数理解析研究所講究録
第 808 巻 1992 年 194-208

Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations

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Abstract. We show that the q-difference systems satisfied by Jackson integrals of Jordan-Pochhammer type give a class of the quantum Knizhnik-Zamolodchikov equation for $U_q(\hat{\mathfrak{sl}}_2)$ in the sense of Frenkel and Reshetikhin.

§1. Introduction

One of the most interesting features of the Knizhnik-Zamolodchikov equation originated in conformal field theory is the relation between its connection matrix and the trigonometric solutions of the quantum Yang-Baxter equation [TK],[K],[D]. It is related to the fact that certain hypergeometric type integrals give solutions to the Knizhnik-Zamolodchikov equation [DJMM], [Ma], [Ch], [SV] etc. This fact is also looked at from the viewpoint of the free field realization, e.g. [Ku], [ATY]. Besides them, the structure of the hypergeometric type integrals had been studied, e.g. [A1],[A2]. Recently it attracts attention to construct a q-analogue of these theories.

The Jackson integrals of Jordan-Pochhammer type are the simplest multivariable generalizations of Heine's basic hypergeometric function which is a q-analogue of Gauss' hypergeometric function. They satisfy a system of first order q-difference equations, whose connection problem was solved by Mimachi [Mi]. Recently Aomoto and others [AKM] showed that the connection matrix determined by Mimachi is related to the ABF-solution of the quantum Yang-Baxter equation. On the other hand, Frenkel and Reshetikhin [FR] studied a q-analogue of the chiral vertex operators of the WZNW model, along the line of Tsuchiya and Kanie [TK]. In particular, they introduced a q-difference system called the quantum Knizhnik-Zamolodchikov equation, and discussed the relation of the connection matrix with elliptic solutions of the quantum Yang-Baxter equation. Then it seems possible to understand the result of [AKM] in the framework of Frenkel and Reshetikhin.

In this article, we shall explicitly give solutions to a certain class of the quantum Knizhnik-Zamolodchikov equation for $U_q(\hat{\mathfrak{sl}}_2)$ by Jackson integrals of Jordan-Pochhammer type. More precisely, we show that the q-difference system for the Jackson integrals of Jordan-Pochhammer type is written in terms of trigonometric quantum R-matrix, and that this equation gives a class of the quantum Knizhnik-Zamolodchikov equation. When q goes to 1, our expression of the solutions go to the integral solutions of the Knizhnik-Zamolodchikov equation given by [Ch] in the trigonometric form.

The paper is organized as follows. In sec.2, we write the q-difference equation for Jackson integrals of Jordan-Pochhammer type, whose proof will be given in sec.4. In sec.3, we identify the equation with the quantum Knizhnik-Zamolodchikov equation. In sec.5, we give some comments on the connection problem according to current literatures.

§2. q-difference system for Jackson integrals

Let p be a fixed complex number such as $0 < |p| < 1$. Let us denote

$$(a)_\infty = \prod_{n=0}^{\infty}(1 - ap^n) \tag{2.1}$$

as usual. For a value $s \in \mathbf{C}^*$ and for a function $\phi(t)$, we define

$$\int_0^{s\infty} \phi(t) d_p t = s(1-p) \sum_{n=-\infty}^{\infty} \phi(sp^n)p^n \tag{2.2}$$

whenever it is convergent. This is called the Jackson integral along a q-interval $[0, s\infty]$, which is a q-analogue of the ordinary integration. The q-difference operator T_k is defined by

$$(T_k F)(x_1, \cdots, x_n) = F(x_1, \cdots, px_k, \cdots, x_n) \tag{2.3}$$

for a function $F(x_1, \cdots, x_n)$.

Now consider the Jackson integral of Jordan-Pochhammer type:

$$F_0(x) = \int_0^{s\infty} t^{\beta-1} \prod_{1 \le j \le n} \frac{(t/x_j)_\infty}{(p^{\beta_j} t/x_j)_\infty} d_p t \tag{2.4}$$

where β_j are complex parameters and $x = (x_1, \cdots, x_n)$ is a variable in $(\mathbf{C}^\times)^n$. We are interested in the q-difference system associated with F_0. Take the set of functions $(F_1, \cdots, F_n)$ defined by

$$F_i(x) = \int_0^{s\infty} \Phi_i d_p t \tag{2.5}$$

where

$$\Phi_i = t^{\beta-1} \frac{\prod_{j=1}^{i}(pt/x_j)_\infty \prod_{j=i+1}^{n}(t/x_j)_\infty}{\prod_{j=1}^{i-1}(p^{\beta_j+1}t/x_j)_\infty \prod_{j=i}^{n}(p^{\beta_j}t/x_j)_\infty}. \tag{2.6}$$

Let us calculate the q-difference system satisfied by F_i. We set

$$x_{ij} = \begin{cases} x_i/x_j & \text{if } i < j, \\ 1 & \text{if } i = j, \\ px_i/x_j & \text{if } i > j. \end{cases} \tag{2.7}$$

Then the result is summarized as the following proposition.

Proposition 1. *We define the $n \times n$ matrix A_k with entries a_{ij}^k as follows.*
(2.8) *If $i = j \neq k$ then*

$$a_{ij}^k = \frac{x_{ki} - 1}{x_{ki} - p^{\beta_k}}.$$

(2.9) *If $i < j \leq k$ or $k \leq i < j$ then*

$$a_{ij}^k = \frac{(1 - p^{\beta_i})x_{ki}}{x_{ki} - p^{\beta_k}} \frac{1 - p^{\beta_k}}{x_{kj} - p^{\beta_k}} \prod_{l=i+1}^{j-1} \frac{p^{\beta_l}x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}}.$$

(2.10) *If $j \leq k \leq i$ then*

$$a_{ij}^k = p^\beta \frac{1 - p^{\beta_k}}{x_{kj} - p^{\beta_k}} \frac{(1 - p^{\beta_i})x_{ki}}{x_{ki} - p^{\beta_k}} \prod_{l=1}^{j-1} \frac{p^{\beta_l}x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}} \prod_{l=i+1}^{n} \frac{p^{\beta_l}x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}}.$$

(2.11) *Otherwise $a_{ij}^k = 0$.*
Then we have

$$(T_k F_1, \cdots, T_k F_n) = (F_1, \cdots, F_n) A_k. \tag{2.12}$$

Remark. For each i,j $(i \neq j)$, let $S_{i,j}$ denote the $n \times n$-matrix defined by

$$(2.13) \qquad \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \frac{p^{\beta_j} x_{ij} - p^{\beta_i}}{x_{ij} - p^{\beta_i}} & & & \frac{1-p^{\beta_i}}{x_{ij}-p^{\beta_i}} & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & \frac{(1-p^{\beta_j})x_{ij}}{x_{ij}-p^{\beta_i}} & & & \frac{x_{ij}-1}{x_{ij}-p^{\beta_i}} & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ i\text{-th} \\ \\ \\ \\ j\text{-th} \\ \\ \\ \\ \end{matrix} .$$

$$\qquad\qquad\qquad i\text{-th} \qquad\qquad\qquad j\text{-th}$$

We also consider the $n \times n$-matrix P_k defined by

$$(2.14) \qquad \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & p^{\beta} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ k\text{-th} \\ \\ \\ \\ \end{matrix} .$$

Then, by an explicit calculation, we see

$$(2.15) \qquad A_i = S_{k,k+1} \cdots S_{k,n} P_k S_{k,1} \cdots S_{k,k-1}.$$

Matrices $S_{i,j}$ form a set of unitary quantum R-matrices. Namely we have

$$(2.16) \qquad S_{i,j}(T_i S_{j,i}) = id, \quad \text{and} \quad S_{1,2}S_{2,3}S_{1,3} = S_{1,3}S_{2,3}S_{1,2}.$$

Finally, let us discuss the relation among $F_0, \cdots, F_n$.

Proposition 2. *We put $\beta_0 = -\beta - (\beta_1 + \cdots + \beta_n)$. Then the following relation holds:*

$$(2.17) \qquad \sum_{i=0}^{n} p^{\beta_{i+1}+\cdots+\beta_n}(1 - p^{\beta_i})F_i = 0.$$

Therefore F_0 is recovered from $F_1, \cdots, F_n$ if $p^{\beta_0} \neq 1$.

Remark. The identity (2.17) is a q-analogue of Aomoto's linear relation in the sense of [A2] and [DJMM].

§3. Comparison with the quantum Knizhnik-Zamolodchikov equations.

Let us briefly review the quantum enveloping algebra and the trigonometric R-matrix in the case of $\hat{\mathfrak{sl}}_2$. The quantum enveloping algebra $\hat{U}_q = U_q(\hat{\mathfrak{sl}}_2)$ is defined as an algebra with the generators:

$$X_0^{\pm},\ X_1^{\pm},\ K_0^{\pm 1},\ K_1^{\pm 1} \tag{3.1}$$

and the relations:

$$\begin{aligned} K_0K_1 &= K_1K_0,\ K_0K_0^{-1} = K_1K_1^{-1} = 1,\\ K_iX_i^{\pm}K_i^{-1} &= q^{\pm 2}X_i^{\pm},\ K_iX_j^{\pm}K_i^{-1} = q^{\mp 2}X_j^{\pm}\ (i\neq j),\\ [X_i^+,\ X_j^-] &= \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}},\\ (X_i^{\pm})^3X_j^{\pm} &- (q^2+1+q^{-2})(X_i^{\pm})^2X_j^{\pm}X_i^{\pm}\\ &+(q^2+1+q^{-2})X_i^{\pm}X_j^{\pm}(X_i^{\pm})^2 - X_j^{\pm}(X_i^{\pm})^3 = 0\ (i\neq j). \end{aligned} \tag{3.2}$$

Here, q denotes a general complex parameter. The comultiplication $\Delta : \hat{U}_q \longrightarrow \hat{U}_q \otimes \hat{U}_q$ is defined by

$$\begin{aligned} \Delta(X_i^+) &= X_i^+ \otimes 1 + K_i^{-1}\otimes X_i^+,\\ \Delta(X_i^-) &= X_i^- \otimes K_i + 1\otimes X_i^-,\ \Delta(K_i) = K_i\otimes K_i. \end{aligned} \tag{3.3}$$

We put $\Delta' = \sigma\circ\Delta$ where $\sigma(a\otimes b) = b\otimes a$ in $\hat{U}_q\otimes\hat{U}_q$. Next we consider the subalgebra $U_q = U_q(\mathfrak{sl}_2)$ generated by $X^{\pm} = X_1^{\pm}, K^{\pm} = K_1^{\pm}$. For each $x\in\mathbf{C}$, we define the algebra homomorphism $\varphi_x : \hat{U}_q \longrightarrow U_q$ by

$$\begin{aligned} \varphi_x(X_0^{\pm}) &= x^{\pm 1}X^{\mp},\ \varphi_x(X_1^{\pm}) = X^{\pm},\\ \varphi_x(K_0) &= K^{-1},\ \varphi_x(K_1) = K. \end{aligned} \tag{3.4}$$

Let (V_i, π_i) be representations of U_q with the highest weights λ_i. Then $(V_i(x), \hat{\pi}_i) = (V_i, \pi_i\circ\varphi_x)$ gives a representation of $\hat{U}_q$ for each $x\in\mathbf{C}$. The operator

$$R_{V_iV_j}(x)\ :\ V_i(x)\otimes V_j(1)\longrightarrow V_i(x)\otimes V_j(1) \tag{3.5}$$

such that

$$\Delta'(a)R_{V_iV_j}(x) = R_{V_iV_j}(x)\Delta(a),\ a\in\hat{U}_q.$$

gives a trigonometric R-matrix. Let v_i be the highest weight vector in V_i. We fix a choice of normalization such that

$$R_{V_iV_j}(x)\, v_i \otimes v_j = v_i \otimes v_j. \tag{3.6}$$

Then $R_{V_iV_j}(x)$ acts as

$$\begin{aligned} R_{V_iV_j}(x)\, X^- v_i \otimes v_j &= \frac{xq^{m_j} - q^{m_i}}{x - q^{m_i+m_j}} X^- v_i \otimes v_j + \frac{1 - q^{2m_j}}{x - q^{m_i+m_j}} v_i \otimes X^- v_j, \\ R_{V_iV_j}(x)\, v_i \otimes X^- v_j &= \frac{x(1 - q^{2m_i})}{x - q^{m_i+m_j}} X^- v_i \otimes v_j + \frac{xq^{m_i} - q^{m_j}}{x - q^{m_i+m_j}} v_i \otimes X^- v_j. \end{aligned} \tag{3.7}$$

Here $m_i = (\lambda_i, \alpha)$, α is the simple root.

Let $\lambda_1, \cdots, \lambda_n, \lambda$ be a set of weights. Let V_i be the irreducible representation of U_q with the highest weight λ_i and the highest weight vector v_i. Let ν be a complex parameter and put $p^\nu = q$. We set $\rho = \alpha/2$, the half sum of the positive roots. For a weight μ, we denote $(q^\mu)_k$ the action of q^μ on the k-th component of the tensor product $V_1 \otimes \cdots \otimes V_n$. For instance,

$$q^\mu(v_k) = q^{(\mu, \lambda_k)} v_k, \quad q^\mu(X^- v_k) = q^{(\mu, \lambda_k - \alpha)} X^- v_k. \tag{3.8}$$

The quantum Knizhnik-Zamolodchikov equation introduced by Frenkel and Reshetikhin [FR] is written as the following system of q-difference equations:

$$\begin{aligned} T_k \mathcal{F} = R_{V_kV_{k-1}}&(px_k/x_{k-1}) \cdots R_{V_kV_1}(px_k/x_1)\, (q^{\lambda+2\rho})_k \\ &q^{-(\lambda,\lambda_k)} R_{V_{k+1}V_k}(x_{k+1}/x_k)^{-1} \cdots R_{V_nV_k}(x_n/x_k)^{-1} \mathcal{F}, \\ k = 1, \cdots&, n, \end{aligned} \tag{3.9}$$

where $\mathcal{F} = \mathcal{F}(x_1, \cdots, x_n)$ is a function valued in $V_1 \otimes \cdots \otimes V_n$.

Let us compare the equations (2.12) and (3.9). Take the weights $\lambda_0, \lambda_\infty$ such that

$$\begin{aligned} \lambda_0 + \cdots + \lambda_n - \lambda_\infty &= \alpha, \\ \lambda_0 + \lambda_\infty &= \lambda, \end{aligned} \tag{3.10}$$

and put the parameters as:

$$\begin{aligned} \beta &= -2(\lambda_\infty + \alpha, \alpha)\nu, \\ \beta_i &= 2(\lambda_i, \alpha)\nu. \end{aligned} \tag{3.11}$$

We set

$$\varphi_i(x_1,\cdots,x_n)=p^{(\beta_{i+1}+\cdots+\beta_n)/2}x_1^{\beta_1}\cdots x_n^{\beta_n}F_i(p^{\beta_1/2}x_1,\cdots,p^{\beta_n/2}x_n), \tag{3.12}$$

for each $i=1,\cdots,n$, and define the $V_1\otimes\cdots\otimes V_n$-valued function $\mathcal{F}$ by

$$\mathcal{F}=\sum_{i=1}^{n}\varphi_i(x_1,\cdots,x_n)\,v_1\otimes\cdots\otimes X^-v_i\otimes\cdots\otimes v_n. \tag{3.13}$$

Then, by rewriting the equation (2.12) in terms of $\mathcal{F}$, we have

Theorem 3. *The system* (2.12) *is equivalent to the restriction of the system* (3.9) *to the weight subspace with the weight* $\lambda_1+\cdots+\lambda_n-\alpha$, *and the function* $\mathcal{F}$ *defined by* (3.13) *is a solution of* (3.9).

Remark. When q goes to 1, $\mathcal{F}$ defined by (3.13) goes to a special case of the integral solutions to the Knizhnik-Zamolodchikov equation obtained by Cherednik [Ch] in the trigonometric form.

We shall give another description of the equation. Let $\lambda_0,\cdots,\lambda_n,\lambda_\infty$ be a set of weights such that

$$\lambda_0+\cdots+\lambda_n-\lambda_\infty=\alpha. \tag{3.14}$$

Let V_i be the irreducible representation of U_q with the highest weight λ_i and the highest weight vector v_i. The quantum Knizhnik-Zamolodchikov equation for a $\mathrm{Hom}_{U_q}(V_\infty, V_0\otimes\cdots\otimes V_n)$-valued function $\mathcal{F}$ is written as:

$$\begin{aligned}T_k\mathcal{F}=&R_{V_kV_{k-1}}(px_k/x_{k-1})\cdots R_{V_kV_1}(px_k/x_1)R_{V_kV_0}(0)\,(q^{2\rho})_k\\&R_{V_\infty^*V_k}(0)^{-1}R_{V_{k+1}V_k}(x_{k+1}/x_k)^{-1}\cdots R_{V_nV_k}(x_n/x_k)^{-1}\mathcal{F}.\end{aligned} \tag{3.15}$$

Here we understand $\mathcal{F}$ as an element of $V_0\otimes\cdots\otimes V_n\otimes V_\infty^*$. Next we consider the set $\mathcal{H}(V_0\otimes\cdots\otimes V_n\,;\,\lambda_\infty)$ of highest weight vectors in $V_0\otimes\cdots\otimes V_n$ with the weight λ_∞. We have an injection

$$\mathrm{Hom}_{U_q}(V_\infty, V_0\otimes\cdots\otimes V_n)\longrightarrow\mathcal{H}(V_0\otimes\cdots\otimes V_n\,;\,\lambda_\infty) \tag{3.16}$$

by evaluating the highest weight vector v_∞. Then the equation (3.15) is regarded as a restriction of the following system:

$$T_k\mathcal{F} = R_{V_kV_{k-1}}(px_k/x_{k-1})\cdots R_{V_kV_1}(px_k/x_1)R_{V_kV_0}(0)\,(q^{\lambda_\infty+2\rho})_k \\ q^{-(\lambda_\infty,\lambda_k)}R_{V_{k+1}V_k}(x_{k+1}/x_k)^{-1}\cdots R_{V_nV_k}(x_n/x_k)^{-1}\mathcal{F}, \tag{3.17}$$

where $\mathcal{F}$ is a $\mathcal{H}(V_0\otimes\cdots\otimes V_n\,;\,\lambda_\infty)$-valued function.

Remarks. (1) If all V_i are the Verma modules or are the finite dimensional modules, then the linear map (3.16) is surjective, and the system (3.15) is same as (3.17).

(2) If $q^{2(\lambda_0,\alpha)}\neq 1$, then the system (3.17) is same as the restriction of the system (3.9) to the weight subspace with the weight $\lambda_1+\cdots+\lambda_n-\alpha$, hence is equivalent to the system (2.12).

We define the $V_0\otimes\cdots\otimes V_n$-valued function $\mathcal{F}$ by

$$\mathcal{F} = \sum_{i=0}^{n}\varphi_i(x_1,\cdots,x_n)\,v_0\otimes\cdots\otimes X^-v_i\otimes\cdots\otimes v_n, \tag{3.18}$$

where φ_i is defined by (3.12) for each $i=0,\cdots,n$. Then, by interpreting the identity (2.17), we have

$$X^+\mathcal{F}=0. \tag{3.19}$$

Therefore $\mathcal{F}$ is one of the highest weight vectors in $V_0\otimes\cdots\otimes V_n$ with the weight λ_∞. Thus we finally obtain:

Theorem 4. *The $\mathcal{H}_{\lambda_\infty}(V_0\otimes\cdots\otimes V_n)$-valued function $\mathcal{F}$ defined by* (3.18) *is a solution of the quantum Knizhnik-Zamolodchikov equation* (3.17).

Notes. (1) In the situation of [FR], V_0 and V_∞ are integrable $\hat{U}_q$-modules and $V_1,\cdots,V_n$ are finite dimensional $\hat{U}_q$-modules, and ν corresponds to $\frac{1}{2(k+g)}$, where k is the fixed level and g is the dual coxeter number. Moreover the quantum Knizhnik-Zamolodchikov equation for the correlation function is written in terms of the image of the universal R-matrix, which differs from our $R_{V_iV_j}$ by a certain scalar factor.

(2) For $n=2$, our expressions of solutions to (3.9) coincide with those given in [FR, sec.7].

§4. Proof of Propositions.

We write $\phi_1(t) \sim \phi_2(t)$ if

$$\int_0^{s\infty} \phi_1(t)d_pt = \int_0^{s\infty} \phi_2(t)d_pt \tag{4.1}$$

holds for any $s \in \mathbf{C}^*$. For example, we have

$$\Phi_i(t) \sim p^\beta \Phi_i(pt). \tag{4.2}$$

Proof of Proposition 1. The following is obvious from the definition:

$$T_kF_i = \int_0^{s\infty} T_k\Phi_i(t)d_pt. \tag{4.3}$$

Therefore the q-difference system (2.12) is equivalent to

$$T_k\Phi_j(t) \sim \sum_{i=1}^{n} a_{ij}^k \Phi_i(t). \tag{4.4}$$

Now, because of (4.2), the following lemma is enough to prove the proposition.

Lemma 5.

(a) *For $j < k$, we have*

$$p^\beta T_k\Phi_j(pt) = p^\beta \sum_{i=1}^{j} a_{ij}^k \Phi_i(pt) + \sum_{i=k}^{n} a_{ij}^k \Phi_i(t).$$

(b) *For $j = k$, we have*

$$p^\beta T_k\Phi_j(pt) = p^\beta \sum_{i=1}^{j-1} a_{ij}^k \Phi_i(pt) + \sum_{i=j}^{n} a_{ij}^k \Phi_i(t).$$

(c) *For $k < j$, we have*

$$T_k\Phi_j(t) = \sum_{i=k}^{j} a_{ij}^k \Phi_i(t).$$

Proof. Since all the cases are treated in a similar way, we will exhibit detailed calculations only for the most difficult case (b). We put $a_{ij} = a^k_{ij}$ for simplicity. Multiplied by appropriate factors, (b) is equivalent to

$$
\begin{aligned}
p^\beta x_j \prod_{l=1}^{j-1}(p^{\beta_l}pt - x_l) \prod_{l=j+1}^{n} (p^{\beta_l}t - x_l) \\
= p^\beta \sum_{i=1}^{j-1} a_{ij} x_i \prod_{l=1}^{i-1}(p^{\beta_l}pt - x_l) \prod_{l=i+1}^{j-1} (pt - x_l) \prod_{l=j}^{n}(p^{\beta_l}t - x_l) \\
+ \sum_{i=j}^{n} a_{ij} x_i \prod_{l=1}^{j-1}(pt - x_l) \prod_{l=j}^{i-1}(p^{\beta_l}t - x_l) \prod_{l=i+1}^{n} (t - x_l).
\end{aligned}
\tag{4.5}
$$

Since both sides are polynomials of degree $n-1$ with respect to t, it suffices to check the equality at n different values of t. Putting $t = x_m/p$, $m \le j-1$, in (4.5), we have

$$
px_j \prod_{l=m}^{j-1}(p^{\beta_l}x_m - x_l) - \sum_{i=m}^{j-1} a_{ij} x_i (p^{\beta_j}x_m - px_j) \prod_{l=m}^{i-1}(p^{\beta_l}x_m - x_l) \prod_{l=i+1}^{j-1} (x_m - x_l) = 0. \tag{4.6}
$$

We put $t = x_j/p^{\beta_j}$, then we have

$$
\begin{aligned}
p^\beta \prod_{l=1}^{j-1}(p^{\beta_l}px_j - p^{\beta_j}x_l) \prod_{l=j+1}^{n} (p^{\beta_l}x_j - p^{\beta_j}x_l) \\
= a_{jj} \prod_{l=1}^{j-1}(px_j - p^{\beta_j}x_l) \prod_{l=j+1}^{n} (x_j - p^{\beta_j}x_l).
\end{aligned}
\tag{4.7}
$$

We finally put $t = x_m/p^{\beta_m}$, $j+1 \le m$, then we have

$$
\sum_{i=j}^{m} a_{ij} x_i \prod_{l=j}^{i-1}(p^{\beta_l}x_m - p^{\beta_m}x_l) \prod_{l=i+1}^{n} (x_m - p^{\beta_m}x_l) = 0. \tag{4.8}
$$

Now let us consider the explicit values of a_{ij} defined by (2.8)-(2.10). Substitute the values of a_{ij} in the left of (4.6) inductively as $i = j-1, j-2, \cdots, N$. Then we have

$$
\begin{aligned}
px_j \prod_{l=N}^{j-1} \frac{p^{\beta_l}px_j - p^{\beta_j}x_l}{px_j - p^{\beta_j}x_l} \prod_{l=m}^{N-1} (p^{\beta_l}x_m - x_l) \prod_{l=N}^{j-1}(x_m - x_l) \\
- \sum_{i=m}^{N} a_{ij} x_i (p^{\beta_j}x_m - px_j) \prod_{l=m}^{i-1}(p^{\beta_l}x_m - x_l) \prod_{l=i+1}^{j-1} (x_m - x_l).
\end{aligned}
$$

When $N = m$, this is zero and (4.6) is verified. (4.7) follows easily from (2.10). To verify (4.8), it suffices to substitute the values of $a_{ij}, i = j, j+1, \cdots, N$ inductively. Hence (4.5) is shown and the proof of (b) is completed. *Q.E.D.*

Proof of Proposition 2. By the relation (4.2), it suffices to show the following lemma.

Lemma 6. *We have the following relation:*

$$p^{\beta_1+\cdots+\beta_n}\Phi(t) - \Phi(pt) = \sum_{i=1}^{n} p^{\beta_{i+1}+\cdots+\beta_n}(p^{\beta_i} - 1)\Phi_i. \tag{4.9}$$

Proof. Multiplied by an appropriate factor, (4.9) is equivalent to

$$\begin{aligned} &p^{\beta_1+\cdots+\beta_n}\prod_{j=1}^{n}(1 - t/x_j) - \prod_{j=1}^{n}(1 - p^{\beta_j}t/x_j) \\ &= \sum_{i=1}^{n} p^{\beta_{i+1}+\cdots+\beta_n}(p^{\beta_i} - 1)\prod_{j=1}^{i-1}(1 - p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1 - t/x_j). \end{aligned} \tag{4.10}$$

The right becomes

$$\begin{aligned} &\sum_{i=1}^{n} p^{\beta_i+\cdots+\beta_n}\prod_{j=1}^{i-1}(1 - p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1 - t/x_j) \\ &\quad \sum_{i=1}^{n} p^{\beta_{i+1}+\cdots+\beta_n}\prod_{j=1}^{i-1}(1 - p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1 - t/x_j) \\ &= \sum_{i=1}^{n} p^{\beta_i+\cdots+\beta_n}\prod_{j=1}^{i-1}(1 - p^{\beta_j}t/x_j)\prod_{j=i}^{n}(1 - t/x_j) \\ &\quad \sum_{i=1}^{n} p^{\beta_{i+1}+\cdots+\beta_n}\prod_{j=1}^{i}(1 - p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1 - t/x_j), \end{aligned}$$

which yields the left of (4.10). *Q.E.D.*

§5. Discussions.

In this paper, we have constructed a Jackson integral representations of solutions to the quantum Knizhnik-Zamolodchikov equation in the simplest case for $U_q(\hat{\mathfrak{sl}}_2)$. Let us briefly review the results of [AKM] and [FR], and discuss the relation of our result and the connection problem of q-difference equations.

Let $F_i' = F_i'(x_1, \cdots, x_n)$ be the function defined by

$$F_i' = \int_0^{s\infty} \frac{t^{\beta-1}}{1-t/x_i} \frac{\prod_{j=1}^n (t/x_j)_\infty}{\prod_{j=1}^n (p^{\beta_j} t/x_j)_\infty} d_q t.$$

Consider the system satisfied by F_i':

$$(T_k F_1', \cdots, T_k F_n') = (F_1', \cdots, F_n') A_k'. \tag{5.1}$$

The asymptotic behavior in

$$\{(x_1, \cdots, x_n)\,;\, |x_{\sigma(1)}| \gg \cdots \gg |x_{\sigma(n)}| \gg 1\}$$

characterizes the fundamental solution $\Xi_\sigma = \Xi_\sigma(x_1, \cdots, x_n)$ for a permutation $\sigma \in \mathfrak{S}_n$. Let e be the identity in $\mathfrak{S}_n$. In the sense of [M], the elementary connection matrix P_i is defined by $\Xi_{\sigma_i} = P_i \Xi_e$ for a transposition $\sigma_i = (i, i+1) \in \mathfrak{S}_n$. Then it is shown in [AKM], for $\beta_1 = \cdots = \beta_n$, that P_i depends only on the ratio x_i/x_{i+1} and satisfies the Yang-Baxter equation:

$$P_i(u) P_{i+1}(uv) P_i(v) = P_{i+1}(v) P_i(uv) P_{i+1}(u).$$

This is equivalent to the Boltzman weights of the eight vertex SOS model, i.e., the ABF-solution of the star-triangle relation (*cf.* [ABF],[JMO]).

On the other hand, Frenkel and Reshetikhin [FR] studied a q-deformed chiral vertex operator along the line of [TK], for a quantum affine algebra $U_q(\hat{\mathfrak{g}})$. They showed that the correlation function satisfies the quantum Knizhnik-Zamolodchikov equation, which is written in terms of the universal R-matrix, and considered the connection matrix as a q-analogue of the braiding matrix in conformal field theory. In some situations, they proved that the connection matrix of the quantum Knizhnik-Zamolodchikov equation for a simple transposition depends only on the ratio of two arguments and it satisfies the quantum Yang-Baxter equation. The most remarkable point of their theory is the factorization property, from which it is possible to determine the connection matrix by computing it for $n = 2$, namely by considering the 4-point function as in the discussion of [TK]. Using this argument and considering Jackson integral solutions for $n = 2$, they calculated the connection matrix in the simplest case for $U_q(\hat{\mathfrak{sl}}_2)$ which includes the ABF-solution [FR, sec.7]. Therefore the connection matrix of the quantum Knizhnik-Zamolodchikov equation for a special case coincides with that of [AKM].

Now our equation (2.12) for the function F_i defined by (2.5) is obviously equivalent to the equation (5.1). In fact, F_i and F_i' are related to each other by a triangular matrix:

$$F_i = \sum_{j=1}^{i} b_{ij} F_j'.$$

The explicit form is given by

$$b_{ij} = \prod_{k=1}^{i} b_{ij}^k, \qquad b_{ij}^k = \begin{cases} \dfrac{p^{\beta_j} x_j - x_k}{x_j - x_k} & (\text{if } k < i) \\ \dfrac{(p^{\beta_i} - 1)x_i}{x_j - x_i} & (\text{if } k = i). \end{cases}$$

Since theorem 3 says that the equation (2.12) is equivalent to the quantum Knizhnik-Zamolodchikov equation (3.9), we have seen the coincidence above explicitly at the level of the q-difference equation before going to the connection matrix. Finally, combined with the discussions in [FR], the results in the present paper enable us to observe the surprising phenomenon revealed by [AKM], that a very rich structure is contained in such a simple expression:

$$\int_0^{s\infty} t^{\beta-1} \prod_{1 \le j \le n} \frac{(t/x_j)_\infty}{(p^{\beta_j} t/x_j)_\infty} d_q t,$$

from the viewpoint of the representation theory of quantum enveloping algebra $U_q(\hat{\mathfrak{sl}}_2)$.

Acknowledgements.

The author is grateful to K. Mimachi, M. Jimbo and A. Tsuchiya for useful discussions and comments.

References.

[A1] K. Aomoto, "On the structure of integrals of power product of linear functions", Sci. Papers. Coll. Gen. Ed. Univ. Tokyo 27 (1977) 49-61.

[A2] K. Aomoto, "Gauss-Manin connection of integrals of difference products", J. Math . Soc. Jpn. 39 (1987) 191-208.

[A3] K. Aomoto, "Finiteness of a cohomology associated with certain Jackson integrals", Tohoku Math. J. 43 (1991) 75-101.

[ABF] G.E. Andrews, R.J. Baxter, P.J. Forrester, "Eight-vertex SOS model and generalized Rogers-Ramanujan-type identities", Jour. Stat. Phys.

[AKM] K. Aomoto, Y. Kato, K. Mimachi, "A solution of Yang-Baxter equation as connection coefficients of a holonomic q-difference system", Preprint.

[ATY] H. Awata, A. Tsuchiya, Y. Yamada, "Integral formulas for the WZNW correlation functions", Preprint KEK-TH-286 (1991).

[Ch] I.V. Cherednik, "Integral solutions to the trigonometric Knizhnik-Zamolodchikov equations", Preprint RIMS-699, to appear in Publ. RIMS.

[DJMM] E. Date, M. Jimbo, A. Matsuo, T. Miwa, "Hypergeometric type integrals and the $\mathfrak{sl}(2, \mathbf{C})$ Knizhnik-Zamolodchikov equations", In: Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory, World Scientific.

[D] V.G. Drinfeld, "Quasi-Hopf algebras", Algebras and Analysis, 1 (1990) 30-46.

[FR] I.B. Frenkel, N.Yu. Reshetikhin, "Quantum affine algebras and holonomic difference equations", Preprint.

[JMO] M. Jimbo, T. Miwa, M. Okado, "Solvable lattice models related to the vector representation of classical simple Lie algebra,", Comm. Math. Phys. 116 (1988) 507-525.

[KZ] V.G. Knizhnik, A.B. Zamolodchikov, "Current algebra and Wess-Zumino models in two dimensions", Nucl. Phys. B247 (1984) 83-103.

[Ko] T. Kohno, "Monodromy representations of braid groups and Yang-Baxter equations", Ann. Inst. Fourier, Grenouble 37 (1987) 139-160.

[Ku] G. Kuroki, "Fock space representations of affine Lie algebras and integral representations in the Wess-Zumino-Witten model", to appear in Comm. Math. Phys.

[Ma] A. Matsuo, "An application of Aomoto-Gelfand hypergeometric functions to SU(n) Knizhnik-Zamolodchikov equation", Comm. Math. Phys. 134 (1990) 1049-1057.

[Mi] K. Mimachi, "Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan-Pochhammer type", Nagoya Math. J. 116 (1989) 149-161.

[SV] V.V. Schechtman, A.N. Varchenko, "Arrangement of hyperplanes and Lie algebra homology", Invent. Math. 106 (1991) 139-194.

[TK] A. Tsuchiya, Y. Kanie, "Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representations of braid groups", Adv. Stud. Pure. Math 16 (1988) 297-372.