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On the vertices of modules in the Auslander–Reiten quiver

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I. Let kG be the group algebra of a finite group G over a field k of characteristic p, where p is a prime. In the theory of modular representations of finite groups, the stable Auslander–Reiten quiver (AR quiver for short) $\Gamma_s(kG)$ of kG is quite important. Probably, one of the main problems is to understand, from the module theoretic point of view, why all the indecomposable kG-modules are distributed into several connected components of $\Gamma_s(kG)$. Since the vertex is an invariant for each indecomposable kG-module, it might be meaningful to investigate how vertices of modules in a single component vary. In this note, we ask when a single connected component Γ of $\Gamma_s(kG)$ has two indecomposable kG-modules with distinct vertices, and give one theorem in this nature, which generalizes some previous results. The detailed version is now prepared by Professor Okuyama and myself ([**OU2**]).

We use the following notations and conventions. For any finite dimensional k-algebra R, we denote its radical by JR. All modules considered here are finite dimensional right modules. If a module X is isomorphic to a direct summand of another module Y, we write $X|Y$ for notational convenience. Also, the direct sum of m copies of X is denoted by mX. The endomorphism algebra of a kG-module X is denoted by $E_G(X)$, or simply by $E(X)$. Also, for an indecomposable module X, we write its vertex by $vx(X)$, and if X is non-projective, then the Auslander–Reiten sequence terminating at X, which is unique up to equivalence, is denoted by

$$A(X) : 0 \longrightarrow \tau X \longrightarrow M(X) \longrightarrow X \longrightarrow 0,$$

where τ is the Auslander–Reiten translate. Notice that since kG is a symmetric algebra, τ is equal to Ω^2, the composite of two Heller operators.

II. We first mention a brief history of this problem.

Local results

(II.1) If $X \to Y$ is a part of Γ, then we have $vx(X) \leq_G vx(Y)$ or $vx(Y) \leq_G vx(X)$ ([**E1**, (2.3)]).

(II.2) Consider relative projectivity of short exact sequences. Green ([**G**]) showed that we can define the vertex of AR sequences analogously. We denote the vertex of $A(X)$ by $vx(A(X))$. Note that all the modules appearing in $A(X)$ are $vx(A(X))$-projective. He also proved the following.

$vx(X) \leq_G vx(A(X)) \leq_G I$, where I is the inertia subgroup of the source of X in $N_G(vx(X))$ ([**G**, (5.12),(7.7)]).

(II.3) $vx(A(X))$ is a maximal element among vertices of modules appearing in $A(X)$. ([**OU1**, Theorem])

(II.4) Write $M(X) = \oplus_i Y_i$. If $vx(X) < vx(A(X))$, then there exists i such that $vx(Y_i) = vx(A(X))$. (See (II.3).) Moreover, if X is not periodic, then such an i is unique. ([**U2**, Theorem 4.1])

Global results

(II.5) $\{vxX | X \in \Gamma\}$ has a minimal element with respect to $\leq_G$. ([**K1**, Lemma 3.1])

(II.6) It is known by Riedtmann that every connected component Γ has a tree T from which Γ can be obtained as a form $\mathbf{Z}T/\pi$, where π is a subgroup of $Aut(\mathbf{Z}T)$.

(See [**B**, 2.29].) The shape of T is called the tree class of Γ, and if Γ is infinite, then T is one of $\tilde{A}_{1,2}$, $\tilde{B}_3$, A_∞, D_∞, B_∞, C_∞ or A_∞^∞. (For the notation used here, see [**B**, 2.30]. See also [**B**, 2.31].) Moreover, $\tilde{A}_{1,2}$ and $\tilde{B}_3$ occur only when the modules in Γ lie in a block whose defect group is a four group, and if k is algebraically closed, then only $\tilde{A}_{1,2}$, A_∞, D_∞ and A_∞^∞ can occur. (See [**B**, 2.31].) Furthermore, if the tree class is A_∞^∞, then we have $\Gamma \cong \mathbf{Z}A_\infty^\infty$ unless the modules in Γ lie in a block whose defect group is a four group. ([**ES**, Sec. 3,4]). See also [**W**] and [**Be**]. In any case, for our question, it suffices to know how vertices vary on T.

(II.7) ([**E2**, Theorem A]) Let k be a perfect field, and G be a p-group. Suppose that Γ is not a tube. Then one of the following holds.

(i) All the modules in Γ have vertices in common.

(ii) We can take $T : X_1 - X_2 - X_3 - \cdots - X_n - \ldots$ in Γ with $\Gamma \cong \mathbf{Z}T$ and $vx(X_1) \leq vx(X_2) \leq vx(X_3) = vx(X_4) = \cdots = vx(X_n) = \ldots$. Moreover, one of the following holds.

(iia) $vx(X_1) < vx(X_2) = vx(X_3)$.

(iib) $vx(X_1) = vx(X_2) < vx(X_3)$ with $|vx(X_3) : vx(X_2)| = p = 2$.

(iii) $p = 2$, $\Gamma = \mathbf{Z}A_\infty^\infty$, and the modules with the smallest vertex Q lie in a subquiver Γ_Q such that both Γ_Q and $\Gamma \setminus \Gamma_Q$ are connected as graphs isomorphic to $\mathbf{Z}A_\infty$. Moreover, all the modules in $\Gamma \setminus \Gamma_Q$ have the same vertex P with $|P : Q| = 2$.

(iv) $p = 2$, $\Gamma \cong \mathbf{Z}A_\infty^\infty$, the modules with the smallest vertex Q lie in two adjacent τ-orbits and all the other modules have the same vertex P. Moreover, Q is a Kleinian four group and P is a dihedral group of order 8.

(II.8) The above is first generalized as follows ([**U2**, Theorem A]). Let k be a perfect field. Suppose that Γ is not a tube and that its minimal vertex is not a four group. Then one of the following holds.

(i) All the modules in Γ have vertices in common.

(ii) The tree class of Γ is A_∞ and we can take $T : X_1 - X_2 - X_3 - \cdots - X_n - \dots$ such that $vx(X_1) \leq vx(X_2) \leq \cdots \leq vx(X_{n-1}) < vx(X_n) = vx(X_{n+1}) = \dots$. Moreover we have;

(iia) If p is odd and $vx(X_i) = vx(X_{i+1})$ for some i, then the above n must be at most i.

(iib) If $p = 2 = |vx(X_{i+1}) : vx(X_i)|$ for some i, then the above n must be $i+1$.

(iii) $p = 2$, $\Gamma = \mathbf{Z}A_\infty^\infty$, and the modules with the smallest vertex Q lie in a subquiver Γ_Q such that both Γ_Q and $\Gamma \setminus \Gamma_Q$ are connected as graphs isomorphic to $\mathbf{Z}A_\infty$. Moreover, all the modules in $\Gamma \setminus \Gamma_Q$ have the same vertex P with $|P : Q| = 2$.

III. The following is the main result of this note.([**OU2**])

Theorem. *Let k be a perfect field, and let Γ be a connected component of $\Gamma_s(kG)$. Suppose that it is not a tube. Then one of the following holds.*

(i) All the modules in Γ have vertices in common.

(ii) We can take $T : X_1 - X_2 - X_3 - \cdots - X_n - \dots$ in Γ with $\Gamma \cong \mathbf{Z}T$ and $vx(X_1) \leq vx(X_2) \leq vx(X_3) \leq vx(X_4) = vx(X_5) = \cdots = vx(X_n) = \dots$. Moreover, one of the following holds.

(iia) $vx(X_1) < vx(X_2) = vx(X_3) = vx(X_4)$.

(iib) $vx(X_1) < vx(X_2) = vx(X_3) < vx(X_4)$ with $|vx(X_4) : vx(X_3)| = p = 2$.

(iic) $vx(X_1) = vx(X_2) < vx(X_3) = vx(X_4)$ with $|vx(X_3) : vx(X_2)| = p = 2$.

(iii) $p = 2$, $\Gamma = \mathbf{Z}A_\infty^\infty$, and the modules with the smallest vertex Q lie in a subquiver Γ_Q such that both Γ_Q and $\Gamma \setminus \Gamma_Q$ are connected as graphs isomorphic to $\mathbf{Z}A_\infty$. Moreover, all the modules in $\Gamma \setminus \Gamma_Q$ have the same vertex P with $|P : Q| = 2$.

(iv) $p = 2$, $\Gamma \cong \mathbf{Z}A_\infty^\infty$, the modules with the smallest vertex Q lie in two or four adjacent τ-orbits and all the other modules have the same vertex P. Moreover,

Q is a Kleinian four group and P is a dihedral group of order 8.

Remark. In Erdmann's theorem (II.7), we do not know that there is an example which satisfies (iib), and similarly, there is no known example of (iib) or (iic) above.

IV. Here we give some devices for proving the above results. First, notice that in order to prove our theorem, we may assume that k is algebraically closed.(See [**B**, 2.33] and [**E2**, Sec.3].) The following are used when proving the results in II.

(IV.1) (Green-Kawata correspondence, [**K1**, Theorem 4.6]) Let Q be a minimal vertex of modules in Γ, H the normalizer $N_G(Q)$ of Q in G, and let X a module in Γ with $vx(X) = Q$, U the Green correspondent of X with respect to (G,Q,H), and Ξ a connected component of $\Gamma_s(kH)$ containing U. Then, there is a vertex preserving graph monomorphism κ from Γ to Ξ. Moreover, if Y in Γ has Q as its vertex, then $\kappa(Y)$ is nothing but the Green correspondent of Y. Furthermore, the image $\kappa(\Gamma)$ consists of modules X' with the property :

There exist kG-modules $X' = X_1, X_2, \ldots X_m = X$ such that X_n and X_{n+1} are connected by an irreducible map for all n with $0 \leq n \leq m-1$ and $vx(X_n) \geq_G Q$ for all n.

(IV.2) By the above result our problem can essentially be reduced to the case where Q is normal in G. Take X in Γ with source V and $vx(X) = Q$. Let I be the inertia subgroup of V in G. Then there is an indecomposable direct summand U of V^I such that $U^G = X$. This U is determined uniquely up to isomorphisms and gives a primitive idempotent e of $E_I(V^I)$. The following yields that the question on $vx(A(X))$ is equivalent to that on the vertex of the simple module $eE_I(V^I)/eJ(E_I(V^I))$ ([**U1**, Theorem 5.4]).

Let $P = vx(A(X))$. (Note that $P \geq Q$ by (II.2).) Then a vertex of the simple $E_I(V^I)$-module $eE_I(V^I)/eJ(E_I(V^I))$ is P/Q.

Notice that $E_I(V^I)$ becomes naturally an I/Q-graded algebra over $E_Q(V)$. Since k is algebraicaly closed, $\overline{E} = E_I(V^I)/(J(E_Q(V))E_I(V^I)$ is a twisted group algebra of I/Q over k.

(IV.3) Here we give an outline of the proof of (II.3) and (II.4). We consider the following exact sequence which is naturally induced from $A(U)$. (Here ${}_I(U, U')$ means $\mathrm{Hom}_{kI}(U, U')$.)

$$0 \to {}_I(V^I, \tau U) \to {}_I(V^I, M(U)) \xrightarrow{\sigma} {}_I(V^I, U) \to e\overline{E}/eJ(\overline{E}) \to 0$$

Then

$$0 \to {}_I(V^I, \tau U) \to {}_I(V^I, M(U)) \to \mathrm{Im}\sigma \to 0$$

splits as a sequence of $E_I(V^I)$-modules ([**OU1**, Prop.4.2]), and thus we get

$${}_I(V^I, M(U)) \cong {}_I(V^I, \tau U) \oplus \mathrm{Im}\sigma.$$

So, first reduce the problem to U and I using (IV.1), [**K2**] etc. If $vx(A(U)) > Q$, then $\mathrm{Im}\sigma$ is non-zero indecomposable and not projective as an $E_I(V^I)$-module. Note that ${}_I(V^I, \tau U)$ and ${}_I(V^I, U)$ are projective as $E_I(V^I)$-modules ([**OU1**, Lemma 3.6]). Hence there must be an indecomposable summand Y of $M(U)$ such that $\mathrm{Im}\sigma |{}_I(V^I, Y)$. Since $\mathrm{Im}\sigma$ and $eE_I(V^I)/eJ(E_I(V^I))$ have the same vertex, this and (IV.2) give $vx(A(U)) = vx(Y)$. For the unicity of such a summand, we need more computations on multiplicities.

(IV.4) The next result gives a decomposition of the restriction of the middle term to Q, and turns out to be a very powerful device. ([**G**, (7.9)], see also [**U2**, Lemma 2.6].)

In the same situation and notation as above, we have

$$M(X)_Q \cong \oplus_{g\in I\backslash G}\{a(X)(M(V)g) \oplus (b(X)-a(X))(Vg \oplus \tau Vg)\},$$

where $a(X) = \dim_k e\overline{E}/eJ(\overline{E})$ and $b(X) = \dim_k eE_I(V^I)/eJ(E_Q(V))E_I(V^I)$. (Note also that $X_Q = b(X) \oplus_g Vg$.)

(IV.5) If $M(X)$ is indecomposable, then X lies at the end of Γ and some computations yield that the tree class is A_∞ and we get (ii) of (II.8). On the other hand, if $M(X)$ is not indecomposable and that $M(X)$ is not Q-projective, then by (II.4) there is an indecomposable direct summand X' of $M(X)$ with $vx(X') = Q$. Letting V' be a source of X', (IV.4) implies that $b(X') \le a(X)$ or $b(X') \le 2a(X)$, and $b(X) \le a(X')$ or $b(X) \le 2a(X')$. Then it follows that $b(X)/a(X)$ is 2, thus (IV.2) and the fact that $\overline{E}$ is a symmetric algebra imply that we have $p = 2$ and $|vx(Y) : vx(X)| = 2$, where Y is an indecomposable direct summand of $M(X)$ with $vx(Y) > Q$. This is a typical argument in the proof of (II.8).

(IV.6) Using this kind of argument and the results of Erdmann (II.6) and Kawata (IV.1), we can prove the following. (See Proof of [**U2**, Theorem A].)

Let $X' \to X \to Y$ be a part of Γ with $Q = vx(X) = vx(X') < vx(Y)$. Suppose that Q is normal and that $vx(X)$ is not a four group. Then $|vx(Y) : vx(X)| = p = 2$ and one of the following holds.

(a) In the tree T, we have a sequence of modules whose vertex Q such that it is isomorphic to A_∞.

(b) Modules in the τ-orbits of X and X' are only modules in Γ having the same vertex as of X.

In each case, modules with bigger vertex have $vx(Y)$ as their vertices.

(IV.7) The above and the next result ([**U2**, Theorem 4.3]) give many restrictions on the structure of Γ.

Let $X \to Y$ be a part of Γ with $|vx(X)| < |vx(Y)|$. Suppose that X and Y are non-periodic. Then in the tree T there is a sequence of modules whose vertex is bigger than or equal to $vx(Y)$ such that it is isomorphic to A_∞.

(IV.8) If Q is not a four group, then the above (IV.6) and (IV.7) easily imply (II.8). To prove the main theorem in the case where Q is a four group, we use essentially the same argument and use the structure theorem of $\Gamma_s(kQ)$. We omit the detail, which is found in [**OU2**].

V. In order to complete the proof of the main theorem, the following new idea is necessary. This is a generalization of the fact in (IV.3) and due entirely to T.Okuyama. We fix the following notation. Q is a normal subgroup of G and V is an indecomposable G-invariant non-projective kQ-module, and for any subgroup H of G with $Q \le H$, we write $E_H = E_H(V^H)$ and $\overline{E}_H = E_H/J(E_Q)E_H$.

(V.1) We now define a category $C(H)$ as follows. Objects of $C(H)$ are short exact sequences of kH-modules

$$0 \to \tau Y \to Z \to X \to 0$$

such that

(i) X and Y are isomorphic to direct summands of direct sum of some copies of V^H, and

(ii) upon the restriction to kQ it is a direct sum of some AR-sequences and a split short exact sequence.

A morphism of $C(H)$ from an object $0 \to \tau Y \to Z \to X \to 0$ to another object $0 \to \tau Y' \to Z' \to X' \to 0$ is a triple (γ, β, α) of kH-homomorphisms which makes the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \gamma \downarrow & & \beta \downarrow & & \alpha \downarrow & & \\ 0 & \longrightarrow & \tau Y' & \longrightarrow & Z' & \longrightarrow & X' & \longrightarrow & 0 \end{array}$$

Notice that by [**B**, 2.17.7] and [**G**, (7.9)], for any indecomposable direct summand X of V^H, the AR-sequence $A(X)$ is an object of $C(H)$. Note that varying subgroups of G/Q, we can define induction, restriction, conjugation functors between suitable categories. We admit usual subscript superscript notations for those notions.

(V.2) We consider those short exact sequences of group modules and modules over $\overline{E}_H$ constructed naturally from those group modules. This construction is closely related to arguments in [**P**] and [**U1**], and essentially due to those ideas. Define an equivalence relation $\equiv$ among objects of $C(H)$ by the relation that $\mathcal{S} \equiv \mathcal{S}'$ if and only if $\mathcal{S} \oplus \mathcal{T} \cong \mathcal{S}' \oplus \mathcal{T}'$ for some split short exact sequences $\mathcal{T}$ and $\mathcal{T}'$. Then we can define a category $\overline{C}(H)$ as follows. Its objects are these equivalence classes. For two objects $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}'}$ of $\overline{C}(H)$, the morphism set $\mathrm{Hom}(\overline{\mathcal{S}}, \overline{\mathcal{S}'})$ is $\mathrm{Hom}(\mathcal{S}, \mathcal{S}')/\mathrm{Ker}\Sigma$, where

$$\mathrm{Ker}\Sigma = \{(\gamma, \beta, \alpha) \in \mathrm{Hom}(\mathcal{S}, \mathcal{S}') | \alpha = \sigma'\delta \text{ for some } \delta \in \mathrm{Hom}_{kH}(X, Z')\}.$$

(V.3) Let $\mathcal{S} : 0 \to \tau Y \to Z \xrightarrow{\sigma} X \to 0$ be an object of $C(H)$. Then ${}_H(V^H, X)J(E_Q) \subseteq Im(\sigma_*)$, where $\sigma_* : {}_H(V^H, Z) \to {}_H(V^H, X)$ is the map naturally induced from σ.

(V.4) Now we define a functor F_H from $\overline{C}(H)$ to $\mathrm{Mod}(\overline{E}_H)$. Let $\mathcal{S} : 0 \to \tau Y \to Z \xrightarrow{\sigma} X \to 0$ be a representative of an object of $\overline{C}(H)$. Then we let $F_H(\overline{\mathcal{S}})$ be the cokernel of σ_*. Namely, $F_H(\overline{\mathcal{S}}) = {}_H(V^H, X)/\mathrm{Im}\sigma_*$. Note that (V.3) implies that $F_H(\overline{\mathcal{S}})$ becomes an $\overline{E}_H$-module. For a morphism (γ, β, α) from $\mathcal{S} : 0 \to \tau Y \to Z \xrightarrow{\sigma} X \to 0$ to $\mathcal{S}' : 0 \to \tau Y' \to Z' \xrightarrow{\sigma'} X' \to 0$, the map $F_H(\overline{(\gamma, \beta, \alpha)})$ from $F_H(\overline{\mathcal{S}})$ to $F_H(\overline{\mathcal{S}}')$ is defined in the obvious way so that the following commutes.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & (V^H, \tau Y) & \longrightarrow & (V^H, Z) & \xrightarrow{\sigma_*} & (V^H, X) & \longrightarrow & F(\mathcal{S}) & \longrightarrow & 0 \\
 & & \gamma_* \downarrow & & \beta_* \downarrow & & \alpha_* \downarrow & & F_H(\overline{(\gamma,\beta,\alpha)}) \downarrow & & \\
0 & \longrightarrow & (V^H, \tau Y') & \longrightarrow & (V^H, Z') & \xrightarrow{\sigma'_*} & (V^H, X') & \longrightarrow & F(\mathcal{S}') & \longrightarrow & 0,
\end{array}$$

Note that for any indecomposable direct summand X of V^H, we have $F_H(A(X)) = eE_I(V^I)/eJ(E_I(V^I))$ in the notation of (IV.2). For an object $\mathcal{S}$ of $C(H)$, we also use $F_H(\mathcal{S})$ to mean the object $F_H(\overline{\mathcal{S}})$ of $\mathrm{Mod}(\overline{E}_H)$.

(V.5) F_H gives a well defined funcor from $\overline{C}(H)$ to $\mathrm{Mod}(\overline{E}_H)$ and commutes with restriction, induction and conjugation functors. Moreover, it gives an isomorphism of categories.

(V.6) Let $\mathcal{S} : 0 \to \tau Y \to Z \xrightarrow{\sigma} X \to 0$ be an object of $C(H)$. Then, we have the following which generalizes the result in (IV.3).

(i) $0 \to {}_H(V^H, \tau Y) \to {}_H(V^H, Z) \to \mathrm{Im}\sigma_* \to 0$ splits as a sequence of E_H-modules.

(ii) If $F_H(\mathcal{S})$ is a non-projective indecomposable $\overline{E}_H$-module with vertex $\overline{P} = P/Q$. Then Z is P-projective and there is a direct sum decomposition $Z = Z_1 \oplus Z_0$ such that

(iia) Z_1 is indecomposable with $vx(Z_1)Q = P$, (iib) $(V^G, Z_1)/\mathrm{Ann}(V^G, Z_1) \cong \Omega(F_H(\mathcal{S})) \oplus B$, where B is a zero or a projective $\overline{E}$-module and Ω is taken among $\overline{E}_H$-modules, and

(iic) $(V^G, Z_0)/\mathrm{Ann}(V^G, Z_0)$ is a projective $\overline{E}$-module.

Here $\mathrm{Ann}(B')$ denotes $\{b \in B' | ba = 0 \text{ for all } a \text{ with } J(E_Q)a = 0\}$ for all E_H-module B'.

(V.7) Let H be a normal subgroup of G with $Q \le H$ and $\mathcal{T} : 0 \to \tau Y \to Z \to X \to 0$ be an object of $C(H)$. Write $\overline{G} = G/Q$. Assume that $F_H(\mathcal{T})$ is $\overline{G}$-invariant, that is, $F_H(\mathcal{T}) \otimes u_{\overline{g}} \cong F_H(\mathcal{T})$ for all $\overline{g}$ in $\overline{G}$. Further, we suppose that $F_H(\mathcal{T})$ is indecomposable and that $vx(F_H(\mathcal{T})) \ne \overline{1}$. Then, by (V.6), there exists a direct sum decomposition $Z = Z_0 \oplus Z_1$ such that Z_1 is indecomposable with ${}_H(V^H, Z_1)/\mathrm{Ann}(\ {}_H(V^H, Z_1)) \cong \Omega(F_H(\mathcal{T})) \oplus Z'$, where Z' is a zero or projective, and that Z_0 has no direct summand isomorphic to Z_1. In particular, it follows that Z_1 is G-invariant since $F_H(\mathcal{T})$ is $\overline{G}$-invariant. Put $B = F_H(\mathcal{T})$. Now recall that $\overline{E}_G$ is a twisted group algebra of $\overline{G}$ over k. Thus $\mathrm{End}_{\overline{E}_H}(B)$ can be regarded as a subalgebra of $\mathrm{End}_{\overline{E}_G}(B \otimes_{\overline{E}_H} \overline{E}_G)$ by identifying f in $\mathrm{End}_{\overline{E}_H}(B)$ with $f \otimes_{\overline{E}_H} Id_{\overline{E}_G}$. Write $E_G(B^G) = \mathrm{End}_{\overline{E}_H}(B \otimes_{\overline{E}_H} \overline{E}_G)$ and $\tilde{E}_G(B^G) = E_G(B^G)/J(\mathrm{End}_{\overline{E}_H}(B))E_G(B^G)$ for convenience. Also, we put $\tilde{E}_G(Z_1^G) = E_G(Z_1^G)/J(\mathrm{End}_{kH}(Z_1))\mathrm{End}_{kG}(L_1^G)$. We define a map Θ from $E_G(B^G)$ to $\tilde{E}_G(Z_1^G)$ in the following way. First, let $\iota : Z_1 \to Z$ and $\pi : Z \to Z_1$ be the injection and the projection with respect to the above decomposition of Z. Then, $\iota^G : Z_1^G \to Z^G$ and $\pi^G : Z^G \to Z_1^G$ give the injection and the projection with respect to the decomposition $Z^G = Z_1^G \oplus Z_0^G$. Recall also that $F_G(\mathcal{T}^G) = B \otimes_{\overline{E}_H} \overline{E}_G$. For each λ in $E_G(B \otimes_{\overline{E}_H} \overline{E}_G)$, choose (γ, β, α) in $\mathrm{End}(\mathcal{T}^G)$ with $F_G((\gamma, \beta, \alpha)) = \lambda$. (Such a triple exists by (V.5).) Then, we define $\Theta(\lambda)$ to be the natural image of the map $\pi^G \beta \iota^G$ in $\tilde{E}_G(Z_1^G)$. Put

$$\widetilde{E_G(B^G)} = \mathrm{End}_{\overline{E}_G}(B \otimes_{\overline{E}_H} \overline{E}_G)/J(\mathrm{End}_{\overline{E}_H}(B))\mathrm{End}_{\overline{E}_G}(B \otimes_{\overline{E}_H} \overline{E}_G) \text{ and}$$
$$\widetilde{E_G(Z_1^G)} = \mathrm{End}_{kG}(Z_1^G)/J(\mathrm{End}_{kH}(Z_1))\mathrm{End}_{kG}(Z_1^G).$$

The above Θ is well defined and induces a k-algebra isomorphism from $\widetilde{E_G(B^G)}$ to $\widetilde{E_G(Z_1^G)}$.

(V.8) Let X be an indecomposable direct summand of V^G. Assume that $P = vx(A(X)) > Q$. Let $\mathcal{T}$ be an object of $C(P)$ such that $B = F_H(\mathcal{T})$ is a source of $eE_G(V^G)/eJ(E_G(V^G))$, and let I be the inertia subgroup of B in $N_G(P)$. Since $eE_G(V^G)/eJ(E_G(V^G))$ is simple, the projective indecomposable $\widetilde{E_I(B^I)}$-module corresponding to $eE_G(V^G)/eJ(E_G(V^G))$ (via Green and Clifford correspondence) is also simple by [**Kn**, Proposition 3.1]. Thus the $\widetilde{E_I(Z_1^I)}$-module corresponding to the above one via (V.7) is also projective and simple. (Here Z_1 is the module as in (V.6) for the object $\mathcal{T}$. Note that I is also the inertia subgroup of Z_1 in $N_G(P)$.) On the other hand, we can also prove that the kP-mosule Z_1 is the source of the indecomposable direct summand Y of $M(X)$ with $vx(Y) = P$. (See (II.4).) Therefore, again by (IV.2), we have the following.

$$vx(Y) = vx(A(X)) = vx(A(Y)).$$

(V.9) Of course, the above says that if $X \to Y$ is a part of the tree T with $vx(X) < vx(Y)$, then the module Z next to Y in T must satisfy $vx(Y) = vx(Z)$. Using this, (II.8) and (IV.6), an inductive argument and the reduction methods yield our main result.

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