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Higher cycles of the moduli space of stable curves

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§1. Introduction and results.

We shall denote the moduli space of stable curves of genus g by $\overline{\mathcal{M}}_g$ and assume $g \geq 3$. $\overline{\mathcal{M}}_g$ is a compactification of the classical moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g [D-M]. It is known that $\overline{\mathcal{M}}_g$ and $\mathcal{M}_g$ are complex V-manifolds of dimension $3g-3$ and the compactification locus $\mathcal{D} = \overline{\mathcal{M}}_g - \mathcal{M}_g$ is the sum of $1+[\frac{g}{2}]$ divisors $\mathcal{D}_0, \ldots, \mathcal{D}_{[\frac{g}{2}]}$. Scott Wolpert showed in [W] that $2+[\frac{g}{2}]$ analytic 2-cycles on $\overline{\mathcal{M}}_g$ can be constructed and they span $H_2(\overline{\mathcal{M}}_g; Q)$ from the result of Harer [H]. Similarly Wolpert defined some analytic $2k$-cycles (for $k < g$) and showed that they are independent in $H_{2k}(\overline{\mathcal{M}}_g; Q)$ by calculating their intersection pairing with components of the strata of $\mathcal{D}$. A surface represented by a point of $\mathcal{D}$ necessarily has nodes (double points). $\mathcal{D}$ is stratified by the number of nodes. Hence the Betti number $b_{2k}(\overline{\mathcal{M}}_g)$ is greater than or equal to n_k, the number of the $2k$-cycles Wolpert constructed. Roughly speaking, n_k is almost equal to $\frac{1}{2}\binom{g-1}{k}$.

The idea for constructing analytic cycles is as follows: Fix a stable curve S with nodes such that $S - \{\text{nodes}\}$ is not connected and choose some components $S_1, \cdots, S_k$ of $S - \{\text{nodes}\}$, then we can get an analytic cycle $\mathcal{A}$ by letting the conformal structure of each S_j vary over all structures represented in its moduli space, while the structure of $S - \bigcup_j S_j$ is kept fixed. Wolpert considers, for S_j, once punctured tori and quadruply punctured spheres when $k = 1$ and considers quadruply punctured spheres when $1 < k < g$.

Because these constructions are only from the moduli space of once-punctured tori and the moduli space of quadruply puncture spheres, we have no information of the Betti numbers of degree more than $2g - 2$ and less than $4g - 4$. In this paper, we shall define analytic fibre spaces having tori as generic fibre and improve Wolpert's estimates. We have estimates of the Betti numbers of all even degrees. We have the following results.

Theorem A.

When $k \geq 2$, $b_{2k}(\overline{\mathcal{M}}_g) = b_{6g-6-2k}(\overline{\mathcal{M}}_g) \geq \max(\alpha_{g,k}, \alpha_{g,3g-3-k})$ where $\alpha_{g,k}$ is a certain kind of permutation numbers.

The number $\alpha_{g,k}$ can be computed by the following formula

Proposition B.

$$f_g(t) = \sum_k \alpha_{g,k} t^k$$

where $f_g(t)$ is a polynomials defined in the section 5.

Roughly speaking, this estimates are more than the square of Wolpert's estimates. In fact we get easily the following inequality.

$$\alpha_{g,k} > \frac{1}{2}\binom{g-1}{k} + \frac{1}{2}\sum_{k',l}\binom{k'-1}{l}\cdot\binom{k-2k'+1}{l+1}\cdot\binom{g-l-2}{k-k'}$$

This paper is organized as follows. In §2 we construct analytic fibrations $\widetilde{\mathcal{Q}}_\ell \to \mathcal{U}_\ell$ which have two analytic cross sections with no intersection and whose general fibre is an elliptic curve. In §3 we construct cycles of $\overline{\mathcal{M}}_g$ from three fibrations; one of them constructed in §2 and the other constructed by Wolpert. In §4 we show that they are independent classes in $H_*(\overline{\mathcal{M}}_g; Q)$ by calculating the intersection pairings between the cycles and components of the strata of $\mathcal{D}$ and show Theorem A. In §5 we give algorithm to compute $\alpha_{g,k}$.

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§2. Twice punctured elliptic curves..

To construct the cycles, we need a fibre space of elliptic curves with the base a compact complex surface which has two cross sections. We require that the sections have no intersections and have no values at nodes of singular fibres. First of all we shall review the definition of the fibre spaces $\mathcal{U}_\ell$, $\mathcal{Q}_\ell$ from [W]. These are fiber spaces of elliptic curves over $\widehat{\mathbf{H}/\Gamma_\ell}$. We assume $\ell \geq 3$. Let $\mathbf{H}$ be the upper half plane, Γ_ℓ the principal congruence subgroup of level ℓ, $\widehat{\mathbf{H}/\Gamma_\ell}$ the compactification of the quotient of $\mathbf{H}$ by Γ_ℓ, and $\mathcal{U}_\ell$ the compactification of the universal elliptic curve with level ℓ structure. The base $\widehat{\mathbf{H}/\Gamma_\ell}$ has $\frac{i}{\ell}$ filled-in punctures (we call them *cusps*), where $i = [PSL(2;Z) : \Gamma_\ell]$. Singular fibres of $\mathcal{U}_\ell$ and $\mathcal{Q}_\ell$ correspond to cusps of $\widehat{\mathbf{H}/\Gamma_\ell}$. The projection $\mathcal{U}_\ell|_{\mathbf{H}/\Gamma_\ell} \to \mathbf{H}/\Gamma_\ell$ is given as follows. Let $\Gamma_\ell L = \Gamma_\ell \ltimes \mathbf{Z}^2$, where for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\ell$, $g(m,n) = (am + cn, bm + dn)$. $\Gamma_\ell L$ acts on $\mathbf{H} \times \mathbf{C}$ by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}(m,n)\right](z,\xi) = \left(\frac{az+b}{cz+d}, \frac{\xi + mz + n}{cz+d}\right), \quad \text{for} \quad (z,\xi) \in \mathbf{H}\times\mathbf{C}$$

Then we define the map to be the projection

$$\mathcal{U}_\ell|_{\mathbf{H}/\Gamma_\ell} = \mathbf{H} \times \mathbf{C}/\Gamma_\ell L \to \mathbf{H}/\Gamma_\ell.$$

In the neighbourhood $D = \{\tau \in \mathbf{C}; |\tau| < 1\}$ of a cusp in $\widehat{\mathbf{H}/\Gamma_\ell}$ the fibration $\pi_1 : \mathcal{U}_\ell \to \widehat{\mathbf{H}/\Gamma_\ell}$ is described as follows. First we consider a fibre over $\{\tau = 0\}$. Consider an ℓ-tuple of projective lines $\mathbf{P}^1_0, \cdots, \mathbf{P}^1_{\ell-1}$ with inhomogeneous coordinate u_k, $v_k = u_k^{-1}$ on $\mathbf{P}^1_k$ (the index k takes values in $Z/\ell Z$). Identify ∞ on $\mathbf{P}^1_k$ with 0 on $\mathbf{P}^1_{k+1}$ to obtain an ℓ-gon of projective lines. Next we consider fibres over $\{\tau \neq 0\}$. Remove

$\{|v_k| \leq |\tau|\}$ on $\mathbf{P}^1_k$ and $\{|u_{k+1}| \leq |\tau|\}$ on $\mathbf{P}^1_{k+1}$ and attach two annuli $\{|\tau| < |v_k| < 1\}$ and $\{|\tau| < |u_{k+1}| < 1\}$ according to $v_k u_{k+1} = \tau$.

From these considerations we can see that $\mathcal{U}_\ell$ is a complex surface and in particular a neighbourhood of a double point of the fibre $\{\tau = 0\}$ is identified with the complex surface $\{uv = \tau\} \cap \{|\tau| < 1, |u| < 1, |v| < 1\}$ in $C^3 = \{(\tau, u, v)\}$ by regarding ∞ on $\mathbf{P}^1_k$ and 0 on $\mathbf{P}^1_{k+1}$ as the double point. $\mathcal{U}_\ell$ has ℓ^2 natural sections $s_1, \cdots, s_{\ell^2}$, the ℓ-division points. The sections form a group isomorphic to $Z/\ell Z \times Z/\ell Z$, acting on $\mathcal{U}_\ell$ by translation in fibres.

We define $\mathcal{Q}_\ell$ by $\mathcal{U}_\ell/(Z/\ell Z \times Z/\ell Z)$.

Let $p : \mathcal{U}_\ell \to \mathcal{U}_\ell/(Z/\ell Z \times Z/\ell Z) = \mathcal{Q}_\ell$ be the natural projection.

A fibre in $\mathcal{Q}_\ell$ over a cusp is a projective line $\mathbf{P}^1$ whose points 0 and ∞ are identified to form a double point. In particular a neighbourhood of a double point of fibre on $\{\tau = 0\}$ is identified with the analytic space $\{uv = \tau^\ell\} \cap \{|\tau| < 1, |u| < 1, |v| < 1\}$ in $C^3 = \{(\tau, u, v)\}$ by regarding ∞ on $\mathbf{P}^1_k$ and 0 on $\mathbf{P}^1_{k+1}$ as the double point. There exists an analytic section s induced from the sections $s_1, \cdots, s_{\ell^2}$. The section s does not take value at the double point. If T' is a line bundle over $\mathcal{Q}_\ell - \{\text{double points}\}$ which consists of the tangent vectors of the fibers of $\pi_2 : \mathcal{Q}_\ell \to \widehat{\mathbf{H}/\Gamma_\ell}$, then $c_1(s^* T') = -\frac{i}{12}$.

Next we shall construct the fibration

$$\widetilde{\pi} : \widetilde{\mathcal{Q}}_\ell \to \mathcal{U}_\ell.$$

We denote $\widehat{\mathcal{Q}_\ell}$ by $\pi_1^* \mathcal{Q}_\ell$.

$$\begin{tikzcd}
\mathcal{Q}_\ell \arrow[d, "\pi_2"'] & \widehat{\mathcal{Q}_\ell} \arrow[l] \arrow[d, "\hat{\pi}"] \\
\widehat{\mathbf{H}/\Gamma_\ell} \arrow[u, bend left=60, "s"] & \mathcal{U}_\ell \arrow[l, "\pi_1"] \arrow[u, bend right=60, "\hat{s}_1,\, \hat{s}_2"']
\end{tikzcd}$$

We shall define two analytic sections $\hat{s}_1$ and $\hat{s}_2$ on $\widehat{\mathcal{Q}_\ell}$. The section $\hat{s}_1$ is the pull-back of s by π_1. The section $\hat{s}_2$ is defined by

$$\hat{s}_2 : \mathcal{U}_\ell \to \widehat{\mathcal{Q}_\ell} : x \mapsto (x, p(x)),$$

where $\widehat{\mathcal{Q}_\ell} = \{(x,\xi) \in \mathcal{U}_\ell \times \mathcal{Q}_\ell | \pi_1(x) = \pi_2(\xi)\}$ and $\hat{\pi}(x,\xi) = x$. But if $s\pi_1(x) = p(x)$, (that is, $x \in \bigcup_j s_j(\widehat{\mathbf{H}/\Gamma_\ell}) \subset \mathcal{U}_\ell$) these two sections have intersections. Furthermore, on a double point of a singular fibre of $\mathcal{U}_\ell$, $\hat{s}_2$ takes value at the node point of a singular fibre. We shall modify this fibration so that the sections do not have any intersection, nor does $\hat{s}_2$ take value at any node point.

Consider a fibre space which is given by removing $s_j(\widehat{\mathbf{H}/\Gamma_\ell})$ from $\mathcal{U}_\ell$, that is,

$$\hat{\pi} : \widehat{\mathcal{Q}_\ell}|\mathcal{U}_\ell^\circ \to \mathcal{U}_\ell^\circ = \mathcal{U}_\ell - \bigcup_j s_j(\widehat{\mathbf{H}/\Gamma_\ell})$$

Let $(\hat{\rho}) := s^*T'$. A metric $\|\quad\|$ is fixed for the line bundle $(\hat{\rho})$, and in local coordinates the absolute value $|\quad|$ is well defined. Let $U = \{x \in (\hat{\rho}); \|x\| < 1\}$ and we may assume that U gives coordinates in a neighbourhood of $s(\widehat{\mathbf{H}/\Gamma_\ell})$ by the section s. Furthermore $p^{-1}(U) = U_1 \amalg \cdots \amalg U_{\ell^2}$ where U_j is a neighbourhood of $s_j(\widehat{\mathbf{H}/\Gamma_\ell})$ in $\mathcal{U}_\ell$ for each j, and U_j can be identified with U by the map $p|_{U_j} : U_j \to U$. Fix j and identify U_j with U. We shall describe an auxiliary fibre space $\mathcal{F}$ over U_j by attaching $\hat{\pi}|_{U_j} : \widehat{\mathcal{Q}_\ell}|U_j \to U_j$ and $U_j \times \mathbf{P}^1 \to U_j$ as follows.

In the fibres over $s_j(\widehat{\mathbf{H}/\Gamma_\ell})$, attach $\widehat{\mathcal{Q}_\ell}|_{s_j(\widehat{\mathbf{H}/\Gamma_\ell})}$ to a trivial bundle $s_j(\widehat{\mathbf{H}/\Gamma_\ell}) \times \mathbf{P}^1 \to s_j(\widehat{\mathbf{H}/\Gamma_\ell})$ by identifying $\hat{s}_1(s_j(\widehat{\mathbf{H}/\Gamma_\ell})) \subset \widehat{\mathcal{Q}_\ell}|_{s_j(\widehat{\mathbf{H}/\Gamma_\ell})}$ and $s_j(\widehat{\mathbf{H}/\Gamma_\ell}) \times \{u = 0\} \subset s_j(\widehat{\mathbf{H}/\Gamma_\ell}) \times \mathbf{P}^1$, where u is one of the inhomogeneous coordinates of $\mathbf{P}^1$. (The other one is denoted by v and satisfies $uv = 1$.) Let (z, ξ) be local coordinates of U (where $z \in \widehat{\mathbf{H}/\Gamma_\ell}$ and $\xi \in C$).

We define D_1, D_2, h as follows.

$$D_1 := \left\{ ((z,\zeta),(z,\xi)) \in U_j \times U \subset \mathcal{U}_\ell \times \mathcal{Q}_\ell \,\middle|\, \zeta \neq 0, |\xi| \leq |\zeta| \right\}$$

$$\subset \widehat{\mathcal{Q}_\ell}|_{U_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})}$$

$$D_2 := \left\{ ((z,\zeta),u) \in U_j \times \mathbf{P}^1 \,\middle|\, \zeta \neq 0, |u| \leq ||\zeta|| \right\}$$

$$\subset (U_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})) \times \mathbf{P}^1$$

$$h : \left\{ ((z,\zeta),u) \in U_j \times \mathbf{P}^1 \,\middle|\, \zeta \neq 0, ||\zeta|| < |u| < 1 \right\}$$

$$\to \left\{ ((z,\zeta),(z,\xi)) \in U_j \times U \subset \mathcal{U}_\ell \times \mathcal{Q}_\ell \,\middle|\, \zeta \neq 0, |\zeta| < |\xi| < \frac{|\zeta|}{||\zeta||} \right\}$$

$$((z,\zeta),u) \quad \mapsto \quad ((z,\zeta),(z,\frac{\zeta}{u}))$$

Then we attach $\widehat{\mathcal{Q}_\ell}|_{U_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})} - D_1$ and $(U_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})) \times \mathbf{P}^1 - D_2$ by the attaching map h, and we have a fibre space $\mathcal{F} \to U_j$. It is easy to check that $\mathcal{F}$ is an analytic fibre space and the map $\mathcal{F} \to U_j$ is holomorphic.

Identify $\mathcal{F}|_{u_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})}$ with $\widehat{\mathcal{Q}_\ell}|_{U_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})}$ by

$$\begin{aligned} (U_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})) \times \mathbf{P}^1 - D_2 &\longrightarrow (U_j - s_j(\widehat{\mathbf{H}/\Gamma_\ell})) \times U_j \\ ((z,\zeta),u) &\longmapsto ((z,\zeta),(z,\frac{\zeta}{u})) \\ ((z,\zeta),v) &\longmapsto ((z,\zeta),(z,\zeta v)) \end{aligned}$$

and denote the identification map by ι_j. For all $j = 1, 2, \cdots, \ell^2$, attach

$\mathcal{F}$ and $\widehat{\mathcal{Q}_\ell}|_{\mathcal{U}_\ell^\circ}$ by ι_j, and denote the analytic fibre space constructed by $\hat{\pi}' : \widehat{\mathcal{Q}_\ell}' \to \mathcal{U}_\ell$.

Next let U' be a neighbourhood of a node on a singular fiber of $\mathcal{U}_\ell$. We shall construct an auxiliary fibre space $\mathcal{F}'$ over U'. We consider coordinates of U' as follows.

$$U' := \{(u,v) \in C^2; |u| < 1, |v| < 1\} \qquad \pi_1(u,v) = uv$$

Let $\mathbf{P}$ be a projective line and D_1, D_2 be two disks, that is,

$$\begin{aligned} \mathbf{P} :=& \mathbf{P}^1 : \text{with inhomogeneous coordinates } u_1,\ v_1 (u_1 v_1 = 1) \\ D_1 :=& \{v_2 \in \mathbf{C} \mid |v_2| < 1\} \\ D_2 :=& \{u_2 \in \mathbf{C} \mid |u_2| < 1\} \end{aligned}$$

Attach $U' \times D_1$, $U' \times D_2$ and $U' \times \mathbf{P}$ as follows.

(i) On $(0,0) \in U'$, identify

$$\begin{aligned} (0,0) \times \{v_2 = 0\} \in& (0,0) \times D_1 \quad \text{and} \quad (0,0) \times \{u_1 = 0\} \in (0,0) \times \mathbf{P}, \\ (0,0) \times \{u_2 = 0\} \in& (0,0) \times D_2 \quad \text{and} \quad (0,0) \times \{v_1 = 0\} \in (0,0) \times \mathbf{P}. \end{aligned}$$

(ii) On $(z_1 \neq 0, 0) \in U'$, remove $\{|v_2| \leq |z_1|^\ell\}$ from D_1, remove $\{|u_1| \leq |z_1|^\ell\}$ from $\mathbf{P}$ and attach $\{|z_1|^\ell < |v_2| < 1\} \subset D_1$ to $\{|z_1|^\ell < |u_1| < 1\} \subset \mathbf{P}$ by identifying $v_2 u_1 = z_1^\ell$. Identify

$$(0,0) \times \{u_2 = 0\} \in (0,0) \times D_2 \quad \text{and} \quad (0,0) \times \{v_1 = 0\} \in (0,0) \times \mathbf{P}.$$

(iii) On $(0, z_2 \neq 0) \in U'$, remove $\{|u_2| \leq |z_2|^\ell\}$ from D_2, remove $\{|v_1| \leq |z_2|^\ell\}$ from $\mathbf{P}$ and attach $\{|z_2|^\ell < |u_2| < 1\} \subset D_1$ to $\{|z_2|^\ell < |v_1| < 1\} \subset \mathbf{P}$ by identifying $v_1 u_2 = z_2^\ell$. Identify

$$(0,0) \times \{v_2 = 0\} \in (0,0) \times D_1 \quad \text{and} \quad (0,0) \times \{u_1 = 0\} \in (0,0) \times \mathbf{P}.$$

(iv) On $(z_1 \neq 0, z_2 \neq 0) \in U'$, remove $\{|v_2| \leq |z_1|^\ell\}$ from D_1, remove $\{|u_1| \leq |z_1|^\ell\}$ from $\mathbf{P}$ and attach $\{|z_1|^\ell < |v_2| < 1\} \subset D_1$ to $\{|z_1|^\ell < |u_1| < 1\} \subset \mathbf{P}$ by identifying $v_2 u_1 = z_1^\ell$. Remove $\{|u_2| \leq |z_2|^\ell\}$ from D_2, remove $\{|v_1| \leq |z_2|^\ell\}$ from $\mathbf{P}$ and attach $\{|z_2|^\ell < |u_2| < 1\} \subset D_2$ to $\{|z_2|^\ell < |v_1| < 1\} \subset P$ by identifying $u_2 v_1 = z_2^\ell$.

This fibre space $\mathcal{F}'$ is an analytic space and the map $\mathcal{F}' \to U'$ is holomorphic.

Finally we shall construct a fibre space $\tilde{\pi} : \widetilde{\mathcal{Q}}_\ell \to \mathcal{U}_\ell$ by attaching $\mathcal{F}'$ to $\widehat{\mathcal{Q}_\ell}' |_{\mathcal{U}_\ell - \{\text{double points}\}}$.

Let B be defined as follows.

$$B = \left\{ ((z_1, z_2), \pi(u,v)) \in U' \times \pi(U') \,\middle|\, z_1 z_2 = uv \right\}$$
$$- \{((0,0), \pi(0,0))\} \subset \widehat{\mathcal{Q}_\ell}'$$

Also we define $\iota' : B \to \mathcal{F}'$ as follows.

$$\iota'((z_1, z_2), \pi(u,v))$$
$$= \begin{cases} (z_1, z_2, \frac{z_1^\ell}{u^\ell}) & \text{in the } (x, y, u_1)\text{-coordinates} \quad (\text{if } z_1 \neq 0,\ u \neq 0) \\ (z_1, z_2, \frac{z_2^\ell}{u^\ell}) & \text{in the } (x, y, v_1)\text{-coordinates} \quad (\text{if } z_2 \neq 0,\ v \neq 0) \\ (z_1, z_2, u^\ell) & \text{in the } (x, y, v_2)\text{-coordinates} \quad (\text{if } |u| \geq |z_1|^\ell) \\ (z_1, z_2, v^\ell) & \text{in the } (x, y, u_2)\text{-coordinates} \quad (\text{if } |v| \geq |z_2|^\ell) \end{cases}$$

This map is well defined and biholomorphic. Attach $\mathcal{F}'$ to $\widehat{\mathcal{Q}_\ell}|_{\mathcal{U}_\ell - \{\text{nodes}\}}$ by the map ι' for all nodes on $\mathcal{U}_\ell$ and we have a new fibre space $\tilde{\pi} : \widetilde{\mathcal{Q}}_\ell \to \mathcal{U}_\ell$.

Lemma 2.1.

$\tilde{\pi} : \widetilde{Q}_\ell \to \mathcal{U}_\ell$ have two analytic cross sections $\tilde{s}_1$, $\tilde{s}_2$ which have no intersection and have no values at nodes on fibers.

Proof

The sections $\tilde{s}_1$ and $\tilde{s}_2$ are defined as follows:

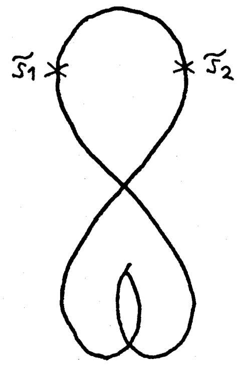
$$\tilde{s}_1 = \begin{cases} \hat{s}_1 \quad \text{in } \widetilde{Q}_\ell|_{\mathcal{U}_\ell - \cup_j s_j(\widehat{\mathbf{H}/\Gamma_\ell})} \\ (z,\xi) \mapsto (z,\xi,0) \in U_j \times \mathbf{P}^1 \quad ((z,\xi,u)\text{-coordinates}) \quad \text{in } \mathcal{F} \end{cases}$$

$$\tilde{s}_2 = \begin{cases} \hat{s}_2 \quad \text{in } \widetilde{Q}_\ell|_{\mathcal{U}_\ell^\circ} \\ (z,\xi) \mapsto (z,\xi,1) \in U_j \times \mathbf{P}^1 \quad ((z,\xi,u)\text{-coordinates}) \quad \text{in } \mathcal{F} \\ (x,y) \mapsto (x,y,1) \in U' \times \mathbf{P}^1 \quad ((z_1,z_2,u_1)\text{-coordinates}) \quad \text{in } \mathcal{F}' \end{cases}$$

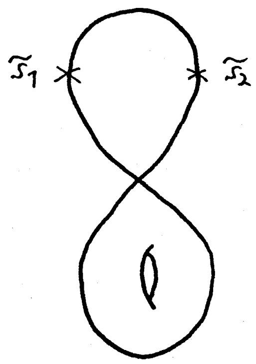
It is easy to show that $\tilde{s}_1$ and $\tilde{s}_2$ are well defined and that they have no intersection. Also they do not take values at nodes on fibres. □

Removing the two sections $\tilde{s}_1$, $\tilde{s}_2$ from $\widetilde{Q}_\ell$ we obtain a family $\widetilde{Q}_\ell^\circ$ of twice punctured tori. Each singular fibres is one of the following four types.

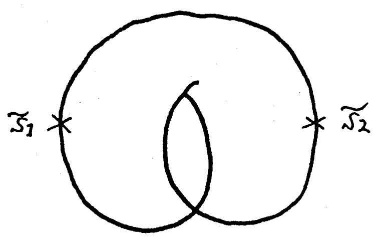
Figure 1.



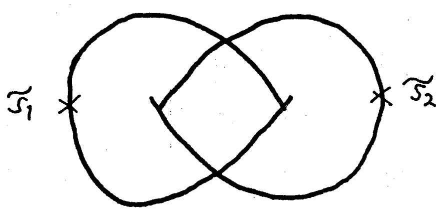
Type 1 On $s(\text{cusps})$



Type 2 On $s(\widehat{\mathbf{H}/\Gamma_\ell} - \text{cusps})$



Type 3 On $\pi_1^{-1}(\text{cusps}) - s(\text{cusps}) - \text{nodes}$



Type 4 On nodes

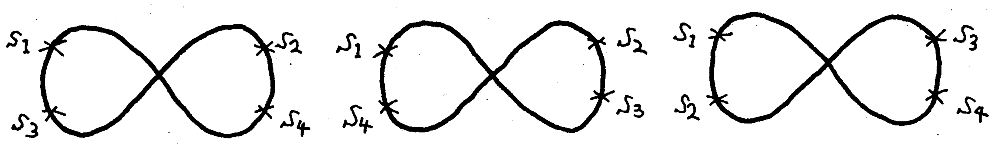
§3. Construction of the cycles.

The locus $\mathcal{D} \subset \overline{\mathcal{M}}_g$ of surfaces having nodes (double points) is a divisor and is stratified by the number of nodes. That is to say, the k-strata is the locus of surfaces with k nodes. Counting the number of components in the k-strata is a combinatorial problem. (The number of 1-stratum components is $1+[\frac{g}{2}]$.) The closure of a k-stratum component represents a $(6g-6-2k)$-dimensional homology class. Wolpert [W] showed that Poincaré dual $[\omega]$ of Weil-Petersson Kähler form ω on $\overline{\mathcal{M}}_g$ and the $1+[\frac{g}{2}]$ cycles above represented by the closure of 1-stratum components span $H_{6g-8}(\overline{\mathcal{M}}_g; \mathbf{Q})$. n_k components of the k-strata of the specific pattern for the nodes are mutually independent in $H_{6g-6-2k}(\overline{\mathcal{M}}_g; \mathbf{Q})$ for $k < g$. Considering another specific pattern for the nodes we construct more homology classes for all even degrees.

In this section first we shall define "k-selections" ($k \leq 2g-2$). After that we shall construct cycles and subvarieties of $\mathcal{M}_g$. To construct cycles we use three fibrations

$$\tilde{\pi} : \widetilde{\mathcal{Q}}^o_\ell \to \mathcal{U}_\ell$$
$$\pi_2 : \mathcal{Q}^o_\ell \to \widehat{\mathbf{H}/\Gamma_\ell}$$
$$\pi : \mathcal{U}^o_2 \to \widehat{\mathbf{H}/\Gamma_2}.$$

where Q_ℓ^o is $Q_\ell - Im(s)$. $\mathcal{U}_2$ is defined by the compactification of $\mathbf{H} \times \mathbf{C}/\Gamma_2 L \to \mathbf{H}/\Gamma_2$ in the same way as $\mathcal{U}_\ell$. Generic fibre of $\mathcal{U}_2$ is a complex projective line with four distinguished points s_1, s_2, s_3, s_4. Singular fibres on filled in cusps are two projective lines connected by a double point.



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$\mathcal{U}_2^o$ is $\mathcal{U}_2 - \cup s_j(\widehat{\mathbf{H}/\Gamma_2})$. Note that $c_1(s_i^* T') = -1$ for $i = 1, 2, 3, 4$.

Fix a surface S and we shall consider the "pattern" for the curves $c_1, c_2, \cdots, c_{g-1}, d_1, d_2, d_2', \cdots, d_{g-1}, d_{g-1}', d_g$ as indicated below. (Figure 2)

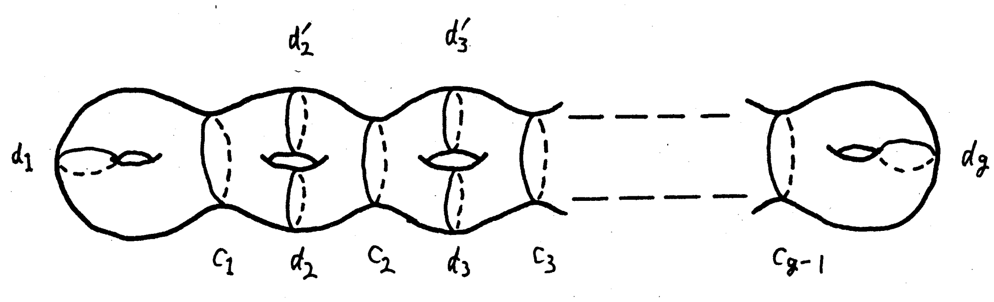

Figure 2

Definition 3.1.. A *$k-selection$* σ is a choice of k homotopy classes from the free homotopy classes $[c_1],\cdots,[c_{g-1}]$, $[d_1]$, $[d_2]$, $[d'_2],\cdots$, $[d_{g-1}]$, $[d'_{g-1}]$, $[d_g]$ satisfying the following two conditions

1) $[d_i]$ is in the k-selection if and only if $[d'_i]$ is in the k-selection.

2) If $[d_i]$, $[d'_i]$ are in the k-selection, then neither $[c_{i-1}]$ nor $[c_i]$ is not in the k-selection (for $2 \leq i \leq g-1$)

The classes $[c_1]$, $\cdots$, $[c_{g-1}]$, $[d_1]$, $[d_2]$, $[d'_2]$, $\cdots$, $[d_{g-1}]$, $[d'_{g-1}]$, $[d_g]$ may be permuted by a homeomorphism of S. We denote the number of distinct k-selections modulo the action of homeomorphisms by $\alpha_{g,k}$. ($\alpha_{g,k}$ is the number of conjugate classes of k-selections. (see definition 4.2))

Fix a k-selection σ, if d_j, d'_j occur in the k-selection σ then collapse c_{j-1}, c_j to nodes and replace the component containing d_j, d'_j by the fibre of a family $\widetilde{\mathcal{Q}}^o_\ell$: remove the component containing d_j, d'_j, identify the node to which c_{j-1} collapses with the first puncture in the fibre of $\widetilde{\mathcal{Q}}^o_\ell$, and identify the node to which c_j collapses with the second puncture in the fibre of $\widetilde{\mathcal{Q}}^o_\ell$.

If c_j $(2 \leq j \leq g-2)$ is in the k-selection σ, we collapse d_j, d'_j, d_{j+1}, d'_{j+1} to nodes and replace the component containing c_j by the fibre of the family $\mathcal{U}^o_2$. If both d_1 and c_1 are in the k-selection σ, we collapse d_2, d'_2 to nodes and replace the component containing c_1, d_1 by the fibre of the family $\widetilde{\mathcal{Q}}^o_\ell$. The construction is similar for the case when both d_g, c_{g-1} are in σ. If c_1 is in the k-selection σ but d_1 is not, we collapse d_1, d_2, d'_2 to nodes and replace the component containing c_1 by the fibre of the family $\mathcal{U}^o_2$. The construction is similar for the case when c_{g-1} is in σ but d_g is not. If d_1 is in the k-selection σ but c_1 is not, we collapse c_1 to a node and replace the component containing d_1 by the fibre of the family $\mathcal{Q}_\ell^o$. The construction is similar for the case when d_g is in σ but c_{g-1} is not.

Thus we have defined an analytic fibre space $\mathcal{A}_\sigma$. $\mathcal{A}_\sigma$ has the Cartesian product $(\mathcal{U}_\ell)^a \times (\widehat{\mathbf{H}/\Gamma_\ell})^b \times (\widehat{\mathbf{H}/\Gamma_2})^c$ as the base and is the connected sum along punctures of the fibres of $(\widetilde{\mathcal{Q}}_\ell^o)^a$, $(\mathcal{Q}_\ell^o)^b$, $(\mathcal{U}_2^o)^c$ and a fixed surface R for some non-negative integers a, b, c. The analytic fibre space $\mathcal{A}_\sigma$ determines a mapping from $(\mathcal{U}_\ell)^a \times (\widehat{\mathbf{H}/\Gamma_\ell})^b \times (\widehat{\mathbf{H}/\Gamma_2})^c$ to $\overline{\mathcal{M}}_g$. Let $[\mathcal{A}_\sigma]$ be the homology class determined by $\mathcal{A}_\sigma$. In this way, we can define a $2k$-cycle $[\mathcal{A}_\sigma]$ for each k-selection σ.

§4. Counting the intersection numbers.

In the Section 3 we showed that for each k-selection we can define a 2k-cycle on $\overline{\mathcal{M}}_g$. On the other hand, for each k-selection τ we can define a subvariety $\mathcal{V}_\tau$ of dimension $6g-6-2k$ of $\overline{\mathcal{M}}_g$ which consists of surfaces whose nodes correspond to the curves selected in the k-selection. Let $\mathcal{V}_k^o \subset \mathcal{D}$ be the locus of surfaces having precisely k nodes. We divide $\mathcal{V}_k^o$ into connected components. The closure of them are distinct subvarieties of $\overline{\mathcal{M}}_g$. They are determined by the choice of k disjoint simple geodesics (which are to be collapsed to nodes) on a surface of genus g.

We shall compute an intersection number of the 2k-cycle $\mathcal{A}_\sigma$ and $6g-6-2k$ subvariety $\mathcal{V}_\tau$ for k-selections σ, τ by pushing off $[\mathcal{A}_\sigma]$ from $\mathcal{D}$.That is, we open up the attaching nodes of replaced components. We shall show that the nodes cannot be opened up for all fibres. To open up the nodes we smoothly perturb sections

$$s : \widehat{\mathbf{H}/\Gamma_\ell} \to \mathcal{Q}_\ell$$

$$s_j : \widehat{\mathbf{H}/\Gamma_2} \to \mathcal{U}_2$$

$$\tilde{s}_1, \quad \tilde{s}_2 : \mathcal{U}_\ell \to \widetilde{\mathcal{Q}}_\ell.$$

The perturbation of s, s_j is given in [W]. Since $c_1(s^*T') = -\frac{i}{12}$, we can choose s', a smooth perturbation of s in U, such that at only one point $p \in \mathcal{Q}_\ell$ it intersects s and $s'(z) = (z, z^{-\frac{i}{12}})$ in (z, ξ) coordinate around p. Since $c_1(s_j^*T') = -1$, we can choose s'_j, a smooth perturbation of s_j, such that at only one point $q \in \mathcal{U}_2$ it intersects s_j and $s'_j(z) = (z, \bar{z})$ in (z, ξ) coordinate around q.

Before defining the perturbation of the sections $\tilde{s}_1$, $\tilde{s}_2$, we give some remarks according to [W].

A V-manifold such as $\overline{\mathcal{M}}_g$ is a rational homology manifold and if cycles intersect at manifold points then their pairing is determined standardly in the intuitive way. Next we shall describe coordinates for the local manifold covers of $\overline{\mathcal{M}}_g$. Let S be a Riemann surface with nodes $p_1, \ldots, p_m$ such that each component of $S - \{p_1, \ldots, p_m\}$ is hyperbolic. Suppose that at the node p_i punctures a_i and b_i are paired. Choose disjoint neighborhood D_i^1, D_i^2 $(i = 1, 2, \ldots, m)$ of the punctures a_i and b_i and let $z_i : D_i^1 \to D = \{u \in C; |u| < 1\}$ and $w_i : D_i^2 \to D$ be local coordinates with $z_i(a_i) = w_i(b_i) = 0$. Fixing an suitable open set $\mathcal{O}$ disjoint from D_i^1 and D_i^2, Beltrami differentials μ_j are chosen with support in $\mathcal{O}$ spanning the Teichmüller space of $S - \{p_1, \ldots, p_m\}$ (the dimension $3g - 3 - m$). If $t = (t_1, \ldots, t_{3g-3-m}) \in C^{3g-3-m}$ is sufficiently close to the origin, the sum $\mu(t) = \sum_j t_j \mu_j$ satisfies $\|\mu(t)\|_\infty < 1$ and thus a μ-conformal solution $\omega^{\mu(t)}$ of the Beltrami equation exists. The Riemann surface $\omega^{\mu(t)}(S) = S_t$ is a quasiconformal deformation of S. The map $\omega^{\mu(t)}$ is conformal on D_i^1 and D_i^2 ; therefore z_i and w_i serve as coordinates for $\omega^{\mu(t)}(D_i^1)$ and $\omega^{\mu(t)}(D_i^2) \subset S_t$ respectively. Given $\tau = (\tau_1, \ldots, \tau_m) \in D^m$, we construct a surface $S_{\tau,t}$ as follows. Remove the discs $\{z_i; |z_i| \leq |\tau_i|\}$ and $\{w_i; |w_i| \leq |\tau_i|\}$ from S_τ. Attach $\{z_i; |\tau_i| < |z_i| < 1\}$ to $\{w_i; |\tau_i| < |w_i| < 1\}$ by identifying z_i and τ_i / w_i

to obtain $S_{\tau,t}$. The couple (τ, t) gives holomorphic coordinates for the local manifold cover of $\overline{\mathcal{M}}_g$ around the point represented by S. The automorphism group Aut(S) acts locally on these coordinates. (see also [B])

Now we construct perturbations of $\tilde{s}_1$, $\tilde{s}_2$. Let $s_1'|_{\mathcal{U}_\ell^o} : \mathcal{U}_\ell^o \to \widetilde{\mathcal{Q}}_\ell$ be the pull back of s' by π_1, and let $s_1'|_{U_j} : U_j \to \widetilde{\mathcal{Q}}_\ell$ be defined by $s_1'(z,\zeta) = (z, \zeta, s'(z)\overline{\zeta})$ in the (z,ζ,v) coordinates of $\widetilde{\mathcal{Q}}_\ell$. By a partition of unity subordinate to $\{\mathcal{U}_\ell^o, U_1, \ldots, U_{\ell^2}\}$, we can construct $\tilde{s}_1' : \mathcal{U}_\ell \to \widetilde{\mathcal{Q}}_\ell$. Note that in U', s_1' is a map to $\widetilde{\mathcal{Q}}_\ell$. The intersection of $\tilde{s}_1$ and $\tilde{s}_1'$ represents the homology class $-\frac{i}{12}[fiber] - \sum_j [s_j(\widehat{\mathbf{H}/\Gamma_\ell})]$. That is, let $\widetilde{\rho_1}$ be the pull back of the tangent vector field along the fibres on $\widetilde{\mathcal{Q}}_\ell$-{ nodes } by $\tilde{s}_1$, then the Poincaré dual of the Euler class of $\widetilde{\rho_1}$ is $-\frac{i}{12}[fibre] - \sum_j [s_j(\widehat{\mathbf{H}/\Gamma_\ell})]$. We can define $\tilde{s}_2'$ in the same way, then for the pull back $\widetilde{\rho_2}$ by $\tilde{s}_2$, the Poincaré dual of the Euler class of $\widetilde{\rho_2}$ is $-\frac{i}{12}[fibre] - \sum_j [s_j(\widehat{\mathbf{H}/\Gamma_\ell})]$.

Using $\tilde{s}_1'$ and $\tilde{s}_2'$ we open up the punctures. Fixing a k-selection σ such that d_j, d_j' are selected but d_{j-1}, d_{j-1}' are not, we open up the first puncture in the fibre of $\widetilde{\mathcal{Q}}_\ell^o$ using $\tilde{s}_1'$. Let a metric $\|\cdot\|$ be fixed for the line bundle $\widetilde{\rho_1}$. In the local coordinate (z_1, z_2, ζ) of $\widetilde{\mathcal{Q}}_\ell$ the absolute value $|\zeta|$ is well defined. Choose a neighborhood $\mathcal{U}$ of the 0-section in $\widetilde{\rho_1}$ mapping injectively to the fibre space $\widetilde{\mathcal{Q}}_\ell$. We identify $\mathcal{U}$ with its image in $\widetilde{\mathcal{Q}}_\ell$. Now $\mathcal{U}$ may be chosen to intersect each fibre in a disk centered at the origin and the section $\tilde{s}_1'$ may be contained in $\mathcal{U}$ and $\|\tilde{s}_1'\| < 1$. Choose a local coordinate chart (z_1, z_2, ζ) of $\mathcal{U}$ in $\widetilde{\mathcal{Q}}_\ell$. The section $\tilde{s}_1'$ is represented in (z_1, z_2, ζ) as $\zeta = \tilde{s}_1'(z_1, z_2)$. Let w be the coordinate disk of neighborhood of fixed side of a node c_j collapsing. We assume the w maps the neighborhood to the unit disk. Remove a disk $\{|w| \leq \|\tilde{s}_1'(z_1, z_2)\|\}$ from S, remove a disk $\{|\zeta| \leq |\tilde{s}_1'(z_1, z_2)\|\}$ from a fibre $F_{(z_1,z_2)}$ over a point (z_1, z_2) of $\widetilde{Q}_\ell$, and form a connected sum of the resulting surfaces by identifying

$$\{\|\tilde{s}'_1(z_1,z_2)\| < |w| < 1\} \subset S$$
$$\to \{|\tilde{s}'_1(z_1,z_2)| < |\zeta| < |\tilde{s}'_1(z_1,z_2)|/\|\tilde{s}'_1(z_1,z_2)\|\} \subset F_{(z_1,z_2)}$$

by setting $w\zeta = \tilde{s}'_1(z_1,z_2)$. It is easy to check that this construction does not depend on the choice of coordinate. In the case when d_j, d'_j are selected but d_{j+1}, d'_{j+1} are not, we can open up the second puncture by using $\tilde{s}'_2$ in the same way as above.

In the case when both d_j, d'_j and d_{j+1}, d'_{j+1} are selected, choose local coordinates (z_1, z_2, ζ_1) around the first puncture of d_{j+1}-component, (x_1, x_2, ζ_2) around the second puncture of d_j-component. Now as above, remove the disc neighborhoods
$\{|\zeta_1| \leq |\tilde{s}'_1(z_1,z_2)|\|\tilde{s}'_2(x_1,x_2)\|\}$ and $\{|\zeta_2| \leq |\tilde{s}'_2(x_1,x_2)|\|\tilde{s}'_1(z_1,z_2)\|\}$ of punctures, and form a connected sum of the resulting surfaces by identifying
$\{|\tilde{s}'_1|\|\tilde{s}'_2\| < |\zeta_1| < |\tilde{s}'_1|/\|\tilde{s}'_1\|\}$ with $\{|\tilde{s}'_2|\|\tilde{s}'_1\| < |\zeta_2| < |\tilde{s}'_2|/\|\tilde{s}'_2\|\}$ by setting $\zeta_1\zeta_2 = \tilde{s}'_1\tilde{s}'_2$. It is easy to show that this construction is independent of the choice of coordinates. In the case when d_1 is selected or c_j is selected, we can open up the node by using s' or $s'_1, \ldots, s'_4$ respectively.

Let $\mathcal{A}^\sharp_\sigma$ be a smooth fibre space of stable curves of genus g constructed from a k-selection σ as above, then $\mathcal{A}^\sharp_\sigma$ and $\mathcal{A}_\sigma$ determine homotopic cycles on $\overline{\mathcal{M}}_g$.

We will consider intersections of $\mathcal{A}^\sharp_\sigma$ and k-selection subvarieties.

Lemma 4.1.

Let σ be a k-selection. Assume that if $g \geq 4$, not all $[d_2]$, $[d_2']$,...,$[d_{g-1}]$, $[d'_{g-1}]$ are selected simultaneously. For any k-selection cycle $[\mathcal{A}_\tau]$, the cycle $[\mathcal{A}_\tau^\sharp]$ can be chosen to intersect the k-selection subvariety $\mathcal{V}_\sigma$ in manifold points of $\overline{\mathcal{M}}_g$.

Proof Consider $Aut(S_\infty)$,where S_∞ is a fibre of $\mathcal{A}_\tau^\sharp$ over an intersection with $\mathcal{V}_\sigma$. Assume the complement of the k-selection nodes in S_∞ has m connected components, $S_1, \ldots, S_m$. From the description of $\mathcal{A}_\tau^\sharp$ each components may be varied arbitrary in a open set of its Teichmüller space. We may divide these components into three types,

i) S_j ; once punctured torus or thrice punctured sphere with two of the punctures identified,

ii) S_j ; twice punctured torus, or quadruply punctured sphere with two of the punctures identified, or quadruply punctured sphere,

iii) S_j ; once punctured surface of genus at least 2 or twice punctured surface of genus at least 2 or thrice punctured surface of genus at least 1 or quadruply punctured surface of genus at least 1.

In the case i) we may assume that $Aut(S_j)$ is generated by an elliptic involution, in the case ii) $Aut(S_j)$ is not trivial but we may assume that the group $Aut_f(S_j)$ of automorphism fixing the punctures is trivial, and in the case iii) we may assume that $Aut(S_j)$ is trivial.

By topological considerations the only homeomorphism of S_∞ which might permute the components S_1,...,S_m is the left-right switch map. Certainly we may assume that the conformal structures for $S_1, \ldots, S_m$are distinct and hence this homeomorphism cannot be represented in $Aut(S_\infty)$.

When $g \geq 4$, since σ does not contain at least one of the curves

$[d_2], [d_2'], \ldots, [d_{g-1}], [d_{g-1}']$, there exists a component of the case iii) in S_∞. Hence any element of $Aut(S_\infty)$ fixes all punctures and $Aut(S_\infty) = \Pi_j Aut_f(S_j)$. Hence we only consider for the case components S_j of case i). There are three possibilities for $Aut(S_\infty)$, that is, 1) $Aut(S_\infty)$ is trivial, or 2) $Aut(S_\infty) = Z/2Z$, or 3) $Aut(S_\infty) = (Z/2Z)^2$. These three cases correspond respectively to 1) none of c_1, c_{g-1}, 2) exactly one of c_1, c_{g-1}, 3) both c_1 and c_{g-1} occur in the k-selection σ. For case 1) S_∞ certainly represents a manifold point. For case 2) assume c_1 is selected in σ, a non- trivial element $k \in Aut(S_\infty)$ is from an elliptic involution. We introduce local manifold cover coordinates (τ_1, t) where τ_1 is for the c_1 node. The elliptic involution is generic for an elliptic curve so that k acts as $k(\tau_1, t) = (-\tau_1, t)$. Hence (τ_1^2, t) give coordinates of $\overline{\mathcal{M}}_g$ around intersection point. For case 3) consider local manifold cover coordinates (τ_1, τ_{g-1}, t), where τ_1 is for the c_1 node, τ_{g-1} is for the c_{g-1} node. $Aut(S_\infty)$ has two generators

$$(\tau_1, \tau_{g-1}, t) \rightarrow (-\tau_1, \tau_{g-1}, t)$$
$$(\tau_1, \tau_{g-1}, t) \rightarrow (\tau_1, -\tau_{g-1}, t).$$

Hence $(\tau_1^2, \tau_{g-1}^2, t)$ give coordinates of $\overline{\mathcal{M}}_g$ around the intersection points.

When $g = 3$, and $[d_2]$ and $[d_2']$ are not selected in σ we can define coordinates as above. When $g = 3$, and $[d_2]$ and $[d_2']$ are selected, each component is twice punctured torus, or quadruply punctured sphere with two of the punctures identified. In this case $Aut(S_\infty) = Z/2Z$ and if (τ_1, τ_2, t) is a local manifold cover coordinate such that τ_1 is for the d_2 node, τ_2 is for the d_2' node, a non-trivial element $k \in Aut(S_\infty)$ acts as

$$k(\tau_1, \tau_2, t) = (\tau_2, \tau_1, t).$$

Hence $(\tau_1+\tau_2,(\tau_1-\tau_2)^2,t)$ gives a coordinate of $\overline{\mathcal{M}}_g$ around intersection points. This completes a proof of lemma 4.1. $\square$

When $g \geq 4$ and all $[d_2],[d_2'],\ldots,[d_{g-1}],[d_{g-1}']$ occur in a k-selection σ, the k-selection cycle $[\mathcal{A}_\tau^\sharp]$ cannot be chosen to intersect $\mathcal{V}_\sigma$ at manifold points of $\overline{\mathcal{M}}_g$. So we need a smooth perturbation $\mathcal{V}_\sigma^\sharp$ of $\mathcal{V}_\sigma$. We consider only the case $\sigma = \{[d_2],[d_2'],\ldots,[d_{g-1}],[d_{g-1}']\}$. The other cases are similar.

Let S_∞ be a fibre of $\mathcal{A}_\tau^\sharp$ over an intersection with $\mathcal{V}_\sigma$, we may assume $Aut(S_\infty) = Z/2Z$. We introduce a local manifold cover $(V,\psi : V \to \overline{\mathcal{M}}_g, \psi(V))$ and coordinates $(\tau_2,\tau_2',\ldots,\tau_{g-1},\tau_{g-1}',t_1,\ldots,t_m)$ of V where τ_j is for the d_j node, τ_j' is for the d_j' node. A non-trivial element $k \in Aut(S_\infty)$ acts as

$$k(\tau_2,\tau_2',\ldots,\tau_{g-1},\tau_{g-1}',t_1,\ldots,t_m) \to (\tau_2',\tau_2,\ldots,\tau_{g-1}',\tau_{g-1},t_1,\ldots,t_m).$$

In this local manifold cover, $\mathcal{V}_\sigma$ is given as the locus $\{\tau_2 = \tau_2' = \cdots = \tau_{g-1} = \tau_{g-1}' = 0\}$ and this locus is mapped injectively to $\overline{\mathcal{M}}_g$. We introduce a local coordinate neighborhood U and a local coordinate chart $(x_1,\ldots,x_m)$ of $\mathcal{V}_\sigma$ around the point represented by S_∞. $(x_1,\ldots,x_m)$ is mapped $(0,\ldots,0,x_1,\ldots,x_m)$ in the local manifold cover. Let $\varepsilon(x)$ be a function on $\mathcal{V}_\sigma$ such that U contains the support of $\varepsilon(x)$ and in a small neighborhood of S_∞ $\varepsilon(x)=\varepsilon$ where ε is a small constant. We set $\mathcal{V}_\sigma^\sharp$ in $\psi(V)$ as follows

$$(x_1,\ldots,x_m) \to \psi(\varepsilon(x),0,\ldots,0,x_1,\ldots,x_m) \in \psi(V).$$

$\mathcal{V}_\sigma^\sharp$ is homotopic to $\mathcal{V}_\sigma$ and we may assume that the cycle $\mathcal{A}_\tau^\sharp$ intersects $\mathcal{V}_\sigma^\sharp$ in manifold points.

For 1-selections, Wolpert [W] showed the intersection pairing between cycles $\{\mathcal{A}_\sigma\}_\sigma$ and varieties $\{\mathcal{V}_\sigma\}_\sigma$ are full rank.

Here we remark the following ; if k-selections σ and τ are in the same class modulo the action of homeomorphisms then the associated subvarieties $\mathcal{V}_\sigma$ and $\mathcal{V}_\tau$ are equal. Thus the number of subvarieties $\{\mathcal{V}_\sigma\}_\sigma$ is $\alpha_{g,k}$. (see §3)

Definition 4.2.

(1) For a k-selection σ, we define its *conjugate* $\bar{\sigma}$ as a k-selection which contains $[d_j]$(resp. $[c_j]$) iff σ contains $[d_{g-j+1}]$(resp. $[c_{g-j}]$).

(2) We call a k-selection σ is *symmetric* iff $\sigma = \bar{\sigma}$.

(3) Assume $k \geq 2$. For two k-selections σ, τ, we define $\widehat{\ }$*-intersection number* $\widehat{[\mathcal{A}_\tau] \cdot [\mathcal{V}_\sigma]}$ as the number of intersections where just selected curves in σ are collapsed to nodes.

Let $[\ \ ] \cdot [\ \ ]$ be an intersection pairing in $\overline{\mathcal{M}}_g$, then it is easy to show the following lemma.

Lemma 4.3.

(1) $\widehat{[\mathcal{A}_\tau] \cdot [\mathcal{V}_\sigma]} = \widehat{[\mathcal{A}_{\bar{\tau}}] \cdot [\mathcal{V}_{\bar{\sigma}}]}$

(2) If σ is not symmetric then

$$[\mathcal{A}_\tau] \cdot [\mathcal{V}_\sigma] = [\mathcal{A}_{\bar{\tau}}] \cdot [\mathcal{V}_\sigma] = \widehat{[\mathcal{A}_\tau] \cdot [\mathcal{V}_\sigma]} + \widehat{[\mathcal{A}_\tau] \cdot [\mathcal{V}_{\bar{\sigma}}]}$$

(3) If σ is symmetric then

$$[\mathcal{A}_\tau] \cdot [\mathcal{V}_\sigma] = [\mathcal{A}_{\bar{\tau}}] \cdot [\mathcal{V}_\sigma] = \widehat{[\mathcal{A}_\tau] \cdot [\mathcal{V}_\sigma]}$$

If and only if τ is equal to either σ or $\bar{\sigma}$, then σ and τ are the same k-selection modulo homeomorphism action, and then $\alpha_{g,k}$ is the number of conjugate classes of k-selections. However, for the present we distinguish them and consider a matrix of $\widehat{\ }$-intersection pairing.

Lemma 4.4.

If the $\widehat{\ }$-intersection pairing matrix is not degenerate then the intersection pairing matrix is not degenerate.

Proof Let $\sigma_1, \ldots, \sigma_m$ be all symmetric k-selections and $\tau_1, \ldots, \tau_n$, $\bar{\tau}_1, \ldots, \bar{\tau}_n$ be all non-symmetric k-selections. From lemma 4.3 (3), the $\widehat{\ }$-intersection pairing matrix is

$$\left(\begin{array}{c|c|c} [\mathcal{A}_{\sigma_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\sigma_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\tau_j}] & [\mathcal{A}_{\sigma_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\bar{\tau}_j}] \\ \hline [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\tau_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\tau_j}] & [\mathcal{A}_{\tau_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\bar{\tau}_j}] \\ \hline [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\bar{\tau}_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\tau_j}] & [\mathcal{A}_{\bar{\tau}_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\bar{\tau}_j}] \end{array}\right).$$

By elementary transformations with respect to the column and lemma 4.3, we can transform it to

$$\left(\begin{array}{c|c|c} [\mathcal{A}_{\sigma_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\sigma_i}] \cdot [\mathcal{V}_{\tau_j}] & [\mathcal{A}_{\sigma_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\bar{\tau}_j}] \\ \hline [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\tau_j}] & [\mathcal{A}_{\tau_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\bar{\tau}_j}] \\ \hline [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\tau_j}] & [\mathcal{A}_{\bar{\tau}_i}] \,\widehat{\cdot}\, [\mathcal{V}_{\bar{\tau}_j}] \end{array}\right).$$

By elementary transformations with respect to the row, we can transform this to

$$\left(\begin{array}{c|c|c} [\mathcal{A}_{\sigma_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\sigma_i}] \cdot [\mathcal{V}_{\tau_j}] & * \\ \hline [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\tau_i}] \cdot [\mathcal{V}_{\tau_j}] & * \\ \hline 0 & 0 & * \end{array}\right).$$

Note that the intersection pairing matrix is

$$\left(\begin{array}{c|c} [\mathcal{A}_{\sigma_i}]\cdot[\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\sigma_i}]\cdot[\mathcal{V}_{\tau_j}] \\ \hline [\mathcal{A}_{\tau_i}]\cdot[\mathcal{V}_{\sigma_j}] & [\mathcal{A}_{\tau_i}]\cdot[\mathcal{V}_{\tau_j}] \end{array}\right)$$

and the lemma follows. □

From Lemma 4.4, it is sufficient to show that the $\widehat{\ }$-intersection pairing matrix is non-degenerate. We prove this by induction. First, we compute the intersection pairing matrix for the two 2-selections $D = \{[d_j], [d'_j]\}$ and $C = \{[c_{j-1}], [c_j]\}$.

Lemma 4.5.

$$\begin{array}{cc} & \begin{array}{cc} \frac{1}{m}\mathcal{V}_D & \frac{1}{n}\mathcal{V}_C \end{array} \\ \begin{array}{c} \frac{1}{i\ell^2}\mathcal{A}_D \\ \mathcal{A}_C \end{array} & \left(\begin{array}{cc} 1 & \frac{1}{12} \\ 2 & 1 \end{array}\right) \end{array}$$

where

$$i = [PSL(2,Z) : \Gamma_\ell]$$

$$m = \begin{cases} 2 & \text{if } g = 3 \\ 1 & \text{if } g \neq 3 \end{cases}$$

$$n = \begin{cases} 4 & \text{if } g = 3 \\ 2 & \text{if } C \text{ contains exactly one of } [c_1],\ [c_{g-1}] \\ 1 & \text{otherwise} \end{cases}$$

Proof

Intersection points between the perturbation $\mathcal{A}_D^\sharp$ of $\mathcal{A}_D$ and $\mathcal{V}_D$ correspond to nodes of singular fibres of $\mathcal{U}_\ell$. Let (z_1, z_2) be coordinates of $\mathcal{U}_\ell$ around a node and let $(\tau, \tau', t \in C^{3g-5})$ be local manifold cover coordinates of $\overline{\mathcal{M}}_g$ around the intersection point p. Then from § 2 we can represent $[\mathcal{A}_D^\sharp]$ as follows.

$$\begin{aligned}&[\mathcal{A}_D^\sharp] : \mathcal{U}_\ell \to \overline{\mathcal{M}}_g\\ &(z_1, z_2) \mapsto [(\tau = z_1^\ell, \tau' = z_2^\ell, t = f(z_1, z_2))]\end{aligned}$$

for some smooth function. When $g \geq 4$, (τ, τ', t) are coordinates of $\overline{\mathcal{M}}_g$ and $\mathcal{V}_D$ is given locally as the locus $\{\tau = \tau' = 0\}$. Since the intersection number at p is ℓ^2 and $\mathcal{U}_\ell$ has i/ℓ singular fibres and one singular fibre has ℓ nodes, the intersection number of $\mathcal{A}_D$ and $\mathcal{V}_D$ is given by $\ell^2 \times i/\ell \times \ell = i\ell^2$. In case $g = 3$, $(\sigma_1, \sigma_2, t) := (\tau + \tau', (\tau - \tau')^2, t)$ give coordinates of $\overline{\mathcal{M}}_g$ around p and only in this case the intersection number is $2i\ell^2$.

Next we shall calculate the intersection of $\mathcal{A}_D$ and $\mathcal{V}_D$. The Poincaré duals of the Euler classes of $\widetilde{\rho}_1$, $\widetilde{\rho}_2$ are both

$$-\frac{i}{12}[fibre] - \sum_j [s_j(\widehat{\mathbf{H}/\Gamma_\ell})]$$

and

$$[s_j(\widehat{\mathbf{H}/\Gamma_\ell})] \cdot [s_k(\widehat{\mathbf{H}/\Gamma_\ell})] = \begin{cases} -\frac{i}{12} & (j = k) \\ 0 & (j \neq k) \end{cases}.$$

Any intersection points of $\mathcal{A}_d^\sharp$ and $\mathcal{V}_C$ correspond to intersections of zero points of $\tilde{s}_1'$ and those of $\tilde{s}_2'$ in $\mathcal{U}_\ell$. Let (τ_1, τ_2, t) be local coordinates of a local manifold cover over an intersection point p in $\overline{\mathcal{M}}_g$ and (z_1, z_2) be local coordinates of corresponding point in $\mathcal{U}_\ell$. Then we have

$$\begin{aligned}
&\mathcal{A}_D^\sharp : \mathcal{U}_\ell \to \overline{\mathcal{M}}_g \\
&(z_1, z_2) \mapsto [(\tau_1 = \tilde{s}_1'(z_1, z_2), \tau_2 = \tilde{s}_2'(z_1, z_2), t = f(z_1, z_2))]
\end{aligned}$$

for some smooth function $f(z_1, z_2)$.

If C does not contain c_1 nor c_{g-1}, the intersection number of $\mathcal{A}_D$ and $\mathcal{V}_C$ is equal to that of $-\frac{i}{12}[fibre] - \sum[s_j(\widehat{\mathbf{H}/\Gamma_\ell})]$ and itself in $\mathcal{U}_\ell$ since (τ_1, τ_2, t) give coordinates of $\overline{\mathcal{M}}_g$.

$$\begin{aligned}
&\left(-\frac{i}{12}[fibre] - \sum_j [s_j(\widehat{\mathbf{H}/\Gamma_\ell})]\right)^2 \\
&= \frac{i}{6}\sum_j [fibre] \cdot [s_j(\widehat{\mathbf{H}/\Gamma_\ell})] + \sum_j [s_j(\widehat{\mathbf{H}/\Gamma_\ell})]^2 \\
&= \frac{i}{6}\ell^2 - \frac{i}{12}\ell^2 = \frac{i}{12}\ell^2.
\end{aligned}$$

In the same way, we obtain that in the case when exactly one of c_1 and c_{g-1} is contained in C, $[\mathcal{A}_D] \cdot [\mathcal{V}_C] = \frac{i\ell^2}{6}$, and that in the case when both c_1 and c_{g-1} are contained in C, $[\mathcal{A}_D] \cdot [\mathcal{V}_C] = \frac{i\ell^2}{3}$. We can calculate the intersection number of $\mathcal{A}_C$ and $\mathcal{V}_C$ using $c_1(s_j^* T') = -1$, that is

$$[\mathcal{A}_C] \cdot [V_D] = \begin{cases} 2 & (g \geq 4) \\ 4 & (g = 3) \end{cases}$$

For the intersection number of $\mathcal{A}_C$ and $\mathcal{V}_C$, note that their intersection points correspond to (∞, ∞) in $(\widehat{\mathbf{H}/\Gamma_2})^2$, and we have

$$[\mathcal{A}_C] \cdot [\mathcal{V}_C] = \begin{cases} 1 & (\text{if no } c_1, c_{g-1} \text{ is contained in } C) \\ 2 & (\text{if exactly one is contained in } C) \\ 4 & (\text{if both are contained in } C) \end{cases}$$

This completes the proof of Lemma 4.5. $\square$

The $\widehat{\ }$-intersection number for 1-selections are not well-defined generally (for instance, for $[\mathcal{V}_{d_1}]$ and $[\mathcal{V}_{d_g}]$). We define them by induction as follows.

Definition 4.6.

$$[\mathcal{A}_\sigma] \widehat{\cdot} [\mathcal{V}_{d_1}] := \begin{cases} [\mathcal{A}_\sigma] \cdot [\mathcal{V}_{d_1}] & (\sigma \neq d_g) \\ 0 & (\sigma = d_g) \end{cases}$$

$$[\mathcal{A}_\sigma] \widehat{\cdot} [\mathcal{V}_{d_g}] := \begin{cases} 0 & (\sigma \neq d_g) \\ [\mathcal{A}_\sigma] \cdot [\mathcal{V}_{d_g}] & (\sigma = d_g) \end{cases}$$

We define A_{k,d_j} as the $\widehat{\ }$-intersection matrix of all k-selections from $[d_1]$, $[c_1]$, $[d_2]$, $[d_2']$, $[c_2]$,..., $[d_j]$, $[d_j']$ and A_{k,c_j} as that of all k-selections from $[d_1]$, $[c_1]$, $[d_2]$ $[d_2']$, ..., $[c_j]$. We shall prove inductively that all A_{k,d_j} and A_{k,c_j} are non-degenerate.

Lemma 4.7.

A_{1,d_j}, A_{1,c_j} are all non-degenerate.

This is immediately from the consequences of §5 in [W].

Lemma 4.8.

A_{2,d_j}, A_{2,c_j} are all non-degenerate.

Proof If $\sigma = \{[d_1],[c_2]\}$ then the $\hat{\ }$-intersection number of $\mathcal{A}_\sigma$ and $\mathcal{V}_\sigma$ is

$$2[\sum_j s_j(\widehat{\mathbf{H}/\Gamma_\ell})] \cdot [\ell(singular \quad fibres)] = 2i\ell^2$$

and we get

$$A_{2,c_1} = (2i\ell^2)$$

and it is non-degenerate.

Considering $\sigma = \{[d_1],[c_1]\}$ and $\tau = \{[d_2],[d_2']\}$ we obtain

$$A_{2,d_2} = \begin{cases} \begin{pmatrix} 2i\ell^2 & \frac{i\ell^2}{12} \\ 0 & i\ell^2 \end{pmatrix} & (g \geq 4) \\ \begin{pmatrix} 2i\ell^2 & \frac{i\ell^2}{6} \\ 0 & 2i\ell^2 \end{pmatrix} & (g = 3) \end{cases}$$

Moreover considering $\eta_1 = \{[c_1],[c_2]\}$, $\eta_2 = \{[d_1],[c_2]\}$ and applying

Lemma 4.5 we have

$$A_{2,c_2} = \left(\begin{array}{c|c} A_{2,d_2} & * \\ \hline * & A_{1,c_1} \end{array}\right)$$

$$= \begin{cases} \begin{pmatrix} 2i\ell^2 & \frac{i\ell^2}{12} & 0 & 0 \\ 0 & i\ell^2 & \frac{i\ell^2}{6} & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & -\frac{i}{6} & i \end{pmatrix} & (g \geq 4) \\ \begin{pmatrix} 2i\ell^2 & \frac{i\ell^2}{6} & 0 & 0 \\ 0 & 2i\ell^2 & \frac{i\ell^2}{3} & 0 \\ 0 & 4 & 4 & -4 \\ 0 & 0 & -\frac{i}{3} & 2i \end{pmatrix} & (g = 3) \end{cases}$$

It is easy to check that A_{2,d_2} and A_{2,c_2} are all non-degenerate.

We consider in the case $g \geq 4$. Assume that up to A_{2,c_j} the statement holds $(2 \leq j \leq g-2)$. $A_{2,d_{j+1}}$ is given by

$$A_{2,d_{j+1}} = \begin{pmatrix} A_{2,c_j} & 0 \\ 0 & i\ell^2 \end{pmatrix}$$

and this is also non-degenerate.

Next assume that up to A_{2,d_j} the statement holds $(3 \leq j \leq g-1)$.

Using lemma 4.5, we have

$$A_{2,c_j} = \left(\begin{array}{c|c} A_{2,d_j} & * \\ \hline * & A_{1,c_{j-1}} \end{array}\right)$$

$$= \begin{cases} \left(\begin{array}{cc|cc} A_{2,c_{j-1}} & 0 & 0 & 0 \\ 0 & i\ell^2 & \frac{i\ell^2}{12} & 0 \\ \hline 0 & 2 & 1 & * \\ 0 & 0 & 0 & A_{1,d_{j-1}} \end{array}\right) \begin{array}{l} \\ \ldots\sigma \\ \\ \ldots\tau \\ \\ \end{array} & (j \neq g-1) \\[2ex] \left(\begin{array}{cc|cc} A_{2,c_{j-1}} & 0 & 0 & 0 \\ 0 & i\ell^2 & \frac{i\ell^2}{6} & 0 \\ \hline 0 & 2 & 2 & * \\ 0 & 0 & 0 & 2A_{1,d_{j-1}} \end{array}\right) \begin{array}{l} \\ \ldots\sigma \\ \\ \ldots\tau \\ \\ \end{array} & (j = g-1) \end{cases}$$

where $\sigma = \{[d_j], [d'_j]\}$, $\tau = \{[c_{j-1}], [c_j]\}$. This implies that A_{2,c_j} is non-degenerate.

Finally we prove that A_{2,d_g} is non-degenerate. We have two steps for that. Let $\eta = \{[c_{g-1}], [d_g]\}$ and let A_{2,d'_g} be a $\widehat{\ }$-intersection matrix of 2-selections except η. We define an equivalent relation $\sim$ to be generated by all elementary transformations.

$$A_{2,d'_g} = \left(\begin{array}{c|c} A_{2,c_{g-1}} & * \\ \hline * & iA_{1,d_{g-1}} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc|c} \multicolumn{2}{c|}{A_{2,d_{g-1}}} & \begin{matrix} 0 \\ \frac{i\ell^2}{6} \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & * \\ \hline \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 0 \end{matrix} & \multicolumn{2}{c|}{2A_{1,c_{g-2}}} & -2A_{1,c_{g-2}} \\ \hline 0 & 0 & \multicolumn{2}{c|}{-\frac{i}{6}A_{1,c_{g-2}}} & iA_{1,c_{g-2}} \end{array}\right)$$

$$
= \left(\begin{array}{cc|cc|cc}
A_{2,c_{g-2}} & 0 & 0 & 0 & * & * \\
0 & i\ell^2 & \frac{i\ell^2}{6} & 0 & -i\ell^2 & * \\ \hline
\begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 0 \end{matrix} & \multicolumn{2}{c|}{2A_{1,c_{g-2}}} & \multicolumn{2}{c}{-2A_{1,c_{g-2}}} \\ \hline
0 & 0 & \multicolumn{2}{c|}{-\frac{i}{6}A_{1,c_{g-2}}} & \multicolumn{2}{c}{iA_{1,c_{g-2}}}
\end{array}\right)
\begin{matrix} \\ \ldots [d_{g-1}] \\ \ldots [c_{g-2}, c_{g-1}] \\ \\ \end{matrix}
$$

$$
\sim \left(\begin{array}{cc|cc|cc}
A_{2,c_{g-2}} & 0 & * & * & * & * \\
0 & i\ell^2 & 0 & * & -i\ell^2 & * \\ \hline
\begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 0 \end{matrix} & \multicolumn{2}{c|}{\frac{5}{3}A_{1,c_{g-2}}} & \multicolumn{2}{c}{-2A_{1,c_{g-2}}} \\ \hline
0 & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{iA_{1,c_{g-2}}}
\end{array}\right)
$$

$$
= \left(\begin{array}{cc|cc|c}
A_{2,c_{g-2}} & 0 & * & * & \multirow{2}{*}{$*$} \\
0 & i\ell^2 & 0 & * & \\ \hline
0 & 2 & \frac{5}{3} & * & \multirow{2}{*}{$-2A_{1,c_{g-2}}$} \\
0 & 0 & 0 & \frac{5}{3}A_{1,d_{g-2}} & \\ \hline
0 & 0 & 0 & 0 & iA_{1,c_{g-2}}
\end{array}\right)
$$

$$
\sim \left(\begin{array}{cc|c|c}
A_{2,c_{g-2}} & * & \multirow{2}{*}{$*$} & \multirow{2}{*}{$*$} \\
0 & i\ell^2 & & \\ \hline
\multicolumn{2}{c|}{0} & \frac{5}{3}A_{1,c_{g-2}} & * \\ \hline
\multicolumn{2}{c|}{0} & 0 & iA_{1,c_{g-2}}
\end{array}\right)
$$

This matrix is non-degenerate and hence A_{2,d'_g} is also non-degenerate.

A_{2,d_g} is given by

$$
A_{2,d_g} = \begin{pmatrix} A_{2,d'_g} & 0 \\ * & 2i\ell^2 \end{pmatrix}
$$

and A_{2,d_g} is also non-degenerate.

In case $g = 3$, we can prove that all A_{2,d_j}, A_{2,c_j} are non-degenerate in the same way. This completes the proof of Lemma 4.8. $\square$

Lemma 4.9.

A_{k,c_j} and A_{k,d_j} are all non-degenerate for all k, j.

Proof

We already prove in the cases $k = 1, 2$. Assume that the statement holds up to the case $k = p - 1$.

If $\{[d_1], [c_1], [d_2], [d_2'], \ldots, [c_{j-1}]\}$ does not have any p-selections but $\{[d_1], [c_1], [d_2], [d_2'], \ldots, [c_{j-1}], [d_j], [d_j']\}$ has p-selections then

$$A_{p,d_j} = i\ell^2 A_{p-2,d_{j-1}}$$

and hence A_{p,d_j} is non-degenerate.

If $\{[d_1], [c_1], [d_2], [d_2'], \ldots, [d_j], [d_j]\}$ does not have any p-selections but $\{[d_1], [c_1], [d_2], [d_2'], \ldots, [d_j], [d_j'], [c_j]\}$ has p-selections then

$$A_{p,c_j} = A_{p-1,c_{j-1}}$$

and hence A_{p,c_j} is non-degenerate.

When $A_{p,c_{j-1}}$ is non-degenerate A_{p,d_j} is given by

$$A_{p,d_j} = \begin{pmatrix} A_{p,c_{j-1}} & 0 \\ 0 & i\ell^2 A_{p-2,d_{j-1}} \end{pmatrix}$$

and hence A_{p,d_j} is non-degenerate.

When A_{p,d_j} is non-degenerate, A_{p,c_j} is given as follows. If there is no p-selection containing both $[c_{j-1}]$,$[c_j]$ then

$$A_{p,c_j} = \begin{pmatrix} A_{p,d_j} & 0 \\ 0 & A_{p-1,c_j} \end{pmatrix}$$

and is non-degenerate. Otherwise

$$A_{p,c_j} = \begin{pmatrix} A_{p,d_j} & * \\ * & A_{p-1,c_{j-1}} \end{pmatrix}$$

and using lemma 4.5 this matrix is conjugate to

$$\begin{pmatrix} B & 0 & 0 & 0 \\ 0 & i\ell^2 A_{p-2,c_{j-2}} & \frac{i\ell^2}{12} A_{p-2,c_{j-2}} & 0 \\ 0 & 2A_{p-2,c_{j-2}} & A_{p-2,c_{j-2}} & C \\ 0 & 0 & D & A_{p-1,d_{j-1}} \end{pmatrix}$$

where

$$\begin{pmatrix} B & 0 \\ 0 & i\ell^2 A_{p-2,c_{j-2}} \end{pmatrix} = A_{p,d_j}$$

and, from the assumption, B is nondegenerate. By elementary transformations we can transform A_{p,c_j} to

$$\begin{pmatrix} B & 0 & 0 & 0 \\ 0 & A_{p-2,c_{j-2}} & 0 & 0 \\ 0 & 0 & \frac{5}{6} A_{p-2,c_{j-2}} & C \\ 0 & 0 & D & A_{p-1,d_{j-1}} \end{pmatrix}$$

When there does not exist any p-selections containing all of $[c_{j-2}]$, $[c_{j-1}]$, $[c_j]$, we have $C = 0, D = 0$ and hence A_{p,c_j} is non-degenerate. Otherwise,

$$\begin{pmatrix} \frac{5}{6}A_{p-2,c_{j-2}} & C \\ D & A_{p-1,d_{j-1}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{6}A_{p-2,d_{j-2}} & \frac{5}{6}E & 0 & 0 \\ \frac{5}{6}F & \frac{5}{6}A_{p-3,c_{j-3}} & 2A_{p-3,c_{j-3}} & 0 \\ 0 & \frac{i\ell^2}{12}A_{p-3,c_{j-3}} & i\ell^2 A_{p-3,c_{j-3}} & 0 \\ 0 & 0 & 0 & G \end{pmatrix}$$

for some E, F, G. We transform this matrix by elementary transformations and have

$$\begin{pmatrix} \frac{5}{6}A_{p-2,d_{j-2}} & \frac{5}{6}E & 0 & 0 \\ \frac{5}{6}F & \frac{4}{6}A_{p-3,c_{j-3}} & 0 & 0 \\ 0 & 0 & A_{p-3,c_{j-3}} & 0 \\ 0 & 0 & 0 & G \end{pmatrix}$$

Since $A_{p-1,d_{j-1}} = \begin{pmatrix} i\ell^2 A_{p-3,c_{j-3}} & 0 \\ 0 & G \end{pmatrix}$ G, is non-degenerate. Then it is sufficient to show that $\begin{pmatrix} A_{p-2,d_{j-2}} & E \\ F & \frac{4}{5}A_{p-3,c_{j-3}} \end{pmatrix}$ is non-degenerate. When there exists no p-selection containing all of $[c_{j-3}]$, $[c_{j-2}]$, $[c_{j-1}]$, $[c_j]$, we have $E = 0$ and $F = 0$ and hence A_{p,c_j} is non-degenerate.

Otherwise (by a certain numbering)

$$\begin{pmatrix} A_{p-2,d_{j-2}} & E \\ F & \frac{4}{5}A_{p-3,c_{j-3}} \end{pmatrix}$$

$$= \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & i\ell^2 A_{p-4,c_{j-4}} & \frac{i\ell^2}{12}A_{p-4,c_{j-4}} & 0 \\ 0 & 2A_{p-4,c_{j-4}} & \frac{4}{5}A_{p-4,c_{j-4}} & \frac{4}{5}I \\ 0 & 0 & \frac{4}{5}J & \frac{4}{5}A_{p-3,d_{j-3}} \end{pmatrix}$$

$$\sim \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & A_{p-4,c_{j-4}} & 0 & 0 \\ 0 & 0 & \frac{19}{24}A_{p-4,c_{j-4}} & I \\ 0 & 0 & J & A_{p-3,d_{j-3}} \end{pmatrix}$$

for some H, I, J. When there exists no p-selection containing all of $[c_{j-4}]$, $[c_{j-3}]$, $[c_{j-2}]$, $[c_{j-1}]$, $[c_j]$, then we have $I = 0$, $J = 0$ and hence A_{p,c_j} is non-degenerate.

Repeating this step it is sufficient to prove that a sequence $\{a_n\}_{n=1,2,\dots}$ such that

$$a_1 = 1, \qquad a_{n+1} = 1 - \frac{1}{6a_n}$$

does not contain zero. It is easy to check $a_n \neq 0$ for any n and hence A_{p,c_j} is non-degenerate for $j \leq g-1$.

To show A_{p,d_g} is non-degenerate we have two steps. Let A_{p,d'_g} be a $\widehat{\ }$-intersection matrix of p-selections containing not both of $[c_{g-1}]$ and $[d_g]$.

$$A_{p,d'_g} = \begin{pmatrix} A_{p,c_{g-1}} & * \\ * & iA_{p-1,d_{g-1}} \end{pmatrix} =$$

$$
\begin{pmatrix}
A_{p,c_{g-2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i\ell^2 A_{p-2,c_{g-3}} & \frac{i\ell^2}{6} A_{p-2,c_{g-3}} & 0 & -i\ell^2 A_{p-2,c_{g-3}} & 0 & \\
0 & 2A_{p-2,c_{g-3}} & & & & & \\
 & & 2A_{p-1,c_{g-2}} & & -2A_{p-1,c_{g-2}} & & 0 \\
0 & 0 & & & & & \\
 & 0 & -\frac{i}{6} A_{p-1,c_{g-2}} & & iA_{p-1,c_{g-2}} & & 0 \\
 & 0 & 0 & & 0 & & i^2\ell^2 A_{p-3,d_{g-2}}
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
A_{p,c_{g-2}} & & & & \\
 & A_{p-2,c_{g-3}} & & * & \\
 & & \frac{5}{3} A_{p-1,c_{g-2}} & & \\
 & 0 & & iA_{p-1,c_{g-2}} & \\
 & & & & i^2\ell^2 A_{p-3,d_{g-2}}
\end{pmatrix}
$$

and hence A_{p,d'_g} is non-degenerate.

A_{p,d_g} is given by

$$
A_{p,d_g} = \begin{pmatrix} A_{p,d'_g} & 0 \\ * & 2i\ell^2 A_{p-2,c_{g-2}} \end{pmatrix}
$$

and hence it is non-degenerate. This completes the proof of Lemma 4.9.
□

From Lemma 4.4 and Lemma 4.9 we get the following theorem.

Theorem A.

When $k \geq 2$, $b_{2k}(\overline{\mathcal{M}}_g) = b_{6g-6-2k}(\overline{\mathcal{M}}_g) \geq \max(\alpha_{g,k}, \alpha_{3g-3-k})$

Remark When $k = 1$, Harer's result shows the following equality.

$$
b_2(\overline{\mathcal{M}}_g) = b_{6g-8}(\overline{\mathcal{M}}_g) = 2 + [\frac{g}{2}](= \alpha_{g,1} + 1)
$$

§5. Number of distinct k-selections.

In this section we introduce certain algorithm to calculate of $\alpha_{g,k}$, the number of the $2k$-cycles we construct. $\alpha_{g,k}$ is a number of distinct k-selections i.e. distinct k homotopy classes of the free homotopy classes $[c_1],\ldots,[c_{g-1}],[d_1],[d_2],[d_2'],\ldots,[d_{g-1}],[d_{g-1}'],[d_g]$ in Figure 2 satisfying 1),2) in Definition 3.1 modulo the action of homeomorphisms. In other words $\alpha_{g,k}$ is the number of conjugate classes of k-selections. The final goal in this section is proposition B.

Let matrices A, B, C in $M_2(Z[t])$ as follows

$$A = \begin{pmatrix} 1 & 1 \\ t & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ t^2 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ t^4 & 0 \end{pmatrix}.$$

Let polynomials $a_g(t)$, $b_g(t)$, $c_g(t)$, $d_g(t)$, $f_g(t)$ be as follows.

$$\begin{pmatrix} a_g(t) & b_g(t) \\ c_g(t) & d_g(t) \end{pmatrix}$$

$$:= \begin{cases} \frac{1}{2}\{A(AB)^{g-2}A^2 + t^2(AB)^{g-3}A^2 + t^2A(AB)^{g-3}A \\ \quad + t^4(AB)^{g-4}A + B(BC)^{\frac{g}{2}-1}A + t^4(BC)^{\frac{g}{2}-2}A\} & (g : \text{even}) \\ \frac{1}{2}\{A(AB)^{g-2}A^2 + t^2(AB)^{g-3}A^2 + t^2A(AB)^{g-3}A \\ \quad + t^4(AB)^{g-4}A + B(BC)^{\frac{g-3}{2}}B^2 + t^4(BC)^{\frac{g-5}{2}}B^2\} & (g : \text{odd}) \end{cases}$$

$$f_g(t) := a_g(t) + c_g(t)$$

where for a matrix X and a negative integer q we denote X^{-1} by the inverse matrix of X in $M_2(Z(t))$ and define $X^q = (X^{-1})^{(-q)}$. It is easy to see that $\begin{pmatrix} a_g(t) & b_g(t) \\ c_g(t) & d_g(t) \end{pmatrix}$ belongs to $M_2(Z[t])$.

Proposition B.

$$f_g(t) = \sum_k \alpha_{g,k} t^k$$

Now for a matrix $X = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in M_2(Z[t])$ we denote $a(t)+c(t)$ by $f(X)$.

Lemma 5.1.

Forgetting homeomorphism action on S, let $\alpha'_{g,k}$ be a number of k-selections containing not both of $[d_1]$ and $[c_1]$ and containing not both of $[d_g]$ and $[c_{g-1}]$. Then

$$f(A(AB)^{g-2}A^2) = \sum_k \alpha'_{g,k} t^k$$

Proof

For each j, we can prove the next claim ($\sharp$) by induction.

CLAIM ($\sharp$).

(1) Let $p_{j,k}$ and $p'_{j,k}$ be defined by

$$(1,1)A(AB)^{j-1} = (\sum_k p_{j,k} t^k, \sum_k p'_{j,k} t^k) \qquad (1 \leq j \leq g-1)$$

then

$$p_{j,k} = \sharp\{\sigma : k - selection | \{[d_1], [c_1]\} \not\subseteq \sigma \subseteq \{[d_1], [c_1], \dots, [d_j], [d'_j]\}\}$$
$$p'_{j,k} = \sharp\{\sigma : k - selection | \{[d_1], [c_1]\} \not\subseteq \sigma \subseteq \{[d_1], [c_1], \dots, [c_{j-1}]\}\}$$

(2) Let $q_{j,k}$ and $q'_{j,k}$ be defined by

$$(1,1)A(AB)^{j-1}A = (\sum_k q_{j,k}t^k, \sum_k q'_{j,k}t^k) \qquad (1 \leq j \leq g-1)$$

then

$$q_{j,k} = \sharp\{\sigma : k-\textit{selection}|\{[d_1],[c_1]\} \not\subseteq \sigma \subseteq \{[d_1],[c_1],\ldots,[c_j]\}\}$$
$$q'_{j,k} = \sharp\{\sigma : k-\textit{selection}|\{[d_1],[c_1]\} \not\subseteq \sigma \subseteq \{[d_1],[c_1],\ldots,[d_j],[d'_j]\}\}$$

Notice that $\alpha'_{g,k} = q_{g-1,k} + q_{g-1,k-1}$ and

$$\begin{aligned}(1,1)A(AB)^{g-2}A^2 &= (\sum_k q_{g-1,k}t^k, \sum_k q'_{g-1,k}t^k)\begin{pmatrix}1 & 1\\ t & 0\end{pmatrix}\\ &= (\sum_k \alpha'_{g,k}t^k, \sum_k q_{g-1,k}t^k)\end{aligned}$$

and we have

$$f(A(AB)^{g-2}A^2) = \sum_k \alpha'_{g,k}t^k.$$

□

Proof of Proposition B

In the similar way to the previous lemma, we have that the number of all k-selections are represented by the coefficients of

$$f(A(AB)^{g-2}A^2 + t^2(AB)^{g-3}A^2 + t^2A(AB)^{g-3}A + t^4(AB)^{g-4}A)$$

and we have that the number of all symmetric k-selections are represented by the coefficients of

$$\begin{cases} f(B(BC)^{\frac{g}{2}-1}A + t^4(BC)^{\frac{g}{2}-2}A) & (g : \text{even})\\ f(B(BC)^{\frac{g-3}{2}}B^2 + t^4(BC)^{\frac{g-5}{2}}B^2) & (g : \text{odd})\end{cases}$$

and we complete the proof. □

We can easily get the number of k-selections only from $[c_1], \ldots, [c_{g-1}]$, $[d_2]$, $[d_2']$, $\ldots$, $[d_{g-1}]$, $[d_{g-1}']$. In fact we can get the following equality by induction.

$$f((AB)^{g-2}A) = \sum_k \{\binom{g-1}{k} + \sum_{k',l} \binom{k'-1}{l} \cdot \binom{k-2k'+1}{l+1} \cdot \binom{g-l-2}{k-k'}\} t^k.$$

Then we get the next inequality.

$$\alpha_{g,k} > \frac{1}{2}\binom{g-1}{k} + \frac{1}{2}\sum_{k',l} \binom{k'-1}{l} \cdot \binom{k-2k'+1}{l+1} \cdot \binom{g-l-2}{k-k'}$$

Finally some examples of f_g shall be described.

(1) When $g = 3$

$$f_3(t) = 1 + 2t + 5t^2 + 3t^3 + 2t^4$$

and we have

$$b_2 = b_{10} = 3$$
$$b_4 = b_8 \geq 5$$
$$b_6 \geq 3$$

(2) When $g = 4$

$$f_4(t) = 1 + 3t + 7t^2 + 9t^3 + 7t^4 + 3t^5 + t^6$$

and we have

$$b_2 = b_{16} = 4$$
$$b_4 = b_{14} \geq 7$$
$$b_6 = b_{12} \geq 9$$
$$b_8 = b_{10} \geq 7$$

(3) When $g = 5$

$$f_5(t) = 1 + 3t + 11t^2 + 16t^3 + 21t^4 + 13t^5 + 8t^6 + 2t^7 + t^8$$

and we have

$$b_2 = b_{22} = 4 \qquad b_4 = b_{20} \geq 11$$
$$b_6 = b_{18} \geq 16 \qquad b_8 = b_{16} \geq 21$$
$$b_{10} = b_{14} \geq 13 \qquad b_{12} \geq 8$$

(4) When $g = 6$

$$f_6(t) = 1 + 4t + 14t^2 + 29t^3 + 43t^4 + 43t^5 + 31t^6 + 16t^7 + 7t^8 + 2t^9 + t^{10}$$

and we have

$$b_2 = b_{28} = 5 \qquad b_4 = b_{26} \geq 14$$
$$b_6 = b_{24} \geq 29 \qquad b_8 = b_{22} \geq 43$$
$$b_{10} = b_{20} \geq 43 \qquad b_{12} = b_{18} \geq 31$$
$$b_{14} = b_{16} \geq 16$$

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