数理解析研究所講究録
第 803 巻 1992 年 107-124

APPLICATIONS OF HODGE MODULES – KOLLÁR CONJECTURE AND KODAIRA VANISHING –

BY
MASA-HIKO SAITO

Department of Mathematics, Faculty of Science, Kyoto University

§0 Hodge Modules.

(0.1). In this note, we will give an exposition of some applications of Morihiko Saito's theory of Hodge modules. All of these applications are due to Morihiko Saito himself.

Though there is a good exposition by Shimizu[Sh], in §0, we will recall quickly the definition of the category $MH(X, \mathbb{Q}, n)$ of Hodge Modules of weight n, mainly for preparing the notations. In §1, we will give the statements of the stability of polarized Hodge modules by projective direct images and the decomposition theorem for the intersection complexes of Beilinson-Bernstein-Deligne-Gabber type, and we will explain how these results imply existence of the natural pure Hodge structures on the intersection cohomology groups.

In §2, we will discuss about Saito's proof of Kollár conjecture on the direct images of the edge components of "generic variation of Hodge structures". §3 is devoted to a generalization of vanishing theorem of Kodaira-type, which follows naturally from the theory of Hodge modules.

(0.2). Let X be a complex manifold. In this note, we will use the filtered *right* $\mathcal{D}_X$-Modules. Let $MF_h(\mathcal{D}_X)$ be the category of filtered $\mathcal{D}_X$-Modules (M, F) such that M is regular holonomic and $Gr^F(M)$ is coherent over $Gr^F\mathcal{D}_X$. By Kashiwara, we have a faithful and exact functor $DR : MF_h(\mathcal{D}_X) \longrightarrow Perv(\mathbb{C}_X)$ (Riemann-Hilbert correspondence), and we define $MF_h(\mathcal{D}_X, \mathbb{Q}_X)$ to be a fiber product of $MF_h(\mathcal{D}_X)$ and $Perv(\mathbb{Q}_X)$ over $Perv(\mathbb{C}_X)$. That is, the objects are $((M, F), K) \in MF_h(\mathcal{D}_X) \times Perv(\mathbb{Q}_X)$ with an isomorphism $\alpha : DR(M) \to K \otimes_{\mathbb{Q}_X} \mathbb{C}_X$, and the morphisms are the pairs of the morphisms compatible with α.

(0.3). Let $i : X \hookrightarrow Y$ be a closed embedding locally defined by $X = \{x_1 = \cdots = x_k = 0\}$ with $(x_1, \cdots, x_m)$ local coordinates of Y. Then for a filtered holonomic $\mathcal{D}_X$-modules (M, F), the direct image $(\tilde{M}, F) = i_*(M, F)$ is defined by $(M, F) \otimes_{\mathcal{D}_X} (\mathcal{D}_{X \to Y}, F)$ (see [Sh]), and locally we have

$$\tilde{M} = M \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \cdots, \partial_k],$$

$$F_p\tilde{M} = \oplus_{\nu\in\mathbb{N}^k} F_{p-|\nu|}M \otimes \partial^\nu,$$

where $\partial^\nu = \prod_{1\leq i\leq k} \partial_i^{\nu_i}, |\nu| = \sum \nu_i, \partial_i = \partial/\partial x_i$. Then we have $DR \circ i_* = i_* \circ DR$ and we get the functor

$$i_* : MF_h(\mathcal{D}_X, \mathbb{Q}) \longrightarrow MF_h(\mathcal{D}_Y, \mathbb{Q}).$$

(0.4). Let g be a holomorphic function on X, and $i_g : X \longrightarrow X \times \mathbb{C}$ the embedding by the graph of g. We say that $(M, F, K) \in MF_h(\mathcal{D}_X, \mathbb{Q})$ is *regular and quasi-unipotent along* g, if the monodromy of $\Psi_g K[-1]$ is quasi-unipotent and $(\tilde{M}, F) = i_{g*}(M, F)$ satisfies

(0.4.1) $$(F_p V_\alpha \tilde{M}) \cdot t \cong F_p V_{\alpha-1}\tilde{M} \qquad \text{for} \qquad \alpha < 0$$

(0.4.2) $$(F_p Gr_\alpha^V \tilde{M}) \cdot \partial_t \cong F_{p+1} Gr_{\alpha+1}^V \tilde{M} \qquad \text{for} \qquad \alpha > -1,$$

where , t is the coordinate of $\mathbb{C}$ and V is the filtration of Kashiwara-Malgrange indexed by $\mathbb{Q}$ such that $t\partial_t - \alpha$ is nilpotent on $Gr_\alpha^V \tilde{M}$. (See [Ka]).

We need the notions of "nearby cycle sheaves" $\Psi_g(K)$ and the "vanishing cycle sheaves" $\Phi_g(K)$, for K a constructible sheaves on X and g a non-constant holomorphic function on X (cf. [SGA7]). They are constructible complexes of sheaves on $g^{-1}(0)$. Gabber proved that, for a non-constant holomorphic function $g : X \longrightarrow \mathbb{C}$, if $K^\cdot$ is a perverse sheaf on X, then $\Psi_g(K)[-1]$ and $\Phi_g(K)[-1]$ are perverse sheaves on $g^{-1}(0)$.

Via the Riemann-Hilbert correspondence, there should exist the corresponding functors Ψ and Φ in the category of holonomic $\mathcal{D}$-modules, and they were constructed explicitly by Malgrange (in the case of $\mathcal{O}_X$), and by Kashiwara [Ka] in the case of regular holonomic $\mathcal{D}$-modules. (For details, see expositions [Sh] and [S.Mu]).

Under the condition (0.4.1-2), we define the nearby cycles functor and the vanishing cycle functor on the level of filtered $\mathcal{D}_X$-modules

$$\Psi_g(M, F, K) = (\oplus_{-1\leq\alpha<0} Gr_\alpha^V(\tilde{M}, F[1]), \Psi_g K)$$

$$\Phi_{g,1}(M, F, K) = (Gr_{-1}^V(\tilde{M}, F), \Phi_{g,1}K),$$

and $can : \Psi_{g,1} \longrightarrow \Phi_{g,1}$ and $Var : \Phi_{g,1} \longrightarrow \Psi_{g,1}(-1)$ are induced respectively by $-\partial_t$ and t, where $F[m]_i = F_{i-m}$. Here $\Psi_{g,1}$ is the unipotent monodromy part of Ψ_g (same for Φ_g). We have

$$\Psi_g(M, F) = 0, \qquad \Phi_{g,1} = (M, F), \qquad \text{if} \qquad suppM \subset g^{-1}(0),$$

because the conditions (0.4.1-2) is equivalent to $(F_pM) \cdot g \subset F_{p-1}M$ in this case.

(0.5) Lemma. *(cf. [S1, 5.1.4]). If $(M, F, K) \in MF_h(\mathcal{D}_X, \mathbb{Q})$ is regular and quasi-unipotent along g for a locally defined holomorphic function g on X, the following conditions are equivalent:*

(0.5.1) *In the category $MF_h(\mathcal{D}_X, \mathbb{Q})$, one has a decomposition*

$$\Phi_{g,1}(M, F) = \mathrm{Im}\ \mathrm{can} \oplus \mathrm{Ker}\ \mathrm{Var},$$

(0.5.2) *One has a unique decomposition in $MF_h(\mathcal{D}_X, \mathbb{Q})$*

$$(M, F, K) = (M_1, F, K_1) \oplus (M_2, F, K_2)$$

where M_2 has a support contained in $X_0 := g^{-1}(0)$ and (M_1, F, K_1) has no sub-object or quotient object supported in X_0.

Let $(M, F, K) \in MF_h(\mathcal{D}_X, \mathbb{Q})$. We say that (M, F, K) has a strict support Z if supp M = supp K = Z and admits no sub-object or quotient object with strictly smaller support.

As a corollary of this lemma, we have the following

(0.6) Proposition. *([S1, 5.1.5]). If $(M, F, K) \in MF_h(\mathcal{D}_X, \mathbb{Q})$ is regular and quasi-unipotent along g, for any g locally defined on X, the following conditions are equivalent:*

(0.6.1) *In the category $MF_h(\mathcal{D}_X, \mathbb{Q})$, one has a decomposition*

$$\Phi_{g,1}(M, F) = \mathrm{Im}\ \mathrm{can} \oplus \mathrm{Ker}\ \mathrm{Var},$$

for any g locally defined on X

(0.6.2) *For any Zariski open set U of X, $(M, F, K)_{|U}$ has the canonical decomposition $\oplus_Z (M_Z, F, K_Z)$ for Z closed irreducible subspaces of U, such that M_Z has strict support Z.*

Moreover M has strict support Z, if and only if supp $M = Z$ and can is surjective, Var is injective for any locally defined g such that $\dim g^{-1}(0) \cap Z < \dim Z$.

(0.7). Let $MF_h(\mathcal{D}_X, \mathbb{Q}_X)_{(0)}$ be the full subcategory of $MF_h(\mathcal{D}_X, \mathbb{Q}_X)$ whose objects are regular and quasi-unipotent along g and satisfies the condition (0.5.1) (or equivalently (0.5.2)), for any g locally defined on X. Moreover, let $MF_h(\mathcal{D}_X, \mathbb{Q})_Z$ be the full subcategory of $MF_h(\mathcal{D}_X, \mathbb{Q}_X)_{(0)}$ whose objects have strict support Z. Then by Proposition (0.6) we have the canonical decomposition (locally finite on X):

(0.7.1) $$MF_h(\mathcal{D}_X, \mathbb{Q})_{(0)} = \oplus_Z MF_h(\mathcal{D}_X, \mathbb{Q})_Z$$

where Z is running over all irreducible subspaces of X.

Let $(M,F,K) \in MF_h(\mathcal{D}_X,\mathbb{Q})_Z$, and g a holomorphic function on X such that $Z \not\subset g^{-1}(0)$ and $can : \Psi_{g,1}(M,F) \to \Phi_{g,1}(M,F)$ is strictly surjective. Then we have

$$F_p\tilde{M} = \sum_i (V_{<0}\tilde{M} \cap j_*j^{-1}F_{p-i}\tilde{M}) \cdot \partial_t^i \tag{0.7.2}$$

with $j : X \times \mathbb{C}^* \hookrightarrow X \times \mathbb{C}$ and $(\tilde{M},F) = i_{g*}(M,F)$ as above. In this case, the filtration on M is uniquely determined by its restriction to the complement of $g^{-1}(0)$.

(0.8) Definition of the Hodge modules.

Now we can define the category of Hodge modules of weight n. First we will give the definition for smooth X, and later mention about the definition for singular X.

(0.8.1) Smooth case. (See [Sh]). Let X be a smooth complex analytic variety. The category $MH(X,\mathbb{Q},n)$ of *Hodge modules of weight* n is the largest full subcategory of $MF_h(\mathcal{D}_X,\mathbb{Q}_X)_{(0)}$ satisfying the following conditions;

(HM1) An object of $MH(X,\mathbb{Q},n)$ with support $\{x\}$ is of the form $(M,F,K) = i_{x*}(H_\mathbb{C},F,H_\mathbb{Q})$ for the inclusion $i_x : \{x\} \hookrightarrow X$, where $(H_\mathbb{C},F,H_\mathbb{Q})$ is a pure $\mathbb{Q}$-Hodge structure of weight n with increasing filtration $F_p = F^{-p}$.

(HM2) If $M \in MH(X,\mathbb{Q},n)$, M is regular and quasi-unipotent along g, and $Gr_i^W\Phi_gM$, $Gr_i^W\Psi_{g,1}M \in MH(U,i)$ for any i, $\Psi_{g,1} = \mathrm{Im}\ (\mathrm{can}) \oplus \mathrm{Ker}\ (\mathrm{Var})$, for any holomorphic function g on an open subset U of X, where W is the monodromy filtration shifted by n - 1 and n.

One can check the well-definedness of this definition by the induction on dim supp M.

(0.8.2) Singular case. Let X be a reduced, separated complex analytic spaces, and take a locally finite covering $X = \cup_i U_i$ and a set of embeddings $U_i \hookrightarrow V_i$ where V_i are smooth varieties. Then a Hodge module of weight n on X can be defined by patching local pieces with compatibility conditions. See Shimizu's exposition [Sh] for detail.

Let

$$MH_Z(X,\mathbb{Q},n) = MH(X,\mathbb{Q},n) \cap MF_h(\mathcal{D}_X,\mathbb{Q})_Z,$$

so that we have the strict support decomposition

$$MH(X,\mathbb{Q},n) = \oplus_Z MH_Z(X,\mathbb{Q},n). \tag{0.8.3}$$

according to (0.7.1).

(0.9). Every morphism in the categories $MH(X,\mathbb{Q},n)$ and $MH_Z(X,\mathbb{Q},n)$ is strict with respect to the filtrations F. Furthermore, these subcategories of $MF_h(\mathcal{D}_X,\mathbb{Q})$ are stable under the operation of taking a direct summand.

(0.10) Objects. In order to see what objects are in $MH(X, \mathbb{Q}, n)$, we will recall the definition of intersection (co-)homology complex. Let X be an irreducible analytic variety of dimension n with a Whitney stratification $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ by analytic subvarieties. The stratums $S_i = X_i - X_{i-1}$ are smooth manifolds of dimension i if it is non-empty. Let $U_k = X - X_k$ be Zariski open sets of X: $U_{-n} \subset U_{-n+1} \subset \cdots \subset U_0 = X$, and let $j_k : U_{k-1} \hookrightarrow U_k$ be the inclusions. Note that $U_{-n} = X - X_{n-1}$ is a smooth Zariski open subset of X. Let L be a local system of $\mathbb{Q}$-vector spaces on U_{-n}. Then we define the intersection (co-)homology complex (with middle perversity) with coefficients in L to be

$$\mathcal{IC}_X(L) = \tau_{\leq -1} Rj_{0*} \cdots \tau_{\leq -n} Rj_{1-n*} L[n] \text{ in } D^b_c(\mathbb{Q}_X) \tag{0.10.1}$$

where τ is the truncation functor. In [BBD], this is denoted by

$$j_{!*}L[n] = \mathrm{Im}(j_! L[n] \longrightarrow j_* L[n]) \tag{0.10.2}$$

where $j : U_{-n} \hookrightarrow X$. It can be proved that $\mathcal{IC}_X(L)$ is independent of stratification.

Let $(\mathbf{V}_{\mathbb{Q}}, F)$ be a variation of Hodge structure of weight n on a smooth complex manifold X (see Usui's exposition [U]), and set $\mathcal{V} = \mathbf{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_X$. We define $(M, F, K) \in MF_h(\mathcal{D}_X, \mathbb{Q}_X)$ for $(\mathbf{V}_{\mathbb{Q}}, F)$ by setting

$$M = \Omega_X^{dimX} \otimes_{\mathcal{O}} \mathcal{V}, \quad F_p M = \Omega_X^{dimX} \otimes_{\mathcal{O}} F^{-p-dimX} \mathcal{V} \tag{0.10.3}$$

$$K = \mathbf{V}_{\mathbb{Q}}[dimX]. \tag{0.10.4}$$

(0.11) Proposition. *([S1, 5.1.10]). Let $(M, F, K) \in MH_Z(X, \mathbb{Q}, n)$, then K is an intersection homology complex $\mathcal{IC}_Z(\mathbf{V}_{\mathbb{Q}})$ and (M, F, K) is generically a variation of Hodge structure of weight $n - d_Z$, i.e. there exists a smooth Zariski dense open set U of Z and a variation of polarized Hodge structure $(\mathbf{V}_{\mathbb{Q}}, F)$ of weight $n - d_Z$ on U such that $(M, F, K)_{|U}$ is isomorphic to $(\Omega_U^{dimU} \otimes_{\mathcal{O}} \mathcal{V}, F, \mathbf{V}_{\mathbb{Q}}[d_Z])$ where the filtration F is given by (0.10.3).*

In order to state the stability of the category of Hodge module under the direct image, one has to introduce the notion of "polarization" of a Hodge module. For $k \in \mathbb{Z}$, Let $\mathbb{Q}(k)$ denote the Hodge structure of Tate of weight $-2k$ and of type $(-k, -k)$.

(0.12) Definition. Assume that $((M, F), K)$ belongs to $MH_Z(X, \mathbb{Q}, n)$ for some irreducible Z. A *polarization* is a pairing

$$S : K \otimes K \longrightarrow a_X^! \mathbb{Q}(-n)$$

which satisfies the following conditions.

(1) If $Z = \{x\}$, there is a polarization S' of Hodge structure M' such that $S = i_{x*}S'$, where i_x and M' as in (i).

(2) S is compatible with the Hodge filtration F, i.e. the corresponding isomorphism $K \simeq (\mathbb{D}K)(-n)$ is extended to an isomorphism $(M, F, K) \cong \mathbb{D}(M, F, K)(-n)$.

(3) For any holomorphic function g on X such that $g^{-1}(0) \not\subset Z$, the induced pairing

$$^{p}\Psi_g S \circ (id \otimes N^i) : Gr^{W}_{n-1+i}\Psi_g K[-1] \otimes Gr^{W}_{n-1+i}\Psi_g K[-1] \longrightarrow a^{!}_{U}\mathbb{Q}(n-1-i)$$

is a polarization on the primitive part $P_N Gr^{W}_{n-1+i}\Psi_g(M, F, K)$. Here, P_N denotes the primitive part with respect to N, and one uses the fact that Ψ commutes with Verdier duality, and the self-duality of the monodromy weight filtration W.

We can give the following examples of polarizable Hodge module.

Let X be a smooth complex manifold of dimension d_X, $(\mathbf{V}_{\mathcal{O}}, F.\mathbf{V}_{\mathbb{Q}})$ a $\mathbb{Q}$-VHS of weight $(n - d_X)$ with the polarization

$$S' : \mathbf{V} \otimes \mathbf{V} \longrightarrow \mathbb{Q}(d_X - n).$$

We define $\mathcal{M} = (M, F, K) \in MF_h(\mathcal{D}_X, \mathbb{Q}_X)$ as in (0.10.3-4), and let S be a polarization on $\mathcal{M}$ induced by S' (see, (2.3.4) of [Sh] or, (5.2.12) of [S1]).

(0.13) Theorem. *([S1, 5.4.3]). Under the above notation, $((M, F, K), S)$ is a polarized Hodge module of weight n.*

Moreover, in relation to (0.11), Saito proved that a polarizable Hodge modules with strict support Z (i.e. its underlying perverse sheaf is an intersection homology complex $\mathcal{IC}_Z(L)$) is a polarizable variation of Hodge structure on a dense Zariski open subset of Z. (See (5.1.10) and (5.2.12) in [S1]). In later article [S2], Saito proved that the converse is also true, i.e. any polarizable variation of Hodge structure with quasi-unipotent local monodromies[1] defined on a smooth dense Zariski open subset of Z can be uniquely and funtorially extended to a polarizable Hodge module with strict support. Therefore, we obtain the following

(0.14) Theorem. *((3.21) in [S2]). For a reduced irreducible separated complex analytic space X of dimension d_X, we have the equivalence of categories:*

$$MH_X(X, \mathbb{Q}, n)^p \cong VHS_{gen}(X, \mathbb{Q}, n - d_X)^p.$$

Here $VHS_{gen}(X, n-d_X)$ is the inductive limit of $VHS(U, \mathbb{Q}, n-d_X)^p$ the categories of polarizable variation of $\mathbb{Q}$-Hodge structures of weight $n - d_X$ on smooth dense Zariski open subsets U. Moreover the polarization corresponds bijectively.

[1]This condition is always satisfied if L has a $\mathbb{Q}$-structure, i.e., if L is a $\mathbb{Q}$-VHS.

(0.15) Direct images. Let $f : X \longrightarrow Y$ be a proper morphism of smooth algebraic varieties, $i_f : X \longrightarrow X \times Y$ the embedding by graph, and $p : X \times Y \longrightarrow Y$ the natural projection. Then the direct image of filtered $\mathcal{D}_X$-module (M, F) is defined by

(0.15.1) $$f_*(M,F) = \mathbf{R}p_* DR_{X\times Y/Y}(i_f)_*(M,F),$$

where $(i_f)_*$ is as in (0.3), $\mathbf{R}p_*$ is the sheaf theoretic direct image. For $(M, F, K) \in MF_{rh}(\mathcal{D}_X, \mathbb{Q})$, we define

$$f_*\mathcal{M} = (f_*(M,F), f_*K), \qquad \mathcal{H}^i f_*\mathcal{M} = (\mathcal{H}^i f_*(M,F), {}^p\mathcal{H}^i f_*K)$$

with the isomorphisms

$$DR(f_*M) = f_*K \otimes_{\mathbb{Q}} \mathbb{C}, \qquad DR(\mathcal{H}^i f_*\mathcal{M}) = {}^p\mathcal{H}^i f_*K \otimes_{\mathbb{Q}} \mathbb{C}$$

induced by $DR \circ f_* = f_* \circ DR$, $DR\mathcal{H}^i = {}^p \mathcal{H}^i \circ DR$.

§1 Stability and Decomposition Theorem.

Now we can state the stability theorem of Hodge modules by the projective direct image, which is one of the main theorems in [S1].

(1.0) Stability Theorem. *(Thèorém (5.3.2) in [S1]). Let $f : X \longrightarrow Y$ be a projective morphism between smooth complex analytic varieties, and l be the first Chern class of a relative ample line bundle. Assume that $((M,F),K) \in MH_Z(X,\mathbb{Q},n)$ is endowed with a polarization S. Then:*

(1.0.1) the complex $f_*(M,F)$ is strict and $\mathcal{H}f_*((M,F),K) \in MH(Y,\mathbb{Q},n+i)$

(1.0.2) the hard Lefschetz theorem holds, i.e.,

$$l^i : \mathcal{H}^{-i} f_*((M,F),K) \xrightarrow{\simeq} \mathcal{H}^i f_*((M,F),K)$$

is an isomorphism;
(1.0.3)

$$(-1)^{i(i-1)/2}\,.^p\mathcal{H}f_*S \circ (id \otimes l^i) : P_l^p\mathcal{H}^{-i} f_*K \otimes P_l^p\mathcal{H}^{-i} f_*K \longrightarrow a_Y^!\mathbb{Q}(-n+i)$$

is a polarization of the primitive part $P_l^p\mathcal{H}^{-i} f_*K (:= Ker l^{i+1} \subset \mathcal{H}^{-i} f_*K)$.

The proof is also due to the induction of dimension supp $M = Z$.

Saito also proved Kähler package of the stabilty theorem for the constant sheaf $(M,F,K) = (\mathcal{O}_X, F, \mathbb{R}_X[d_X])$ with $Gr_i^F \mathcal{O}_X = 0$ for $i \neq 0$.

(1.1) Theorem. *(Theorem (3.1) in [S3]). Let $f : X \longrightarrow Y$ be a proper morphism of complex analytic spaces. Assume that X is smooth Kähler with Kähler class l. Then we have the stability theorem (1.0.1-3) for the constant sheaf $(M, F, K) = (\Omega^{d_X}{}_X, F, \mathbb{R}_X[d_X])$.*

Let X be an irreducible *smooth* complex projective variety, L a polarized variation of Hodge structure over a Zariski dense open subset of X, and (M, F, K) a Hodge module corresponding to $\mathcal{IC}(L)$ (see theorem (0.14)). In case Y is a point, the assertion that the differential of $f_*(M, F)$ is strict with the filtration F is equivalent to say that

$$E^{p,q} = H^{p+q}(X, Gr^F_{-p}(\mathcal{IC}(L))) \Rightarrow IH^{p+q}(X, L) = H^{p+q}(X, \mathcal{IC}(L)) \tag{1.1.4}$$

degenerates at E_1. This is a generalization of the E_1-degeneration of Hodge to de Rham spectral sequence, and this gives the canonical Hodge filtration of the intersection cohomology group $IH^{p+q}(X, L)$, and from (1.0.2) one can obtain the primitive decomposition of $IH^{p+q}(X, L)$. And primitive part $PIH^{\cdot}(X, L)$ has a natural a polarization induced from the polarizations of X and L. In order to obtain the canonical Hodge structure on $IH^{\cdot}(X, L)$, when X is projective and irreducible, but not necessarily smooth, one needs the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber type.

(1.2) Decomposition Theorem. *Let $f : X \longrightarrow Y$ be a projective morphism between analytic manifolds, L a local system which underlies the variation of Hodge structure on a Zariski open set U on X. We have the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber type for $f_*\mathcal{IC}_X L$ the direct image of intersection complex, i.e.*

$$f_*\mathcal{IC}_X L \simeq \oplus_j ({}^p\mathcal{H}^j f_*\mathcal{IC}_X L)[-j] \quad in \quad D^b_c(\mathbb{Q}_Y), \tag{1.2.1}$$

$${}^p\mathcal{H}^j f_*\mathcal{IC}_X L = \oplus_{Z'} \mathcal{IC}_{Z'} L^l_{Z'} \quad in \quad Perv(\mathbb{Q}_Y), \tag{1.2.2}$$

where Z' are irreducible closed subvarieties of Y and $L^j_{Z'}$ are local systems on smooth Zariski open sets of Z'.

The assertion (1.2.1) follows from the hard Lefschetz theorem (1.0.2), and the decomposition (1.2.2) was induced by the decomposition by strict support (0.8.1) and theorem (0.14).

There is also a Kähler package of the decomposition theorem (Theorem (0.6), [S3]). We say that a variation of $\mathbb{R}$-Hodge structure L is "geometric" if L is a direct factor of the restriction of $R^j\pi_*\mathbb{R}_{\tilde{X}}$ to a smooth Zariski open subset for some proper surjective holomorphic map $\pi : \tilde{X} \longrightarrow X$ between analytic varieties with $\tilde{X}$ smooth Kähler.

(1.3) Theorem. *Let $f : X \longrightarrow Y$ be a proper morphism between irreducible analytic spaces. Assume that there is a proper surjective morphism $\pi : \tilde{X} \to X$ with $\tilde{X}$ smooth Kähler. Assume that L is "geometric" variation of $\mathbb{R}$-Hodge structure on a Zariski open subset U on X. Then we have the decomposition theorem for $f_* \mathcal{IC}_X L$ as in (1.2.1-2) (with replacing the coefficient $\mathbb{Q}$ by $\mathbb{R}$).*

(1.4) The canonical Hodge structure on the intersection cohomology.

Let X be an irreducible complex projective variety. First we will show how one can show the existence of the "canonical" Hodge structure on $IH^{\cdot}(X, \mathbb{Q}_X) := H^{\cdot}(X, \mathcal{IC}_X(\mathbb{Q}_X))$.

Let $\pi : \tilde{X} \longrightarrow X$ be a resolution of singularities, so that π is a projective morphism and $\tilde{X}$ is a irreducible smooth projective variety. The decomposition theorem implies that the perverse Leray spectral sequence

(1.4.1) $$E_2^{i,j} = H^{\cdot}(X, {}^p\mathcal{H}^j \pi_* \mathbb{Q}_{\tilde{X}}) \Rightarrow H^{i+j}(\tilde{X}, \mathbb{Q}_{\tilde{X}})$$

degenerates at E_2. Moreover from (1.2.1) one has the strict support decomposition

(1.4.2) $$\pi_*(\mathbb{Q}_{\tilde{X}}) = \mathcal{IC}_X(\mathbb{Q}_X) \oplus T \quad \text{a direct sum}$$

where T is a sum of perverse sheaves whose strict supports Z are proper irreducible subvarieties of X. From E_2 degeneration of (1.4.1), $H^{\cdot}(X, \pi_* \mathbb{Q}_{\tilde{X}})$ can be written as $Gr^G(H^{\cdot}(\tilde{X}, \mathbb{Q}_{\tilde{X}}))$ where G is the filtration induced by the Leray spectral sequence. Moreover from (1.4.2), $H^{\cdot}(X, \mathcal{IC}_X \mathbb{Q}_X)$ is a direct factor of $Gr^G(H^{\cdot}(\tilde{X}, \mathbb{Q}_{\tilde{X}})) = H^{\cdot}(\tilde{X}, \mathbb{Q}_{\tilde{X}})$. Since the filtration G and the decomposition (1.4.2) respect the Hodge filtration F, cohomology groups $Gr^G(H^{\cdot}(\tilde{X}, \mathbb{Q}_{\tilde{X}}))$ and $H^{\cdot}(X, \mathcal{IC}_X \mathbb{Q}_X)$ admit the canonical Hodge structures induced from $H^{\cdot}(\tilde{X}, \mathbb{Q}_{\tilde{X}})$. This result can be generalized to the case of compact complex analytic space in class $\mathcal{C}$ in the sense of Fujiki by using (1.3). Furthermore, by using a result in [KK2] Saito proved the following

(1.5) Theorem. *Let X be an irreducible analytic variety in the class $\mathcal{C}$, L a local system of $\mathbb{R}$-modules on a Zariski dense open subset of X which underlies a polarized variation of $\mathbb{R}$-Hodge structure of weight n. Then the intersection cohomology group $IH^i(X, L) = H^i(X, \mathcal{IC}(L))$ admits the canonical Hodge structure of weight $n+i+d_X$[2]. Moreover, one has a primitive decomposition $IH^i(X, L)$, and its primitive parts carry natural polarized Hodge structures.*

[2]The index i is shifted by $-d_X$, so it varies from $-d_X$ to d_X

§2 Kollár's conjecture.

In [Kol], Kollár showed the following torsion-freeness of higher direct images of dualizing sheaves and the vanishing theorem, which are powerful tools in the classification theory of higher dimensional projective varieties.

(2.0) Theorem. *([Ko1], Theorem 2.1). Let X and Y be a complex projective varieties and assume that X is smooth. Let $f : X \longrightarrow Y$ be a surjective map and L an ample line bundle on Y, and $\omega_X = \Omega_X^{dimX}$ the dualizing sheaf of X. The we have*

(1) *$R^i f_* \omega_X$ is torsion-free for $i \geq 0$,*

(2) *$H^j(Y, R^i f_* \omega_X \otimes L) = 0$ for $j > 0$.*

In [Ko2], he proceeded to study the sheaves $R^i f_* \omega_X$ more deeply, a1 d obtained locally freeness of the sheaves $R^i f_* \omega_{X/Y}$ under certain conditions. In o1 uer to explain this result more explicitly, we introduce the following notations.

Let $f : X^{n+r} \longrightarrow Y^n$ be a surjective map from X to Y, where X is a smooth projective variety of dimension $n+r$ and Y is a projective variety of dimension n. Let $Y^0 \subset Y$ be the smooth locus, $X^0 = f^{-1}(Y^0)$ and $f^0 = f_{|X^0}$. Then $f^0 : X^0 \longrightarrow Y^0$ is a smooth morphism, hence a topological fiber bundle. Therefore, the topological sheaves $R^i f_* \mathbb{C}_{X^0}$ are local systems, and they underlie variations of Hodge structures. If Y is smooth and the branch locus of f is a divisor with normal crossings in Y, then

$$R^i f_* \omega_{X/Y} \simeq {}^u\mathcal{F}^{r-i}(R^{n-k+i} f_*^0 \mathbb{C}) \tag{2.0.1}$$

and

$$R^i f_* \mathcal{O}_X \simeq {}^l\mathcal{G}r^0(R^i f_*^0 \mathbb{C}), \tag{2.0.2}$$

where we set $\omega_{X/Y} = \omega_X \otimes_{\mathcal{O}_X} f^* \omega_Y^{-1}$. Here, the sheaves ${}^u\mathcal{F}^{r-i}$ a1 d ${}^l\mathcal{G}r^0$ denote the Deligne's upper and lower canonical extensions of $\mathcal{F}^{r-i}(R^{n-k+i} f_*^0 \mathbb{C})$ and $\mathcal{G}r^0(R^i f_* \mathbb{C})$ on Y^0 respectively. These are locally free sheaves on Y, hence so are $R^i f_* \omega_{X/Y}$ and $R^i f_* \mathcal{O}_X$.

Moreover, he obtained the following decomposition theorem of $\mathbf{R} f_* \omega_X$.

(2.1) Theorem. *([Ko2], Theorem 3.1). Let $f : X \longrightarrow Y$ be as in Theorem (2.0). Then we have the following isomorphism in the derived category $D(\mathcal{O}_Y)$.*

$$\mathbf{R} f_* \omega_X \simeq \sum_i R^i f_* \omega_X. \tag{2.1.1}$$

This theorem yields, for example,

(2.2) Corollary. *Under the same assumption, one has*

$$h^p(X, \omega_X) = \sum_{i=0}^{p} h^i(Y, R^{p-i} f_* \omega_X).$$

(2.3) The conjectures and the results.

In [Ko2], he also explained about the relation between the sheaves $R^i f_* \omega_{X/Y}$ and the intersection complex $\mathcal{IC}_Y(\mathbf{R}^{r+i} f^0_* \mathbb{C}^0_X)$, and also obtained conjectures about abstract (not necessarily geometric) variation of Hodge structures (see Ch.4 and 5 of [Ko2]), which are natural generalizations of Theorem (2.0) and (2.1).

A proof of these conjectures are given by Morihiko Saito by using his theory of polarized Hodge modules. After getting the definition of Hodge modules and the result like theorem (0.14) and decomposition theorem (1.2), torsion freeness of $Rf_*\omega_X$ and the decomposition theorem (2.1) naturally follow from them. (Of course, all of these results are rather deep.)

Let X be an irreducible complex algebraic variety (assumed always separated and reduced) of dimension d_X, and

$$\mathbf{V} = (\mathcal{V}, F^{\cdot}, \mathbf{V}_{\mathbb{Q}})$$

a polarizable variation of $\mathbb{Q}$-Hodge structure of weight w on a dense Zariski open set U of the smooth locus of X. Then, by Theorem (0.14), $\mathbf{V}$ extends uniquely to a polarizable Hodge module $\mathcal{M} = (M, F, \mathbf{K}_{\mathbb{Q}})$ on X where $\mathbf{K}_{\mathbb{Q}} = \mathcal{IC}_X(\mathbf{V}_{\mathbb{Q}}[d_X])$. (See (0.10.3-4)). For simplicity, assume that X is a closed subvariety of a smooth complex variety X'. Then $\mathcal{M} = (M, F, K_{\mathbb{Q}})$ belongs to $MH_X(X', \mathbb{Q}, n + d_X)$, and M is obtained as the regular holonomic $\mathcal{D}_{X'}$-modules corresponding to $\mathbf{K} \otimes_{\mathbb{Q}} \mathbb{C}$. The Hodge filtration $F.M$ on M is determined by its restriction to any open dense subset using the filtration V of Kashiwara-Malgrange and the formula (0.7.2).

Let

(2.3.1) $$p' = min\{p : F_pM \neq 0\}.$$

Then p' depends only on $\mathbf{V}$, and $F_{p'}M$ depends on $\mathbf{V}$ and X (i.e. independent of embedding X into smooth varieties) as an $\mathcal{O}_X$-module. We denote them by

(2.3.2) $$p(\mathcal{M}) = p' = min\{p : F_pM \neq 0\}, \qquad S_X(\mathcal{M}) = F_{p(\mathcal{M})}M.$$

Set moreover

(2.3.3) $$q(\mathbf{V}) = \max\{p, Gr^p_F \mathcal{V} \neq 0\}, \quad S_X(\mathbf{V}) := S_X(\mathcal{M})$$

and $q'(\mathbf{V}) = min\{p : Gr_F^p \mathcal{V} \neq 0\}$. Comparing (2.3.3) with (2.3.2) and (0.10.3), we have the relation

$$(2.3.4) \qquad p(\mathcal{M}) = -d_X - q(\mathbf{V}),$$

and

$$S_X(\mathbf{V})_{|U} = \Omega_U^{d_X} \otimes F^{q(\mathbf{V})}\mathcal{V}.$$

We also define $Q_X(\mathbf{V}) = \mathbb{D}(S_X(\mathbf{V}^*)) \in D^b_{coh}(\mathcal{O}_X)$, where $\mathbb{D}$ is the dual functor for $\mathcal{O}$-Modules and $\mathbf{V}^* = Hom(\mathbf{V}, \mathbb{C})$ the dual VHS of $\mathbf{V}$.

(2.4) Lemma. *Under the same notations and assumption as above, assume moreover that X is smooth and the $D = X - U$ the singularity of $\mathbf{V}$ is a normal crossing divisor. Then we have*

$$(2.4.1) \qquad S_X(\mathbf{V}) = \Omega_X^{d_X} \otimes_{\mathcal{O}} (\mathcal{V}_X^{>-1} \cap j_* F^{q(\mathbf{V})}\mathbf{V}),$$

$$(2.4.2) \qquad Q_X(\mathbf{V}) = \mathcal{V}^{\geq 0} / \mathcal{V}^{\geq 0} \cap j_* F^{q'(\mathbf{V})+1}\mathbf{V}[d_X].$$

Here $\mathcal{V}_X^{>\alpha}$ (resp. $\mathcal{V}_X^{\geq\alpha}$) denotes Deligne's extension of $\mathcal{V}$ with eigenvalues of residue of connection in $(\alpha, \alpha+1]$ (resp. $[\alpha, \alpha+1)$). In particular, the sheaves $S_X(\mathbf{V})$ and $Q_X(\mathbf{V})$ are locally free.

(2.5) Remark. Even if one has no assumptions on X and the singularity of V, one can show that $S_X(\mathbf{V})$ is a torsion-free sheaf by using (0.7.2).

(2.6). Let $f : X \longrightarrow Y$ be a proper surjective morphism of irreducible varieties with $r = dimX - dimY$, and $\mathcal{M} = (M, F, K) \in MH_X(X, n)^p$. Then by Theorem (0.14) there exists a variation of Hodge structure $\mathbf{V}$ of weight w on a smooth dense Zariski open set U on X such that $K = \mathcal{IC}_X(\mathbf{V})$. Here we have $w = n - d_X$. Then from (2.3.3), one has

$$(2.6.1) \qquad q(\mathbf{V}) = -p(\mathcal{M}) - d_X, \quad q'(\mathbf{V}) = w - q(\mathbf{V}) = p(\mathcal{M}) + d_X.$$

Taking the direct image, one obtains $f_*\mathcal{M} = (f_*(M,F), f_*K) \in MF_{rh}(\mathcal{D}_X, \mathbb{Q})$ so that

$$(2.6.2) \qquad F_{p(\mathcal{M})}(f_*\mathcal{M}) = \mathbf{R}f_*S_X(\mathcal{M}) = \mathbf{R}f_*S_X(\mathbf{V}),$$

and $\mathcal{H}^i f_*\mathcal{M} \in MH(Y, n+i)$ from the stability theorem (0.1).

Moreover from the decomposition theorem (1.2), one has a decomposition $f_*\mathcal{M} = \oplus_j \mathcal{H}^j f_*\mathcal{M}[-j]$, which induces the decompositions $f_*(M, F)$ and

$$(2.6.3) \qquad f_*\mathcal{IC}_X(\mathbf{V}) \cong \oplus_j ({}^p\mathcal{H}^j f_*\mathcal{IC}_X(\mathbf{V}))[-j] \quad in \quad D_c^b(\mathbb{Q}_Y).$$

Let

$$(2.6.4) \qquad \mathcal{H}^j f_*\mathcal{M} = \oplus_{Z\subset Y}\mathcal{M}_Z^j$$

be the decomposition by strict supports. Then, by theorem (0.14), the Hodge modules $\mathcal{M}_Y^j \in MH_Y(Y, n+j)$ corresponds to a variation of Hodge structure $\mathbf{V}^j$ on a dense smooth Zariski open set U of Y.

The following is a key lemma of the proof of Kollár conjecture.

(2.7) Lemma. *(Proposition 2.6 in [SK]). Under the notations and the assumptions as above, we have*

$$p(\mathcal{M}^j_Z) > p(\mathcal{M}), \tag{2.7.1}$$

if $Z \subset Y$ is a proper irreducible subvariety of Y.

Now we can state Saito's theorem, which is a generalization of Kollár's results.

(2.8) Theorem. *(Theorem (3.2), [SK]). Under the same notation and assumption as in (2.6), we have the canonical isomorphisms in $D^b_{coh}(\mathcal{O}_Y)$:*

$$\mathbf{R}f_*S_X(\mathbf{V}) = \bigoplus_{q(\mathbf{V}^i)=q(\mathbf{V})+r} S_Y(\mathbf{V}^i)[-i], \tag{2.8.1}$$

$$\mathbf{R}f_*Q_X(\mathbf{V}) = \bigoplus_{q'(\mathbf{V}^i)=q'(\mathbf{V})} Q_Y(\mathbf{V}^i)[-i], \tag{2.8.2}$$

where we set $q'(\mathbf{V}) = n - q(\mathbf{V})$. Moreover one has canonical isomorphisms $R^if_*S_X(\mathbf{V}) = S_Y(\mathbf{V}^i)$ for $q(\mathbf{V}^i) = q(\mathbf{V}) + r$, and ${}^d\mathcal{H}^i\mathbf{R}f_*Q_X(\mathbf{V}) = Q_Y(\mathbf{V}^i)$ for $q'(\mathbf{V}^i) = q'(\mathbf{V})$.

Sketch of proof. Since the decomposition (2.6.3) respects the Hodge filtration $F_.$, from (2.6.2), one has

$$\mathbf{R}f_*S_X(\mathbf{V}) = F_{p(\mathcal{M})}f_*\mathcal{M} = \bigoplus_j F_{p(\mathcal{M})}\mathcal{H}^jf_*\mathcal{M}[-j].$$

We also have the decomposition by strict supports (2.6.4), and this implies that

$$F_{p(\mathcal{M})}\mathcal{H}^jf_*\mathcal{M}[-j] = F_{p(\mathcal{M})}\mathcal{M}^j_Y[-j] \oplus (\bigoplus_{Z \subsetneq Y} F_{p(\mathcal{M})}\mathcal{M}^j_Z[-j]).$$

Lemma (2.7) shows that $F_{p(\mathcal{M})}\mathcal{M}^j_Z = 0$ unless $Z = Y$, and the Hodge module $\mathcal{M}^j_Y[-j]$ corresponds to a variation of Hodge structure $\mathbf{V}^j$ on a smooth dense Zariski open set of Y. Thus we obtain the assertion on $\mathbf{R}f_*S_X(\mathbf{V})$. Since $\mathbb{D}(\mathcal{M}^j_Y) = (\mathbb{D}\mathcal{M})^{-i}_Y$ by duality so that $S_Y((\mathbf{V}^*)^j) = S_Y((\mathbf{V}^{-j})^*)$ and $q(\mathbf{V}^j) + q(\mathbf{V}^{-j}) = w + d_X$, the assertion on $\mathbf{R}f_*Q_X(\mathbf{V})$ follows from this by taking dual.

Together with remark (2.5), this yields the following

(2.9) Corollary. *Under the same notations and assumptions as in (2.6), the higher direct images $R^if_*S_X(\mathbf{V})$ are torsion-free sheaves.*

(2.10) Example. Under the same notations as in (2.6), assume moreover X is smooth and of dimension d. Let $\mathbf{V} = \mathbb{Q}_X$ denote the trivial variation of Hodge structure of rank one and type $(0,0)$. If $f : X \longrightarrow Y = pt$ is the structure morphism, one have $\mathbf{V}^i = H^{i+d}(X, \mathbb{C}_X)$, $S_X(\mathbf{V}) = \omega_X = \Omega_X^d$, $Q_X(\mathbf{V}) = \mathcal{O}_X[d]$, $q(\mathbf{V}) = q'(\mathbf{V}) = 0$, and $S_{pt}(\mathbf{V}^i) = H^i(X, \omega_X)$ for $q(\mathbf{V}^i) = d$, $Q_{pt}(\mathbf{V}^i) = H^{i+d}(X, \mathcal{O}_X)$ for $q'(\mathbf{V}^i) = 0$. Let $f : X \longrightarrow Y$ be as in (2.6), $\mathbf{V} = \mathbb{Q}_X$ as above, and assume X is smooth and Y is arbitrary. Then one has $\mathbf{R}f_* S_X(\mathbb{Q}_X) \cong \mathbf{R}f_*\omega_X$ and $\mathbf{V}^i = R^i f^0_* \mathbb{Q}_X$, where $f^0 : X^0 \longrightarrow Y^0$ is the smooth part of f.

Then we have canonical isomorphisms

$$R^i f_* \omega_X \cong S_Y(R^i f^0_* \mathbb{Q}_X).$$

(2.11) Remark. (1) If X is embeddable into the smooth variety, we have

$$(2.11.1) \qquad Q_X(\mathbf{V}) = Gr^F_{-p(\mathcal{M})-n} DR(M),$$

$$(2.11.2) \qquad Gr^F_p DR(\mathcal{M}) = 0 \qquad \text{for } p > -p(\mathcal{M}) - n,$$

by $Gr^F DR \circ \mathbb{D} = \mathbb{D} Gr^F DR$, $\mathbb{D}(\mathcal{M}) = \mathcal{M}(n)$, and we get canonical morphisms

$$S_X(\mathbf{V}) \longrightarrow DR(M), \qquad DR(M) \longrightarrow Q_X(\mathbf{V}).$$

(2) Theorem (2.8) can be generalized to the analytic case as in (1.3).

§3 Kodaira vanishing.

(3.1) Mixed Hodge Modules.

Let X be a complex manifold. We denote by $MHM(X)^p$ the category of polarizable Mixed Hodge Modules. An object in $MHM(X)^p$ can be written as $\mathcal{M} = ((M,F),K;W)$ where $((M,F),K)$ belongs to $MF_h(X,\mathbb{Q})$ and W is a filtration of $((M,F),K)$ such that $Gr_i^W(M,F,K) \in MH(X,i)^p$. These objects have to satisfy more conditions, but we will not mention the details here. (See [S2] or [Sh]). An Mixed Hodge Module $\mathcal{M} \in MHM(X)^p$ is called smooth if K is a local system. A variation of Mixed Hodge structure is called admissible if it is graded polarizable and for any morphism $HFS \to X$ with $dim S = 1$, its pull-back by f is admissible in the sense of Steenbrink-Zucker. (See [SZ] or (3.1) in [U]). Then a smooth Mixed Hodge module on X corresponds to an admissible variation of Mixed Hodge structure. We also have the decomposition of a Mixed Hodge Module by strict support. For an irreducible subvariety $Z \subset X$, we denote by $MHM_Z(X, \mathbb{Q})^p$ the full subcategory of $MHM(X, \mathbb{Q})^p$ whose objects have strict support Z.

The following theorem is a generalization of Kodaira vanishing theorem.

(3.2) Theorem. *([S2], Proposition (2.33)). Let Z be a (reduced) irreducible projective variety with an ample invertible sheaf L and $i : Z \hookrightarrow X = \mathbb{P}^r$ the embedding by L^m for some positive integer m. Then for $\mathcal{M} = ((M, F), K; W) \in MHM_Z(X)^p$ (or $\mathcal{M} = ((M, F), K) \in MH_Z(X, \mathbb{Q}, n)^p$),*

(1) $Gr_p^F DR_X(M, F)$ belongs to $D^b(\mathcal{O}_Z)$ and it is independent of the embedding of Z into a complex manifold.

(2) We have the Kodaira vanishing theorem

$$H^i(Z, Gr_p^F DR_X(M, F) \otimes L) = 0 \quad \text{for} \quad i > 0, \tag{3.2.1}$$

$$H^i(Z, Gr_p^F DR_X(M, F) \otimes L^{-1}) = 0 \quad \text{for} \quad i < 0. \tag{3.2.2}$$

Sketch of Proof. The first assertion of (1) follows from (3.2.6) in [S1], and since the direct image is compatible with DR and Gr^F the independence of embedding of Z into a smooth variety follows from the argument like (5.1.9) in [S1].

From this fact, we may assume that $m \geq 2$ to prove the Kodaira vanishing (2). Since $Gr^F DR$ is exact, we may also assume that $\mathcal{M} \in MH_Z(X, n)$. By duality, it is enough to show (3.2.2).

Let Y be a generic hyperplane of $X = \mathbb{P}^r$, strictly non-characteristic to (M, F) (cf. (3.5.1), [S1]), and take a section s of $H^0(Z, L^m)$ such that s defines $Y \cap Z$. Then we define the $\mathcal{O}_Z$-algebra structure on $\oplus_{0 \leq i < m} L^i$ by $(\oplus_{0 \leq i} L^i t^i)/Im(t^m - s)$, and obtain a finite covering

$$\pi : \tilde{Z} := Specan_Z(\oplus_{0 \leq i < m} L^i) \longrightarrow Z$$

ramified along $Y \cap Z$. Let $j : U = X \backslash Y \hookrightarrow X$ be the natural inclusion. Set

$$\begin{aligned} j^* j^{-1} \mathcal{M} &= ((M(*Y), F), j_* j^* K; W) \in MHW_Z(X)^p \\ \tilde{M} &= (\tilde{M}, F, \tilde{K}) = Coker(\mathcal{M} \to \pi_* \pi^* \mathcal{M}) \in MH_Z(X, n)^p \\ \tilde{L} &= Coker(\mathcal{O}_Z \longrightarrow \pi_* \mathcal{O}_{\tilde{Z}}) \end{aligned}$$

so that L^{-1} is a direct factor of $\tilde{L}$. Here $\pi_* \pi^* M$ can be regarded as the unique extension to Z of its restriction to the smooth open set U' where π is unramified.

We can see that $\mathcal{M}$ is a direct factor of $\pi_*\pi^*\mathcal{M}$, and we have a natural injection $\mathcal{M} \longrightarrow \pi_*\pi^*\mathcal{M}$ induced by its restriction to U'. Moreover one has an exact sequence

$$(3.2.3) \qquad 0 \longrightarrow \mathcal{M} \longrightarrow j_*j^{-1}\mathcal{M} \longrightarrow \mathcal{H}^1 i^! \mathcal{M} \longrightarrow 0$$

so that $\mathcal{H}^1 i^! \mathcal{M} \in MH_{Z\cap Y}(Y, n+1)$ by the non-charactericity, where $i : Y \hookrightarrow X$ is the natural inclusion, (cf. [S2] 2.11 and [S1], (3.5.9)).

Now applying the stability theorem (1.0) for $Z \longrightarrow pt$, we have the following

(3.2.4) Lemma. *The spectral sequence*

$$E_1^{p,q} = H^{p+q}(Z, Gr^F_{-q} DR_X \tilde{M}) \Rightarrow H^{p+q}(Z, DR_X \tilde{M}) \simeq H^{p+q}(Z, \tilde{K} \otimes \mathbb{C})$$

degenerates at E_1.

This yields the following implication

$$(3.2.5) \qquad H^i(Z, \tilde{K}) = 0 \quad \Rightarrow \quad H^i(Z, Gr^F_p DR_X(\tilde{M})) = 0.$$

On the other hand, by the non-charactericity, we have

$$H^i(Z, \tilde{K}) = H^i(Z, j_! j^{-1}\tilde{K}) = H^i(Z, j_* j^{-1}\tilde{K}).$$

Since $U" = Z - Z \cap Y$ is an affine variety, one has

$$H^i(Z, j_* j^{-1}\tilde{K}) \simeq H^i(U", j^{-1}\tilde{K}) = 0, \quad \text{for} \quad i > 0$$

and by duality

$$H^i(Z, j_! j^{-1}\tilde{K}) = 0 \quad \text{for} \quad i < 0.$$

Therefore one has

$$H^i(Z, \tilde{K}) = 0 \quad \text{for} \quad i \neq 0,$$

and from (3.2.5) we get

$$(3.2.6) \qquad H^i(Z, Gr^F_p DR_X(\tilde{M})) = 0 \quad \text{for} \quad i \neq 0.$$

(3.2.7) Lemma. *Under the same notation and the assumption as above, we have the following isomorphism*

$$(3.2.8) \qquad Gr^F_p \tilde{M} \simeq Gr^F_p M(*Y) \otimes \tilde{L}$$

In particular, we get

$$(3.2.9) \qquad H^i(Z, Gr^F_p M(*Y) \otimes L^{-1}) = 0, \quad \textit{for} \quad i \neq 0$$

One can check the assertion (3.2.8) by considering the structure of $\mathcal{D}_X$-modules of $\tilde{M}$ and $M(*Y)\otimes\tilde{L}$, and using the V-filtration to give the filtration F. (See (2.33) in [S2]). Since L^{-1} is a direct factor of $\tilde{L}$, we have the second assertion.

Because the functor $Gr_p^F DR_X$ is exact, from (3.2.3), we obtain an exact sequence (3.2.10)

$$0 \longrightarrow Gr_p^F DR_X(M)\otimes L^{-1} \longrightarrow Gr_p^F DR_X(M(*Y))\otimes L^{-1} \longrightarrow Gr_p^F DR_X(\mathcal{H}^1 i^! M)\otimes L^{-1} \longrightarrow 0.$$

Together with this and (3.2.9), we obtain isomorphisms

$$H^{i-1}(Z\cap Y, Gr_p^F DR_X(\mathcal{H}^1 i^! M)\otimes L^{-1}) \simeq H^i(Z, Gr_p^F DR_X(M)\otimes L^{-1}) \quad \text{for} \quad i<0,$$

then induction on $dimZ$ finishes the proof of (3.2.2). q.e.d.

We have many corollaries of Theorem (3.2). For example, setting $\mathcal{M}=\mathbb{Q}_Z[d_Z]$, we obtain the following

(3.3) Kodaira-Nakano Vanishing Theorem. *Let Z be a projective smooth complex variety, and L an ample invertible sheaf on Z. Then we have*

$$H^q(Z,\Omega_Z^p\otimes L)=0 \quad \textit{for} \quad p+q>dimZ,$$

$$H^q(Z,\Omega_Z^p\otimes L^{-1})=0 \quad \textit{for} \quad p+q<dimZ.$$

Moreover if we apply theorem (3.2) for the edge components of Hodge modules (cf. (2.3)), we obtain the following

(3.4) Theorem. *([S5]). Let Z be a projective variety, $\mathbf{V}$ a variation of Hodge structure defined on a dense smooth Zariski open subset of Z, and $S_Z(\mathbf{V})$ and $Q_Z(\mathbf{V})$ as in (2.3). For an ample invertible sheaf L on Z, we have*

$$H^i(Z,S_Z(\mathbf{V})\otimes L)=0 \quad \textit{for} \quad i>0,$$

$$H^i(Z,Q_Z(\mathbf{V})\otimes L^{-1})=0 \quad \textit{for} \quad i<0.$$

Let $f:Y\longrightarrow Z$ be a projective morphism such that Y is smooth, and L an ample invertible sheaf on Z. If we set $\mathcal{M}=\mathcal{H}^j f_*\mathbb{Q}_Y[d_Y]$ and use theorem (2.8), we obtain the following theorem as a special case of (3.4).

(3.5) Ohsawa-Kollár vanishing. *([Kol]). Under the notations and assumption as above, we have*

$$H^i(Z,R^j f_*\omega_Y\otimes L)=0 \quad \textit{for} \quad i>0.$$

References

[BBD] Beilinson, A.A., Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque, vol. 100, 1982, pp. 5-171.

[SGA7] P. Deligne, *Le formalisme des cycle évanscents*, SGA 7 II, Springer Lecture Notes in Math., vol. 340, 1973, pp. 82-115.

[Ka] Kashiwara, Masaki, *Vanishing cycle sheaves and the holonomic systems of differential equations*, Springer Lecture Notes in Math., vol. 1016, 1983, pp. 134-142.

[K-K1] Kashiwara, Masaki, & Kawai, Takahiro, *The Poincaré lemma for a variation of Hodge structure*, Publ. RIMS **23** (1987), 345-407.

[K-K2] _____, *Hodge structure and holonomic systems*, Proc. Japan Acad. **62A** (1985), 1-4.

[Ko1] Kollár, J., *Higher direct images of dualizing sheaves, I*, Ann. of Math. **123** (1986), 11-42.

[Ko2] _____, *Higher direct images of dualizing sheaves, II*, Ann. of Math. **124** (1986), 171-202.

[S1] Saito, Morihiko, *Modules de Hodge polarisables*, Publ. RIMS. Kyoto University **24** (1988), 849-995.

[S2] _____, *Mixed Hodge Modules*, Publ. RIMS. Kyoto University **26** (1990), 221-333.

[S3] _____, *Decomposition Theorem for Proper Kähler morphisms*, Tohoku Math. J. **42** (1990), 127-148.

[S4] _____, *Introduction of mixed Hodge Modules*, Astérisque **179-180** (1989), 145-162.

[S5] _____, *On Kollár's conjecture*, Proceedings of Symposia in Pure Math. **52 , Part 2** (1991), 509-517.

[S.Mu] Saito, Mutsumi, *A short course on b-functions and vanishing cycles*, In this volume.

[Sh] Shimizu, Yuji, *An introduction to Morihiko Saito's Theory of Mixed Hodge Modules*, In this volume.

[SZ] Steenbrink, J., Zucker, S., *Variation of mixed Hodge strcuture I.*, Invent. math **80** (1985), 485-542.

[U] Usui, Sampei, *Period maps and their extensions*, In this volume.

[Z] Zucker, S., *Hodge theory with degenerating coefficients: L_2-cohomology in the Poincaré metric*, Annals of Math. **109** (1979), 415-476.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto, 606, Japan

E-mail: mhsaito@kusm.kyoto-u.ac.jp