)$, one should have

$$\mathrm{ch}\, \mathfrak{L}(\mu) = \sum_{y \in W_p} (-1)^{\ell(yw_\mu)} P_{y,w_\mu}(1)\mathrm{ch}\, \overline{L}(\mu_y).$$

5° Induction

(5.1) In a large part the representation theory of $\mathfrak{G}$ is the study of the cohomologies $H^\cdot(\mathfrak{G}/\mathfrak{B}, \mathscr{L}(\lambda))$ of the $\mathfrak{G}$-linearized invertible sheaves $\mathscr{L}(\lambda)$ induced from the 1-dimensional $\mathfrak{B}$-module $(\mathbb{F}_p)_\lambda$, $\lambda \in X$, on the flag scheme $\mathfrak{G}/\mathfrak{B}$, where $\mathfrak{B}$ is the Borel subgroup of $\mathfrak{G}$ whose roots are $-R^+$. Algebraically, $H^\cdot(\mathfrak{G}/\mathfrak{B}, \mathscr{L}(\lambda))$ are obtained as the right derived functors of the induction functor $\mathrm{ind}_{\mathfrak{B}}^{\mathfrak{G}}$ from $\mathfrak{B}$<u>Mod</u> into $\mathfrak{G}$<u>Mod</u> defined by $M \longmapsto (M \otimes_{\mathbb{F}_p} \mathbb{F}_p[\mathfrak{G}])^{\mathfrak{B}}$ the fixed points of $M \otimes_{\mathbb{F}_p} \mathbb{F}_p[\mathfrak{G}]$ under the action of $\mathfrak{B}$ on the coordinate algebra $\mathbb{F}_p[\mathfrak{G}]$ of $\mathfrak{G}$ (resp. on M) by the right regular action (resp. as given) with $\mathfrak{G}$ acting on the resulting set by the left regular action on $\mathbb{F}_p[\mathfrak{G}]$. In [APW] H.H. Andersen, P. Polo, and Wen K. quantize the induction functor, on which we will briefly touch in what remains.

(5.2) In this section we assume $\ell = p$ a prime. Let $\mathfrak{m} = (p, v-1)$ a maximal ideal of $\mathscr{A}$, and set $\mathscr{A}' = \mathscr{A}_{\mathfrak{m}}$. We will abbreviate $U_{\mathscr{A}} \otimes_{\mathscr{A}} \mathscr{A}'$, $U_{\mathscr{A}}^{\pm} \otimes_{\mathscr{A}} \mathscr{A}'$, $U_{\mathscr{A}}^0 \otimes_{\mathscr{A}} \mathscr{A}'$ as U, $U^{\pm}$, and U^0, respectively. An advantage of working over $\mathscr{A}'$ is that any projective $\mathscr{A}'$-module is free as $\mathscr{A}'$ is local.

If $M \in U\underline{\mathrm{Mod}}$, $\sum_{\lambda\in X} M_\lambda = \coprod_{\lambda\in X} M_\lambda$. We set

$$F(M) = \{ m \in \coprod_\lambda M_\lambda \mid E_i^{(r)}m = 0 = F_i^{(r)}m \ \forall i \in [1,n] \text{ and } r >> 0 \}.$$

Let $\mathscr{C}$ be the full subcategory of $U\underline{\mathrm{Mod}}$ consisting of those M such that $M = F(M)$. Define also $\mathscr{C}_f = \{ M \in \mathscr{C} \mid M \text{ is } \mathscr{A}'\text{-finite} \}$.

(5.3) Set $U^\# = U^0U^+$. If $M \in U^\#\underline{\mathrm{Mod}}$, put

$$F^\#(M) = \{ m \in \coprod_\lambda M_\lambda \mid E_i^{(r)}m = 0 \quad \forall r >> 0 \text{ and } i \},$$

and let $\mathscr{C}^\#$ the full subcategory of $U^\#\underline{\mathrm{Mod}}$ consisting of those M such that $F^\#(M) = M$. Let also $\mathscr{C}_f^\# = \{ M \in \mathscr{C}^\# \mid M \text{ is } \mathscr{A}'\text{-finite} \}$.

If $M \in \mathscr{C}_f^\#$, $U \otimes_{U^\#} M$ has a unique minimal U-submodule M' such that $(U \otimes_{U^\#} M)/M'$ is $\mathscr{A}'$-finite. We will denote $(U \otimes_{U^\#} M)/M'$ by $D(M)$. In particular, we will abbreviate $D(\mathscr{A}'_\lambda)$ as $D(\lambda)$. After A.Joseph the correspondence $M \longmapsto D(M)$ defines a right exact functor from $\mathscr{C}_f^\#$ into $\mathscr{C}_f$. The functor D is left adjoint to the restriction functor $\mathscr{C}_f \longrightarrow \mathscr{C}_f^\#$: $\forall M \in \mathscr{C}_f^\#$ and $V \in \mathscr{C}_f$, $\mathscr{C}_f^\#(M, V) \xrightarrow{\sim} \mathscr{C}_f(D(M), V)$ via

$$\varphi \longmapsto \tilde{\varphi} \quad \text{with} \quad u \otimes m \longmapsto u\varphi(m).$$

In particular, $D(\lambda)$ is the universal $\mathscr{A}'$-finite highest weight U-module of highest weight λ.

We have that $D(\lambda) \neq 0$ iff $\lambda \in X^+$. Further, $D(\lambda)$ is $\mathscr{A}'$-free with

$\mathcal{A}$'ch $D(\lambda)$ = ch $\overline{L}(\lambda)$, the proof of which makes use of the fact that $\tilde{U}_{\mathbb{F}_p} \simeq \overline{U}_{\mathbb{F}_p}$ (1.6).

(5.4) If $M \in \mathcal{A}\underline{\mathrm{Mod}}$, put $\mathcal{H}(M) = \mathcal{A}\underline{\mathrm{Mod}}(U, M)$. $\mathcal{H}(M)$ carries two compatible structures of U-modules γ and δ defined by

$$(\gamma(u)\theta)(x) = \theta(S(u)x) \quad \text{and} \quad (\delta(u)\theta)(x) = \theta(xu), \quad \theta \in \mathcal{H}(M),\ u,x \in U.$$

Accordingly we define $F_\gamma(\mathcal{H}(M))$ and $F_\delta(\mathcal{H}(M))$.

On the other hand, let $\mathcal{I}$ be the set of ideals I of U such that U/I is $\mathcal{A}'$-finite and that I contains a finite intersection of $\ker(\chi_\lambda)$, $\lambda \in X$. Then [APW],(1.30)

$$(1) \qquad \{\varphi \in \mathcal{H}(M) \mid \varphi(I) = 0 \ {}^{\exists}I \in \mathcal{I}\} = F_\gamma(\mathcal{H}(M)) = F_\delta(\mathcal{H}(M)).$$

We define a functor $H : \mathcal{A}'\underline{\mathrm{Mod}} \longrightarrow \mathcal{C}$, called the induction from $\mathcal{A}'\underline{\mathrm{Mod}}$ into $\mathcal{C}$, via $M \longmapsto F_\delta(\mathcal{H}(M))$. The functor H is right adjoint to the restriction $\mathcal{C} \longrightarrow \mathcal{A}'\underline{\mathrm{Mod}}$: ${}^{\forall}V \in \mathcal{C}$ and $M \in \mathcal{A}'\underline{\mathrm{Mod}}$,

$$\mathcal{A}'\underline{\mathrm{Mod}}(V, M) \xrightarrow{\sim} \mathcal{C}(V, H(M)) \quad \text{via} \quad \varphi \longmapsto \varphi' \quad \text{such that } \varphi'(x) = \varphi(\_x)$$

with inverse $\psi \longmapsto \mathcal{E}v\circ\psi$, where $\mathcal{E}v(\theta) = \theta(1)$. In particular, H sends injective $\mathcal{A}'$-modules into injectives in $\mathcal{C}$.

Further, H is exact, commutes with taking arbitrary direct sums, and $H(\mathcal{A}')$ is $\mathcal{A}'$-free. We set $\mathcal{A}'[U] = H(\mathcal{A}')$, and call it the quantum coordinate algebra of U. $\mathcal{A}'[U]$ comes naturally equipped with a structure of Hopf algebra over $\mathcal{A}'$. If $M \in U\underline{\mathrm{Mod}}$, then [APW],(1.31)

(2) $$H(M) \simeq \mathscr{A}[U] \otimes_{\mathscr{A}} M \simeq \mathscr{A}[U] \otimes_{\mathscr{A}} M_{triv},$$

where M_{triv} is the U-module M under ε. In particular, if Γ is an $\mathscr{A}'$-algebra under $v \longmapsto 1$ that is a field of characteristic 0, we have an isomorphism of Hopf algebras over Γ (cf.Appendix)

(3) $$\mathscr{A}'[U] \otimes_{\mathscr{A}'} \Gamma \simeq \Gamma[\mathfrak{G}_\Gamma] \quad \text{the coordinate algebra of } \mathfrak{G}_\Gamma.$$

(5.5) Let $U^\flat = U^- U^0$. If $M \in U^\flat\underline{\mathrm{Mod}}$, set

$$F^\flat(M) = \{ m \in \coprod_\lambda M_\lambda \mid F_i^{(r)} m = 0 \ {}^\forall r >> 0 \text{ and } i \},$$

and let $\mathscr{C}^\flat$ be the full subcategory of $U^\flat\underline{\mathrm{Mod}}$ consisting of those M such that $F^\flat(M) = M$. Considering a functor $H^\flat : \mathscr{A}\underline{\mathrm{Mod}} \longrightarrow \mathscr{C}^\flat$ defined just like H, one finds that $\mathscr{C}^\flat$ has enough injectives.

We now define a functor, called the induction from $\mathscr{C}^\flat$ into $\mathscr{C}$, by

$$M \longmapsto F_\delta(U^\flat\underline{\mathrm{Mod}}(U, M)).$$

The functor is right adjoint to the restriction from $\mathscr{C}$ into $\mathscr{C}^\flat$, hence left exact, so we can consider its right derived functors. We denote the j-th derived functor by $H^j(U/U^\flat, \_)$.

We have ${}^\forall M \in \mathscr{C}^\flat$,

$$F_\delta(U^\flat\underline{\mathrm{Mod}}(U, M)) \simeq H^0(U/U^\flat, M) \simeq (M \otimes_{\mathscr{A}'} \mathscr{A}'[U])^{U^\flat} \quad \text{in } \mathscr{C},$$

where U (resp. $U^\flat$) acts on $M \otimes_{\mathcal{A}'} \mathcal{A}'[U]$ trivially on M and by δ on $\mathcal{A}'[U]$ (resp. by Δ with γ on $\mathcal{A}'[U]$). Comparing with the classical induction from $\mathfrak{B}$Mod into $\mathfrak{G}$Mod (5.1), one may regard $H^0(U/U^\flat, \_)$ as a quantization of $\mathrm{ind}_{\mathfrak{B}}^{\mathfrak{G}}$.

(5.6) Let $w_0 \in W$ with $w_0R^+ = -R^+$. If $\lambda \in X^+$,

$$H^0(U/U^\flat, \mathcal{A}'_\lambda) \simeq D(-w_0\lambda)^* \quad \text{in} \quad U\underline{\mathrm{Mod}},$$

where the U-module strudture on the right hand side is given by

$$(u\theta)(x) = \theta(\Psi(u)x), \ u \in U, \ \theta \in D(-w_0\lambda)^*, \ x \in D(-w_0\lambda)$$

with $\Psi \in \mathcal{A}'\underline{\mathrm{Alg}}(U, U^{\mathrm{op}})$ such that $E_i \longmapsto E_i$, $F_i \longmapsto F_i$, $K_i \longmapsto K_i^{-1}$. In particular, $\mathcal{A}'\mathrm{ch}\, H^0(U/U^\flat, \mathcal{A}'_\lambda) = \mathrm{ch}\, \overline{L}(\lambda)$.

(5.7) Let Γ be a field that is an $\mathcal{A}'$-algebra, eg., $\mathbb{Q}$, $\mathbb{F}_p$ under $v \longmapsto 1$, or $\mathcal{B}'$, $\mathbb{Q}(v)$. If $\lambda \in X^+$, $H^0(U_\Gamma/U_\Gamma^\flat, \Gamma_\lambda)$ has a simple socle, denoted $L_\Gamma(\lambda)$, of highest weight λ, and every simple object in $\mathcal{C}_\Gamma$ arises in this way. The notation is compatible with the one in (3.5). Also $L_{\mathbb{F}_p}(\lambda) \simeq \mathfrak{L}(\lambda)$.

Assume now that $v = 1$ in Γ. Then

$$\begin{array}{ccc} U_\Gamma/(K_i-1)_i & \simeq & \mathrm{Dist}(\mathfrak{G}_\Gamma) \\ \uparrow & \circlearrowleft & \uparrow \\ U_\Gamma^\flat/(K_i-1)_i & \simeq & \mathrm{Dist}(\mathfrak{B}_\Gamma), \end{array}$$

hence $\mathcal{C}_\Gamma \simeq \mathfrak{G}_\Gamma\underline{\mathrm{Mod}}$, $\mathcal{C}_\Gamma^\flat \simeq \mathfrak{B}\underline{\mathrm{Mod}}$, and one finds $^\forall j \in \mathbb{N}$ and $\lambda \in X$,

$$H^j(U_\Gamma/U_\Gamma^\flat, \Gamma_\lambda) \simeq H^j(\mathfrak{G}_\Gamma/\mathfrak{B}_\Gamma, \mathcal{L}_\Gamma(\lambda)),$$

where $\mathcal{L}_\Gamma(\lambda)$ is the induced sheaf on $\mathfrak{G}_\Gamma/\mathfrak{B}_\Gamma$.

(5.8) Many results on $H^\cdot(\mathfrak{G}/\mathfrak{B}, \mathcal{L}(\lambda))$ as described in [J] carry over to $H^\cdot(U/U^\flat, \mathcal{A}'_\lambda)$. In particular, let $\lambda \in X^+$ and write $w_0 = s_{j_N}\ldots s_{j_2}s_{j_1}$ with $N = |R^+|$. There is a natural U-homomorphism

$$H^{k+1}(s_{j_{r+1}}\ldots s_{j_1}\cdot\lambda) \longrightarrow H^k(s_{j_r}\ldots s_{j_1}\cdot\lambda) \quad \forall k,\ r \in \mathbb{N},$$

where we abbreviate $H^k(U/U^\flat, \mathcal{A}'_\mu)$ as $H^k(\mu)$. Let T_{w_0} be the composite

$$H^N(w_0\cdot\lambda) \longrightarrow H^k(s_{j_{N-1}}\ldots s_{j_1}\cdot\lambda) \longrightarrow \ldots \longrightarrow H^1(s_{j_1}\cdot\lambda) \longrightarrow H^0(\lambda).$$

Andersen, Polo and Wen conjecture [APW],(10.15) that T_{w_0} is diagonizable over $\mathcal{A}'$. Their conjecture will imply

$$\mathcal{B}'\mathrm{ch}\, L_{\mathcal{B}'}(\lambda) = \mathrm{ch}\, \mathfrak{L}(\lambda) \quad \forall \lambda \in X_p.$$

Appendix

(A.1) We attempt to give a proof to (5.4)(3). A similar assertion appears in [LQG],(8.17).

For that it may be worthwhile to start with reviewing the proof of (5.4)(1), i.e., [APW],Cor.1.30: $\forall M \in \mathcal{A}'$<u>Mod</u>,

$$(1) \qquad F_\delta(\mathcal{A}'\underline{\mathrm{Mod}}(U, M)) = \{ \varphi \in \mathcal{A}'\underline{\mathrm{Mod}}(U, M) \mid \varphi(I) = 0 \ {}^{\exists}I \in \mathfrak{I} \},$$

where $\mathcal{I} = \{ I \trianglelefteq U \mid U/I$ is $\mathcal{A}'$-finite and $I \supseteq \bigcap_{\text{finite}} \ker(\chi_\lambda)\}$.

That the left hand side contains the right hand side is the content of [APW],(1.9): Let f be an element of the right hand side with $f(I) = 0$, $I \in \mathcal{I}$. Then im f is $\mathcal{A}'$-finite, so therefore is $\delta(U)f$ as $\delta(U)f \subseteq \mathcal{A}'\underline{\mathrm{Mod}}(U/I, \text{im } f)$ and as $\mathcal{A}'$ is noetherian.

If $I \supseteq \ker(\chi_\lambda) \cap \ker(\chi_\mu)$ with $\lambda \neq \mu$, one finds $u \in U^0$ such that

$$(2) \qquad (u - \chi_\lambda(u)) - (u - \chi_\mu(u)) \in \mathcal{A}'^{\times}.$$

Put $I^0(\lambda) = U(\ker \chi_\lambda)$, $U(\lambda) = U/I^0(\lambda)$, and define likewise $I^0(\mu)$, $U(\mu)$. Then

$$U(\lambda) \oplus U(\mu) \longrightarrow U/I \quad \text{via} \quad (x, y) \longmapsto x(u - \chi_\lambda(u)) - y(u - \chi_\mu(u))$$

is a surjective homomorphism of U^0-modules, hence

$$\begin{aligned}\mathcal{A}'\underline{\mathrm{Mod}}(U/I, M) &\leq \mathcal{A}'\underline{\mathrm{Mod}}(U(\lambda), M) \oplus \mathcal{A}'\underline{\mathrm{Mod}}(U(\mu), M)\\ &= \mathcal{A}'\underline{\mathrm{Mod}}(U, M)_\lambda \oplus \mathcal{A}'\underline{\mathrm{Mod}}(U, M)_\mu ,\end{aligned}$$

where the weight spaces are taken with respect to $\delta(U^0)$. In general, repeat the argument to find that $\mathcal{A}'\underline{\mathrm{Mod}}(U/I, M)$ admits a weight space decomposition with respect to $\delta(U^0)$, so therefore does $\delta(U)f$. Then $\delta(U)f \in \mathcal{C}_f$ as $\delta(U)f$ is $\mathcal{A}'$-finite, hence f belongs to the left hand side of (1).

To see the reverse inclusion, we need [APW],Prop.1.27 and Prop. 1.29: if $\nu \in X$, let $\Omega(\nu) = \{ (\lambda, \mu) \in X^+ \times X^+ \mid \mu - \lambda = \nu \}$, $J(\lambda, \mu) = I^0(\nu) + \sum_{\substack{i\in[1,n] \\ r_i > \lambda_i}} U E_i^{(r_i)} + \sum_{\substack{i\in[1,n] \\ s_i > \mu_i}} U F_i^{(s_i)}$, and $D(\lambda, \mu) = U/J(\lambda,\mu)$. Then

$$(3) \qquad D(\lambda, \mu) \in \mathscr{C}_f.$$

Also under the identification of $\mathscr{A}'\underline{\mathrm{Mod}}(D(\lambda,\mu), M)$ with $\{ \varphi \in \mathscr{A}'\underline{\mathrm{Mod}}(U, M) \mid \varphi(J(\lambda,\mu)) = 0 \}$ we have

$$(4) \qquad F_\delta(\mathscr{A}'\underline{\mathrm{Mod}}(U, M))_\nu = \bigcup_{(\lambda,\mu)\in\Omega(\nu)} \mathscr{A}'\underline{\mathrm{Mod}}(D(\lambda,\mu), M)$$

Now let φ be an element of the left hand side of (1). We may assume φ has a weight ν relative to $\delta(U^0)$. Then $\varphi \in \mathscr{A}'\underline{\mathrm{Mod}}(D(\lambda,\mu), M)$ for some $(\lambda, \mu) \in \Omega(\nu)$ by (4), hence $\mathrm{im}(\varphi)$ is $\mathscr{A}'$-finite by (3). Then again by (3) one gets $\varphi \in \mathscr{A}'\underline{\mathrm{Mod}}(D(\lambda,\mu), \mathrm{im}\ \varphi) \leq F_\gamma(\mathscr{A}'\underline{\mathrm{Mod}}(U, M))$, hence one can write

$$(5) \qquad \varphi = \sum_i \varphi_i \quad \text{with} \quad \varphi_i \in F_\gamma(\mathscr{A}'\underline{\mathrm{Mod}}(U, M))_{\eta^i}.$$

On the other hand, define a $\mathscr{A}'$-linear automorphism $\mathfrak{s}$ of $\mathscr{A}'\underline{\mathrm{Mod}}(U, M)$ by $\varphi \longmapsto \varphi \circ S^{-1}$. Then $\forall x \in U$ and $\varphi \in \mathscr{A}'\underline{\mathrm{Mod}}(U, M)$,

$$(6) \qquad \gamma(x)\mathfrak{s}(\varphi) = \mathfrak{s}(\delta(x)\varphi),$$

from which one finds that $\forall \eta \in X$ and $(\xi, \zeta) \in \Omega(\eta)$,

$$(7)\qquad F_{\gamma}(\mathscr{A}'\underline{\mathrm{Mod}}(U,\ M))_{\eta} = \bigcup_{(\xi,\zeta)\in\Omega(\eta)} \mathscr{A}'\underline{\mathrm{Mod}}(U/S(J(\xi,\zeta)),\ M).$$

Hence each φ_i annihilates an $\mathscr{A}'$-cofinite right ideal of U, so therefore does φ. Then $\delta(U)\varphi$ is $\mathscr{A}'$-finite, hence

$$(8)\qquad \delta(U)\varphi \in \mathscr{C}_f.$$

Write $\delta(U)\varphi = \sum_j \mathscr{A}'\psi_j$ with ψ_j having weight ν^j with respedt to $\delta(U^0)$. By (3) each ψ_j annihilates some $J(\lambda^j,\ \mu^j)$, $(\lambda^j,\ \mu^j) \in \Omega(\nu^j)$. If we let J to be the ideal of U generated by $\bigcap_j J(\lambda^j,\ \mu^j)$, then J belongs to $\mathscr{I}$ and is annihilated by φ, hence φ lies in the right hand side of (1), as desired.

(A.2) Let Γ be an $\mathscr{A}'$-algebra under $v \longmapsto 1$ that is a field of arbitrary characteristic. Repeating the argument one checks that [APW],Cor.1.30 remains to hold with U replaced by U_Γ: with the obvious notational changes $^\forall M \in \Gamma\underline{\mathrm{Lin}}$,

$$F_{\Gamma,\delta}(\Gamma\underline{\mathrm{Lin}}(U_\Gamma,\ M)) = \{\ \varphi \in \Gamma\underline{\mathrm{Lin}}(U_\Gamma,\ M)\ |\ \varphi(I) = 0\ ^\exists I \in \mathscr{I}_\Gamma\ \},$$

where $\mathscr{I}_\Gamma = \{\ I \trianglelefteq U_\Gamma\ |\ \dim U_\Gamma/I < \infty$ and $I \supseteq \bigcap_{\text{finite}} \ker \chi_{\lambda,\Gamma}\ \}$ with $\chi_{\lambda,\Gamma} = \chi_\lambda \otimes_{\mathscr{A}'} \Gamma$.

(A.3) We now consult [H],(XVI.3) for some generalities of the Hopf algebra duals over a field K. If B is a K-algebra, set $B' = \{\ \varphi \in K\underline{\mathrm{Lin}}(B,\ K)\ |\ \varphi(I) = 0\ ^\exists I \trianglelefteq B$ with $\dim(B/I) < \infty\ \}$. Then

$$(1)\qquad B' \otimes_K B' \xrightarrow{\ \sim\ } (B \otimes_K B)' \quad \text{in} \quad K\underline{\mathrm{Lin}}$$

in a natural way, which induces a structure of K-coalgebra on B' with the comultiplication given by $K\underline{\text{Lin}}(\text{mult}, K)$. In case B is even a Hopf algebra over K with the comultiplication Δ_B, then the multiplication on B' defined by $K\underline{\text{Lin}}(\Delta_B, K)$ makes B' into a Hopf algebra over K, called the dual Hopf algebra of B.

Let us fix the notation. With the comultiplication (resp. counit, antipode) of $K[\mathfrak{G}_K]$ denoted by $\Delta_{\mathfrak{G}}$ (resp. $\varepsilon_{\mathfrak{G}}$, $\sigma_{\mathfrak{G}}$) we will write $\Delta'_{\mathfrak{G}} : \text{Dist}(\mathfrak{G}_K) \longrightarrow \text{Dist}(\mathfrak{G}_K) \otimes_K \text{Dist}(\mathfrak{G}_K)$ for the comultiplication on $\text{Dist}(\mathfrak{G}_K)$: $\Delta'_{\mathfrak{G}}(\mu)(a \otimes b) = \mu(ab)$, $\mu \in \text{Dist}(\mathfrak{G}_K)$, $a, b \in K[\mathfrak{G}_K]$. Recall also that the multiplication on $\text{Dist}(\mathfrak{G}_K)$ is given by $(\mu\nu)(a) = \sum_i \mu(a_i)\nu(b_i)$ if $\Delta_{\mathfrak{G}}(a) = \sum_i a_i \otimes b_i$. Let $\text{Dist}(\mathfrak{G}_K)'$ be the Hopf algebra dual to $\text{Dist}(\mathfrak{G}_K)$.

We now show a variation of [H], Th.XVI.3.1 and Th.XVIII.5.1.

Theorem. Let K be a field. Define $\theta : K[\mathfrak{G}_K] \longrightarrow K\underline{\text{Lin}}(\text{Dist}(\mathfrak{G}_K), K)$ via $\theta(a)(\mu) = \mu(a)$, $\mu \in \text{Dist}(\mathfrak{G}_K)$, $a \in K[\mathfrak{G}_K]$. Then θ induces an injective homomorphism of Hopf algebras over K from $K[\mathfrak{G}_K]$ into $\text{Dist}(\mathfrak{G}_K)'$. If K is algebraically closed of characteristic 0, then $\text{im}\,\theta = \text{Dist}(\mathfrak{G}_K)'$, hence $K[\mathfrak{G}_K] \simeq \text{Dist}(\mathfrak{G}_K)'$.

Proof. If $a \in K[\mathfrak{G}_K]$, write $\Delta_{\mathfrak{G}}(a) = \sum_i a_i \otimes b_i$. Then $V := \bigcap_i \{\ker(\theta(a_i)) \cap \ker(\theta(b_i))\}$ is a K-subspace of $\text{Dist}(\mathfrak{G}_K)$ having a finite codimension. But $\forall \mu \in V$ and $\nu \in \text{Dist}(\mathfrak{G}_K)$,

$$(\mu\nu)(a) = \sum \mu(a_i)\nu(b_i) = 0 = \sum \nu(a_i)\mu(b_i) = (\nu\mu)(a).$$

Hence $\theta(a)$ annihilates the ideal of Dist$(\mathfrak{G}_K)$ generated by V, so im(θ) lies in Dist$(\mathfrak{G}_K)'$. If $\mu(a) = 0$ $\forall\mu \in$ Dist$(\mathfrak{G}_K)$, then

$$a \in \bigcap_{m\in\mathbb{N}} (\ker \varepsilon_{\mathfrak{G}})^{m+1} = 0 \quad \text{by Krull's intersection theorem,}$$

hence θ is injective. Also from the definitions given above it follows that θ preserves the structure of Hopf algebras.

Assume finally that K is algebraically closed of characteristic 0. Let $\mathfrak{G}'$ be the K-group with the coordinate algebra Dist$(\mathfrak{G}_K)'$. As θ is injective, we get a surjective morphism $\theta^\# : \mathfrak{G}' \longrightarrow \mathfrak{G}$ of K-groups. Then $\mathfrak{G}'$ is also semisimple [SC],Prop.18.1 with Lie$(\mathfrak{G}')$ = Lie$(\mathfrak{G})$ by [H],Th.XVIII.5.1. If $\mathfrak{G}'$ has the weight lattice X', the fundamental group of $\mathfrak{G}'$ is X/X' [St],Ex., p.45. But $\mathfrak{G}'$ is centrallly closed in the category of reduced algebraic K-groups by [H],Th.XVIII.5.1 again, hence $X' = X$, so $\mathfrak{G}' \simeq \mathfrak{G}$. Then $\mathfrak{G}_K$ is centrally closed, hence $\theta^\#$ must be invertible as K is algebraically closed of characteristic 0. In turn, θ induces a bijection from $K[\mathfrak{G}_K]$ onto Dist$(\mathfrak{G}_K)'$, as desired.

(A.4) We are now ready to show

<u>Theorem</u>. If Γ is an $\mathcal{A}'$-algebra under $v \longmapsto 1$ that is a field of characteristic 0, then

$$\mathcal{A}'[U] \otimes_{\mathcal{A}'} \Gamma \simeq \Gamma[\mathfrak{G}_\Gamma] \quad \text{as Hopf algebras over } \Gamma.$$

<u>Proof</u>. We have

$$\mathcal{A}'[U] \otimes_{\mathcal{A}'} \Gamma \simeq F_\delta(\mathcal{A}'\underline{\mathrm{Mod}}(U, \Gamma)) \quad \text{by (5.4)(2)}$$

$$(1) \qquad \simeq F_{\delta}(\Gamma\underline{\mathrm{Lin}}(U_{\Gamma}, \Gamma)) = F_{\Gamma,\delta}(\Gamma\underline{\mathrm{Lin}}(U_{\Gamma}, \Gamma))$$

$$= \{ \varphi \in \Gamma\underline{\mathrm{Lin}}(U_{\Gamma}, \Gamma) \mid \varphi(I) = 0 \ {}^{\exists}I \in \mathfrak{I}_{\Gamma} \} \quad \text{by (A.2)}.$$

We will write $H_{\Gamma,\delta}(\Gamma)$ for the last term. If $\varphi \in H_{\Gamma,\delta}(\Gamma)_{\nu}$ and $x \in U_{\Gamma}$, then $\varphi(xK_i) = (\delta(K_i)\varphi)(x) = \varphi(x)$, hence $\varphi(x(K_i-1)) = 0$. As $K_i - 1$ is central in U_{Γ} , however, φ annihilates the ideal $(K_i-1)_i$ of U_{Γ} generated by $K_i - 1$, $i \in [1, n]$, hence

$$(2) \qquad H_{\Gamma,\delta}(\Gamma) = \{ \varphi \in \Gamma\underline{\mathrm{Lin}}(U_{\Gamma}, \Gamma) \mid \varphi(I) = 0 \ {}^{\exists}I \in \mathcal{J}'_{\Gamma} \},$$

where $\mathcal{J}'_{\Gamma} = \{ I \in \mathcal{J}_{\Gamma} \mid I \supseteq (K_i-1)_i \}$.

Now recall from (1.6) that $U_{\Gamma}/(K_i-1)_i \simeq \overline{U}_{\Gamma} \simeq \mathrm{Dist}(\mathfrak{G}_{\Gamma})$, through which define $\overline{\lambda}_{\Gamma} \in \Gamma\underline{\mathrm{Alg}}(\overline{U}^0_{\Gamma}, \Gamma)$ such that $h_i \longmapsto \lambda_i$ corresponding to $\chi_{\lambda,\Gamma}$, $\lambda \in X$, where $\overline{U}^0_{\Gamma}$ is the subalgebra of $\overline{U}_{\Gamma}$ generated by $\binom{h_i}{t}$, $i \in [1, n]$, $t \in \mathbb{N}$. Corresponding also to $\mathcal{J}'_{\Gamma}$ let $\overline{\mathcal{J}}_{\Gamma} = \{ I \trianglelefteq \overline{U}_{\Gamma} \mid \dim(\overline{U}_{\Gamma}/I) < \infty$ and $I \supseteq \bigcap_{\text{finite}} \ker(\overline{\lambda}_{\Gamma}) \}$. Then by (1) and (2)

$$\mathcal{A}'[U] \otimes_{\mathcal{A}'}\Gamma \simeq \{ \varphi \in \Gamma\underline{\mathrm{Lin}}(\overline{U}_{\Gamma}, \Gamma) \mid \varphi(I) = 0 \ {}^{\exists}I \in \overline{\mathcal{J}}_{\Gamma} \}.$$

Finally, $\overline{U}^0_{\Gamma}$ is the polynomial algebra in $h_1, \ldots, h_n$, hence $\ker \overline{\lambda}_{\Gamma} = (h_i-\lambda_i)_i$. If J is an ideal of $\overline{U}_{\Gamma}$ of finite codimension, then $\overline{U}_{\Gamma}/J$ is naturally a finite dimensional $\overline{U}_{\Gamma}$-module, hence admits a weight space decomposition with respect to $\overline{U}^0_{\Gamma}$ by integral weights. Write in $\overline{U}_{\Gamma}/J$

$$1 = m_1 + \dots + m_r \quad \text{with } m_i \text{ having weight } \bar{\lambda}^i_\Gamma,\ \lambda^i \in X.$$

Then $h_j \cdot 1 = \sum_{i=1}^{r} \lambda^i_j m_i \ \forall j \in [1, n]$, so

(3)
$$\bigcap_{i=1}^{r} \ker(\bar{\lambda}^i_\Gamma) = \prod_{i=1}^{r} \ker(\bar{\lambda}^i_\Gamma) \quad \text{as the } \ker(\bar{\lambda}^i_\Gamma) \text{ are pairwise coprime}$$
$$\subseteq J.$$

Hence $\bar{\mathcal{I}}_\Gamma = \{ I \trianglelefteq \bar{U}_\Gamma \mid \dim \bar{U}_\Gamma/I < \infty \}$. Then

(4)
$$\mathcal{A}'[U] \otimes_{\mathcal{A}'} \Gamma \simeq \bar{U}_\Gamma' \simeq \mathrm{Dist}(\mathbb{G}_\Gamma)',$$

hence we have an injective homomorphism $\theta_\Gamma : \Gamma[\mathbb{G}_\Gamma] \longrightarrow \mathcal{A}'[U] \otimes_{\mathcal{A}'} \Gamma$ of Hopf algebras over Γ by (A.3). If $\bar{\Gamma}$ is the algebraic closure of Γ, then

(5)
$$\begin{array}{ccc} \Gamma[\mathbb{G}_\Gamma] \otimes_\Gamma \bar{\Gamma} & \xrightarrow{\theta_\Gamma \otimes_\Gamma \bar{\Gamma}} & \mathcal{A}'[U] \otimes_{\mathcal{A}'} \Gamma \otimes_\Gamma \bar{\Gamma} \\ \wr\| & \circlearrowleft & \wr\| \\ \bar{\Gamma}[\mathbb{G}_{\bar{\Gamma}}] & \xrightarrow[\theta_{\bar{\Gamma}}]{} & \mathcal{A}'[U] \otimes_{\mathcal{A}'} \bar{\Gamma} \end{array}$$

with $\theta_{\bar{\Gamma}}$ invertible by (4) and (A.3). Hence θ_Γ is already invertible, as desired.

(A.5) Remark. By (A.4)(5) we see that for any field K of characteristic 0

$$K[\mathfrak{G}_K] \xrightarrow{\sim} \mathrm{Dist}(\mathfrak{G}_K)'$$

in Th.A.3.

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