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Three-dimensional hypersurface purely elliptic singularities of (0, 1)-type

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Introduction.

In the theory of normal two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational double points. They are characterized as two-dimensional purely elliptic singularities of (0, 1)-type and of (0, 0)-type, respectively. What are natural generalizations in three-dimensional case of simple elliptic singularities. The notion of a simple K3 singularity was defined in [IW1, IW2] as a three-dimesnional isolated Gorenstein purely elliptic singularity of (0, 2)-type. A simple K3 singularity is characterized as a normal three-dimensional isolated singularity such that the exceptional set of any $\mathbb{Q}$-factorial terminal modification is a normal K3 surface (see [IW2]). Here we are interested in a three-dimensional hypersurface purely elliptic singularities of (0, 1)-type. Let $f \in \mathbb{C}[z_0, z_1, z_2, z_3]$ be a polynomial which is nondegenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [V], and whose zero locus $X =$

$\{ f = 0 \}$ in $\mathbb{C}^4$ has an isolated singularity at the origin $0 \in \mathbb{C}^4$. Then the condition for (X, x) to be a purely elliptic singularity of $(0, 1)$-type is given by a property of the Newton boundary of $\Gamma(f)$ of f.

In this paper, we classify the principal parts of defining equations, which define three-dimensional hypersurface purely elliptic singularities of $(0, 1)$-type.

1. Preliminaries. In this section, we recall some definitions and results from [I1], [IW2], [W1] and [W2].

First we define the plurigenera δ_m, $m \in \mathbb{N}$, for normal isolated singularities and define purely elliptic singularities. Let (X, x) be a normal isolated singularity in an n-dimensinal analytic space X, and $\pi : (M, E) \longrightarrow (X, x)$ a good resolution. In the following, we assume that X is a sufficiently small Stein neighbourhood of x.

Definition 1.1 ([W1]). Let (X, x) be a normal isolated singularity. For any positive integer m,

$$\delta_m(X, x) := \dim_{\mathbb{C}} \Gamma(X - \{x\}, \mathcal{O}(mK)) / L^{2/m}(X - \{x\}),$$

where K is the canonical line bundle on $X - \{x\}$, and $L^{2/m}(X - \{x\})$ is the set of all $L^{2/m}$-integrable (at x) holomorphic m-ple n-form on $X - \{x\}$.

Then δ_m is finite and does not depend on the choice of a Stein neighbourhood X.

Definition 1.2 ([W1]). A singularity (X, x) is said to be purely elliptic if $\delta_m = 1$ for every $m \in \mathbb{N}$.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there is a nowhere vanishing holomorphic 2-form on $X - \{x\}$ (see [88]). But in higher dimension, purely elliptic singularities are not always quasi-Gorenstein (see [IW1]).

In the following, we assume that (X, x) is quasi-Gorenstein. Let $E = \cup E_i$ be the decomposition of the exceptional set E into its irreducible components, and write

$$K_M = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_j$$

with $m_i \geq 0$, $m_j > 0$. Ishii [I1] defined the essential part of the exceptional set E as $E_J = \sum_{j \in J} m_j E_j$, and showed that if (X, x) is purely elliptic, then $m_j = 1$ for all $j \in J$.

Definition 1.3 (Ishii[I1]). A quasi-Gorenstein purely elliptic singularity (X, x) is of $(0, i)$-type if $H^{n-1}(E_J, \mathcal{O}_E)$ consists of the $(0, i)$-Hodge component $H^{0, i}(E_J)$, where

$$\mathbb{C} \simeq H^{n-1}(E_J, \mathcal{O}_E) = Gr_F^0 H^{n-1}(E_J) = \bigoplus_{i=1}^{n-1} H^{0, i}(E_j).$$

n-dimensional quasi-Gorenstein purely elliptic singularities are classified into 2n classes, including the condition that the singularity is Cohen-Macaulay or not.

Next we consider the case where (X, x) is a hypersurface singularity defined by a nondegenerate polynomial $f = \sum a_\nu z^\nu \in$

$\mathbb{C}\{z_0, z_1, \ldots, z_n\}$, and $x = 0 \in \mathbb{C}^{n+1}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_\nu \neq 0}(\nu + \mathbb{R}_0^{n+1})$ in $\mathbb{R}^{n+1}$. For any face Δ of $\Gamma_+(f)$, set $f_\Delta := \sum_{\nu \in \Delta} a_\nu z^\nu$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial z_0} = \frac{\partial f_\Delta}{\partial z_1} = \cdots = \frac{\partial f_\Delta}{\partial z_n} = 0$$

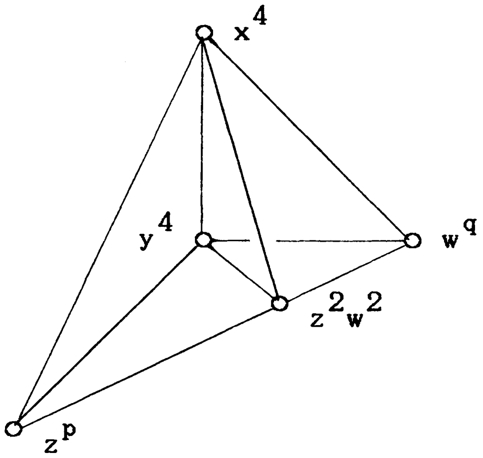
has no solution in $(\mathbb{C}^*)^{n+1}$ for any face Δ. When f is nondegenerate, the condition for (X, x) to be a purely elliptic singularity of $(0, i)$-type is given as follows:

Theorem 1.4. Let f be a nondegenerate polynomial and suppose $X = \{ f = 0 \}$ has an isolated singularity at $x = 0 \in \mathbb{C}^{n+1}$.

(1) (X, x) is purely elliptic if and only if $(1, 1, \cdots, 1) \in \Gamma(f)$.

(2) Let $n = 3$ and let Δ_0 be the face of $\Gamma(f)$ containing the point $(1, 1, 1, 1)$ in the relative interior of Δ_0. Then (X, x) is a singularity of $(0, 1)$-type if and only if $\dim_{\mathbb{R}} \Delta_0 = 2$.

Example 1.5. Let f be a polynomial of the form $f = x^4 + y^4 + (zw)^2 + z^p + w^q$ ($p, q \geq 5$). Then $\{ f = 0 \}$ has a purely elliptic singularity of $(0, 1)$-type at the origin in $\mathbb{C}^4$. The two-dimensional face, spanned by $(4, 0, 0, 0)$, $(0, 4, 0, 0)$ and $(0, 0, 2, 2)$, contains the point $(1, 1, 1, 1)$ in its interior with respect to the relative topology. The principal parts of the polynomial is $f_{\Delta_0} = x^4 + y^4 + (zw)^2$. Then the weights of f is $\alpha(f) = (\frac{1}{4}, \frac{1}{4}, \alpha, \frac{1}{2} - \alpha)$, not uniquely determined.



$$f = x^4 + y^4 + (zw)^2 + z^p + w^q$$
$$f_{\Delta_0} = x^4 + y^4 + (zw)^2$$

$$\alpha(f) = (\frac{1}{4}, \frac{1}{4}, r, \frac{1}{2} - r)$$

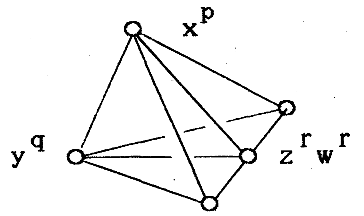
Thus if f is nondegenerate and defines a three-dimendional purely elliptic singularity of (0,1)-type, then f_{Δ_0} is a quasi-homogeneou polynomial of a not uniquely determined set α of weights. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Q}_+^4$ be one of such sets of weights. Then $\deg_\alpha(\nu) := \sum_{i=1}^4 \alpha_i \nu_i = 1$ for any $\nu \in \Delta_0$. In particular, $\sum_{i=1}^4 \alpha_i = 1$, since (1,1,1,1) is always contained in Δ_0.

2. Principal parts of hypersurface purely elliptic singularities of (0,1)-type

In this section, we calculate the principal parts of hypersurface purely elliptic singularities of (0,1)-type defined by nondegenerate polynomials.

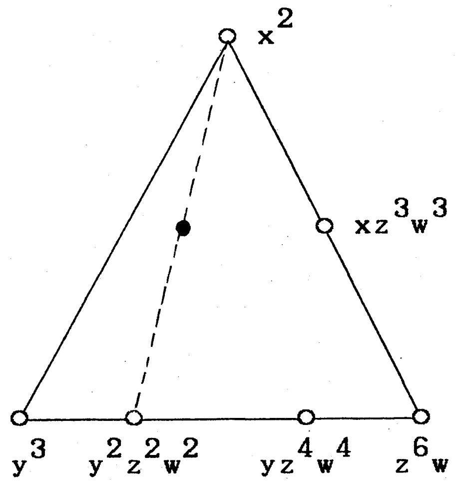
In the following can be found the complete list of the principal parts of defining equations:

I.

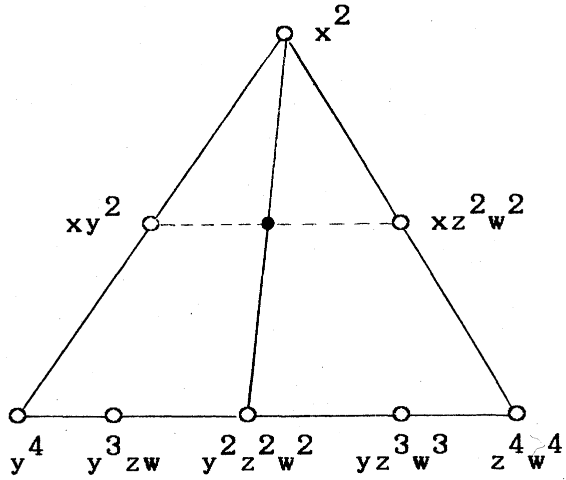


$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad (p \leq q)$$

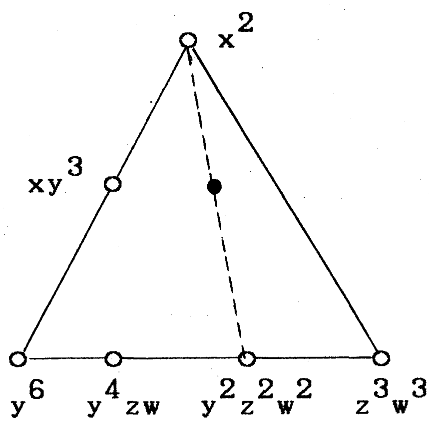
1. $\left(\frac{1}{2}, \frac{1}{3}, \alpha, \frac{1}{6} - \alpha\right)$



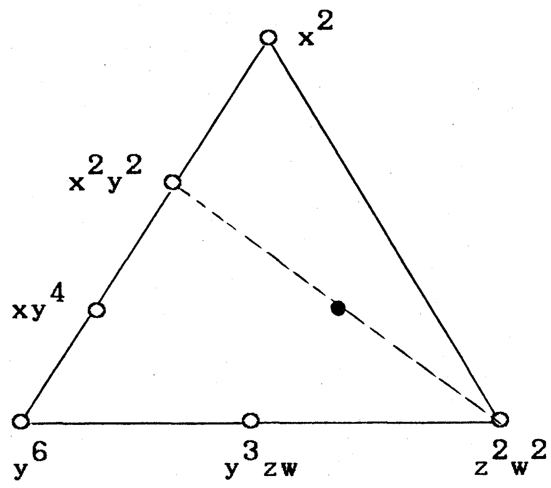
2. $\left(\frac{1}{2}, \frac{1}{4}, \alpha, \frac{1}{4} - \alpha\right)$



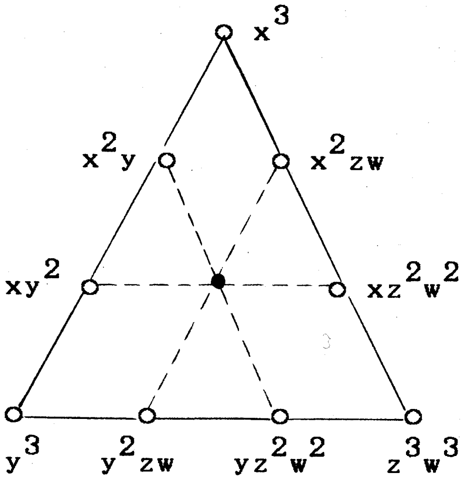
3. $\left(\frac{1}{2}, \frac{1}{6}, \alpha, \frac{1}{3} - \alpha\right)$



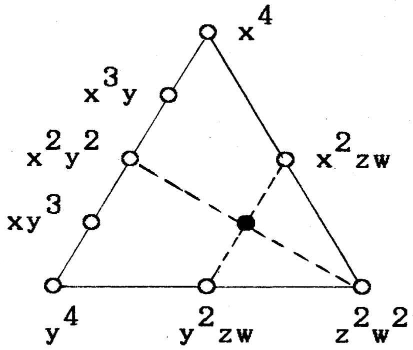
4. $\left(\frac{1}{3}, \frac{1}{6}, \alpha, \frac{1}{2} - \alpha\right)$



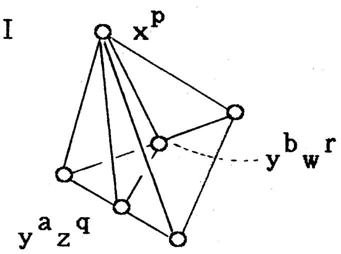
5. $\left(\frac{1}{3} , \frac{1}{3} , \alpha , \frac{1}{3} - \alpha \right)$



6. $\left(\frac{1}{4} , \frac{1}{4} , \alpha , \frac{1}{2} - \alpha \right)$

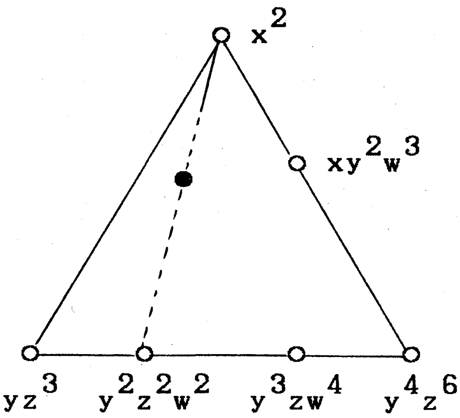


II

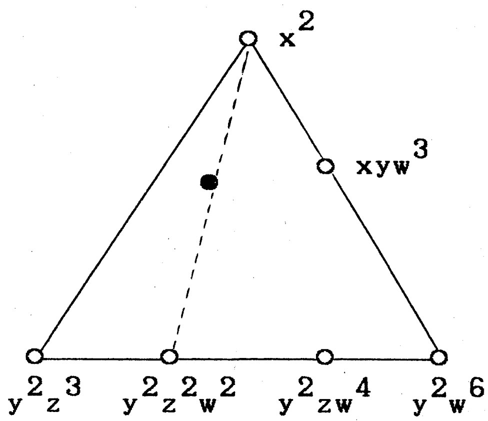


$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \quad \frac{a}{q} + \frac{b}{r} = 1 \quad (a \leq b)$$

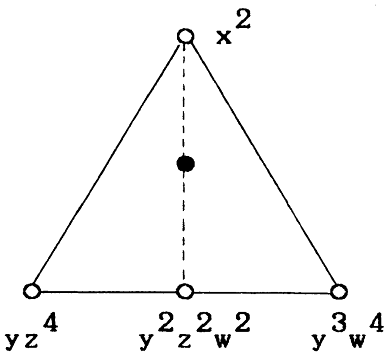
1. $\left(\frac{1}{2} , \alpha , \frac{1-\alpha}{3} , \frac{1-4\alpha}{6} \right)$



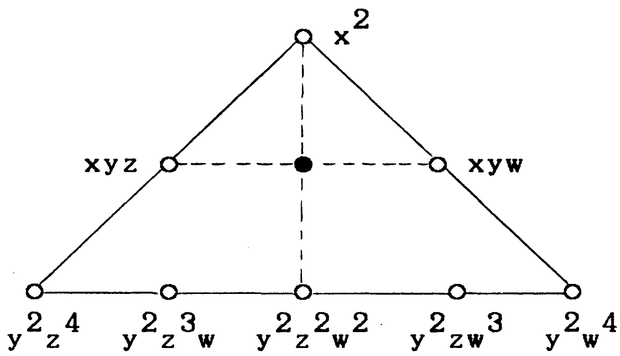
2. $\left(\frac{1}{2} , \alpha , \frac{1-2\alpha}{3} , \frac{1-2\alpha}{6} \right)$



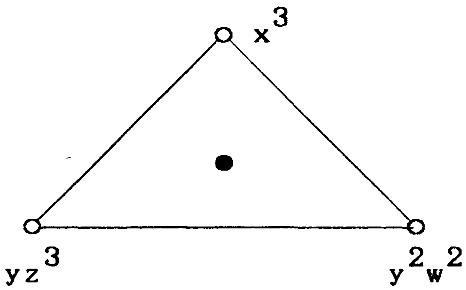
3. $\left(\frac{1}{2} , \alpha , \frac{1-\alpha}{4} , \frac{1-3\alpha}{4} \right)$



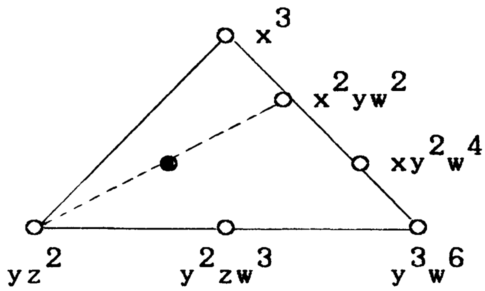
4. $\left(\frac{1}{2} , \alpha , \frac{1-2\alpha}{4} , \frac{1-2\alpha}{4} \right)$



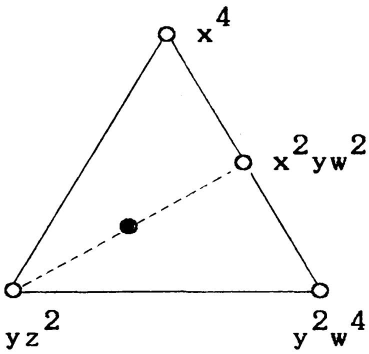
5. $\left(\frac{1}{3} , \alpha , \frac{1-\alpha}{3} , \frac{1-2\alpha}{3} \right)$



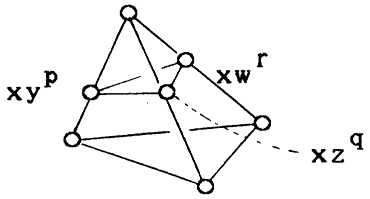
6. $\left(\frac{1}{3} , \alpha , \frac{1-\alpha}{2} , \frac{1-2\alpha}{6} \right)$



7. $\left(\frac{1}{4} , \alpha , \frac{1-\alpha}{2} , \frac{1-2\alpha}{4} \right)$

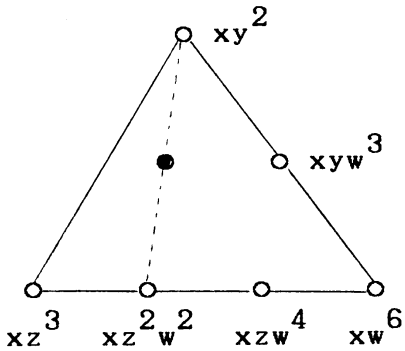


III

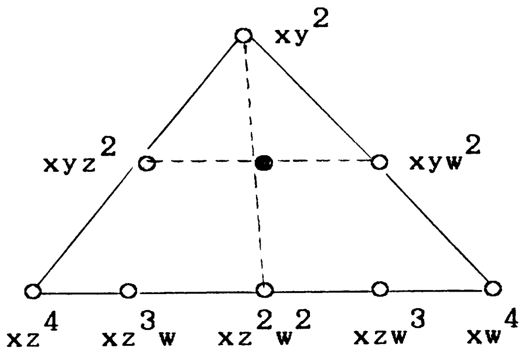


$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad (p \le q \le r)$$

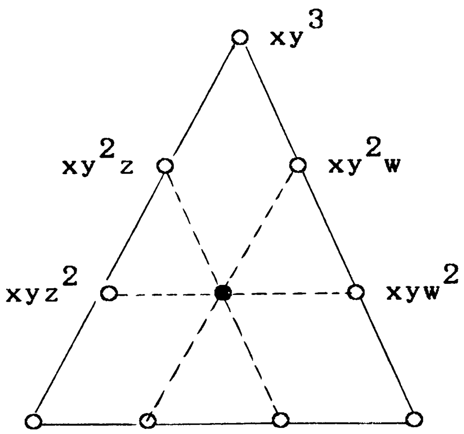
1. $\left(\alpha , \frac{1-\alpha}{2} , \frac{1-\alpha}{3} , \frac{1-\alpha}{6} \right)$



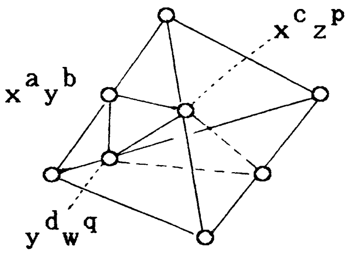
2. $\left(\alpha , \frac{1-\alpha}{2} , \frac{1-\alpha}{4} , \frac{1-\alpha}{4} \right)$



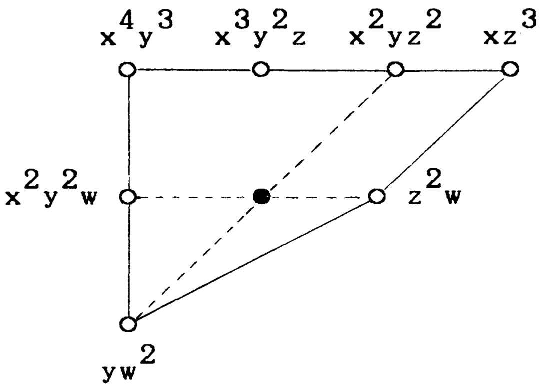
3. $\left(\alpha , \frac{1-\alpha}{3} , \frac{1-\alpha}{3} , \frac{1-\alpha}{3} \right)$



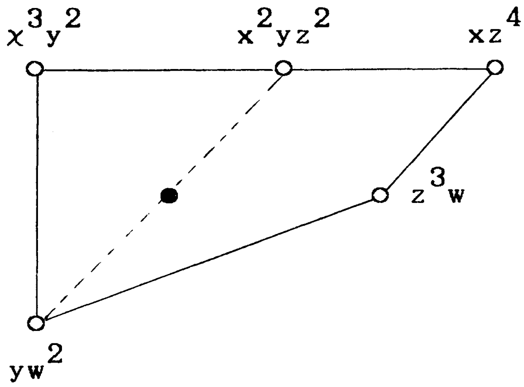
IV



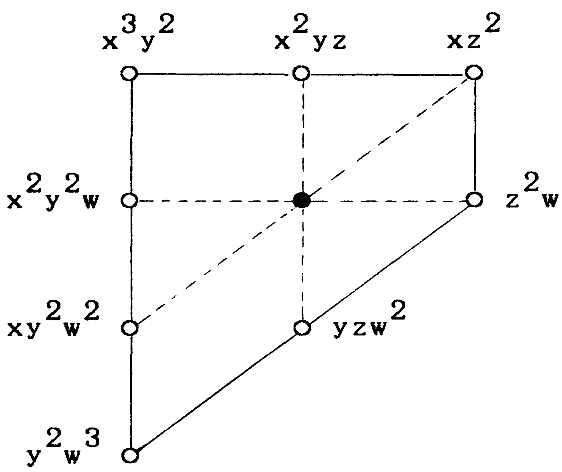
1. $\left(\alpha ,\ \frac{1-4\alpha}{3} ,\ \frac{1-\alpha}{3} ,\ \frac{1+2\alpha}{3} \right)$



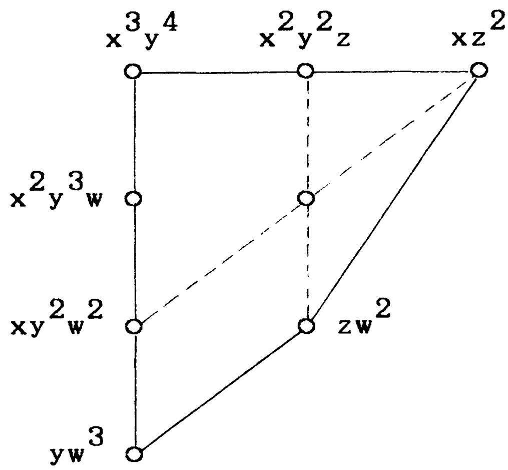
2. $\left(\alpha ,\ \frac{1-3\alpha}{2} ,\ \frac{1-\alpha}{4} ,\ \frac{1-3\alpha}{4} \right)$



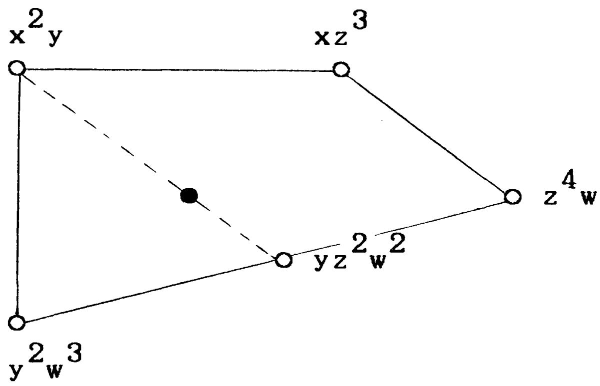
3. $\left(\alpha ,\ \frac{1-3\alpha}{2} ,\ \frac{1-\alpha}{2} ,\ \frac{1+2\alpha}{3} \right)$



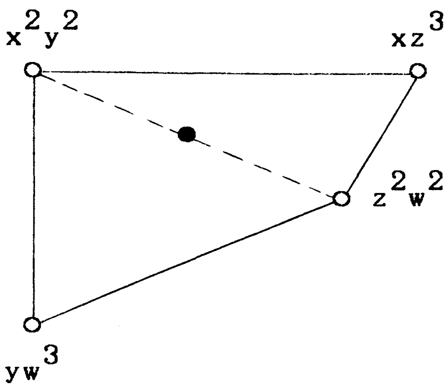
4. $\left(\alpha ,\ \frac{1-3\alpha}{4} ,\ \frac{1-\alpha}{2} ,\ \frac{1+\alpha}{4} \right)$



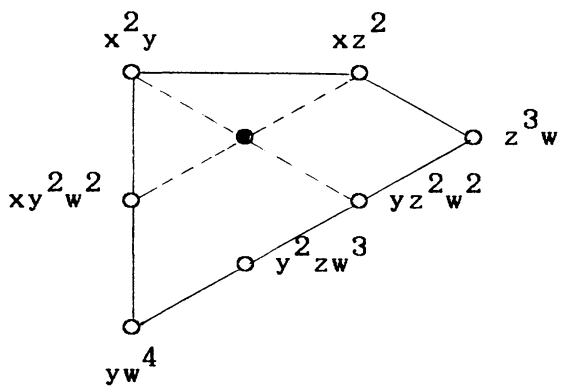
5. $\left(\alpha \,,\ 1-2\alpha \,,\ \frac{1-\alpha}{3} \,,\ \frac{4\alpha-1}{3} \right)$



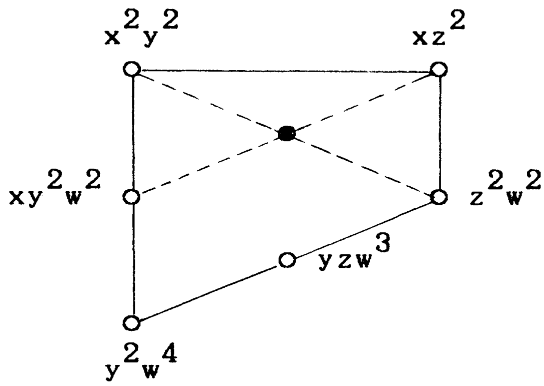
6. $\left(\alpha \,,\ \frac{1-2\alpha}{2} \,,\ \frac{1-\alpha}{3} \,,\ \frac{1+2\alpha}{6} \right)$



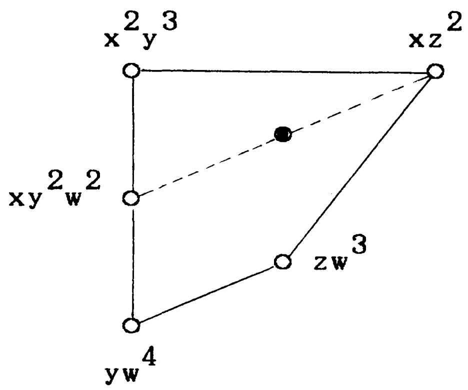
7. $\left(\alpha \,,\ 1-2\alpha \,,\ \frac{1-\alpha}{2} \,,\ \frac{3\alpha-1}{2} \right)$



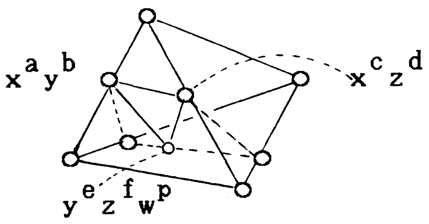
8. $\left(\alpha \,,\ \frac{1-2\alpha}{2} \,,\ \frac{1-\alpha}{2} \,,\ \frac{\alpha}{2} \right)$



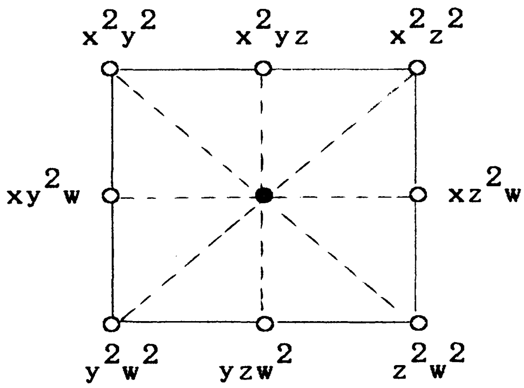
9. $\left(\alpha \,,\ \frac{1-2\alpha}{3} \,,\ \frac{1-\alpha}{2} \,,\ \frac{\alpha+1}{2} \right)$



V



1. $\left(\alpha , \frac{1-2\alpha}{2} , \frac{1-2\alpha}{2} , 2 \right)$



REFERENCES

[I1] S. Ishii, On isolated Gorenstein singularities, Math. Ann. 270(1985), 541-554.

[I2] S. Ishii, On the classification of two dimensional singualrities by the invariant κ, preprint, 1988.

[IW] S. Ishii and K. Watanabe, On simple K3 singularities (in Japanese), Notes appearing in the Proceedings of the Conference on Algebraic Geometry at Tokyo Metropolitan Univ. 1988, 20-31.

[IW2] S. Ishii and K. Watanabe, A geometric characterization of a simple K3 singularity, Tôhoku Math. J. 44(1992), 19-24.

[V] A. N. Varchenko, Zete-Function of monodromy and Newton's diagram, Invent. Math. 37(1976), 253-262.

[W1] K. Watanabe. On plurigenera of normal isolated singularities, I, Math. Ann. 250(1980), 65-94.

[W2] K. Watanabe, On plurigenera of normal isolated singularities, II, in Complex Analytic Singularities (T. Suwa and P. Wagreich, eds.), Advanced Studies in Pure Math. 8, Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, Oxford, 1986, 671-685.