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Stable mappings with trivial monodromies and application to inactive log-transformations

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0. Introduction.

(0.0) In this article we state two main results, based on the same idea. In the first one (Theorem B), we classify the diffeo-types of closed, oriented, smooth four-manifolds which are domains of certain stable maps $f : M \to \mathbf{R}^2$. In the second one (Theorem C), we give examples of such maps $h : M^4 \to S^2$ as follows; if the two log-transformations at distinct torus fibres of h preserve the homeo-type of the domain, then they preserve the diffeo-type also, for any pair of multiplicities. Such log-transformations are called inactive by Viro [Vi]. It is a remarkable contrast that the same log-transformations often provide exotic manifolds for some pairs of multiplicities ([FM],[Theorem 1, Vi]). Our examples contain the Viro's and certain types of good torus fibrations of Y. Matsumoto [Mt].

To prove these results, we use a result (Theorem A) on removing a certain connected component of the singular set $S(f)$ or $S(h)$ by performing a surgery. By this, Theorem B and C are reduced to the cases that $S(f)$ and $S(h)$ consist of two, and one connected components, respectively. To prove Theorem C in the reduced case, we shall construct a stable map g from the manifold obtained from M by the two log-transformations, into $\mathbf{R}^2$, and to g we shall apply Theorem B.

(0.1) Here we recall some definitions and notations given in [Kb]. Let $f : M^4 \to P^2$ be a stable map where $P^2 = \mathbf{R}^2$ or S^2. By $S(f)$, we mean the singular set of f, the set of points in M where the Jacobian is not surjective. Note that $S(f)$ consists of smooth closed curves in M. Let $q_f : M \to W_f$ be the quotient map which collapses each connected component of f-fibres to a point. In the case that the Euler numbre $\chi(M)$ is even, we call a stable map $f : M \to \mathbf{R}^2$ *simple* if (1) W_f is homeomorphic to D^2, a closed 2-disc, and (2) $q_f|S(f)$ is an embedding. The symbol $g_f \leq 1$ means that each regular fibre of q_f is a torus T^2 or a sphere S^2. Let R be a connected component of $W_f \backslash q_f(S(f))$. We say R is a *0-region* if the regular fibre over a point in R is a sphere, and a *1-region* if it is a torus. For a simple map $f : M \to \mathbf{R}^2$ with $g_f \leq 1$, we say f is *configuration trivial* if there is no 1-region inside of any 1-region.

1. Simple maps with trivial monodromies.

(1.1) Let $f : M \to \mathbf{R}^2$ be a simple map with $g_f \leq 1$. Note that $\partial W_f \subset q_f(S(f))$ and that the q_f-preimage of a thin collar neighbourhood of ∂W_f is a trivial D^3-bundle over ∂W_f. For other connected components C_i of $q_f(S(f))$, the q_f-preimages are $\mathbf{T}'$-bundles over C_i's where $\mathbf{T}'$ is a solid torus with an open 3-disc removed. Therefore we can regard M as some T^2-bundles, S^2-bundles, $\mathbf{T}'$-bundles, and a $D^3 \times S^1$ pasted together along their boundaries. We call the isomorphism on $H_1(\partial_T \mathbf{T}', \mathbf{Z})$ induced by C_i the *monodromy of* q_f *over* C_i where $\partial_T \mathbf{T}'$ is the torus in $\partial \mathbf{T}'$. The monodromies over C_i's determine the bundle structures of the rest (see Proposition 3.2, [Kb]).

(1.2) By changing the glueings, compatible with the restrictions of q_f to each boundary, one obtains another simple stable map $f' : M' \to \mathbf{R}^2$ with $g_f \leq 1$. Now assume that all monodromies are trivial. Then such glueings are ample, hence there are many right-equivalent classes of the pair (M, f) with f a simple map of $g_f \leq 1$. To the contrast, we get the following result, which states that the diffeo-types of the domains are strictly restricted.

THEOREM B. *Let* $f : M \to \mathbf{R}^2$ *be a simple map with* $g_f \leq 1$. *Assume that* f *is configuration trivial and has trivial monodromies. Then* M *is diffeomorphic to either* (a) $L(a_1)\sharp \ldots \sharp L(a_n) \sharp l(S^3 \times S^1) \sharp m(S^2 \times S^2)$ or, (b) $L(a_1)\sharp \ldots \sharp L(a_n) \sharp l(S^3 \times S^1) \sharp m(S^2 \tilde{\times} S^2)$. *Here* l, m *are non negative integers, and for an integer* a_i, $L(a_i)$ *means the Pao's manifold defined in [Pa].*

Conversely, the manifolds listed above admit such maps f*'s.*

Remark ([Pa]). $L(1) = S^4$ and $L(0) = S^3 \times S^1 \sharp S^2 \times S^2$.

2. Inactive log-transformations.

(2.1) Let $h : M \to S^2$ be a stable map with the following conditions.

(1) $Im(h) = S^2$.

(2) Each regular fibre of h is a torus or a sphere.

(Note that (1) and (2) implies $q_h = h$)

(3) $S(h)$ is not empty and $h|S(h)$ is an embedding.

(4) h has a unique 1-region R.

(5) The monodromies over $h(S(h))$ are trivial.

In addition, assume that (6) h has a smooth cross-section, in the case of $\chi(M) = 2$.

(2.2) In the following, we shall define a C^∞-log-transformation. For a pair of co-prime integers (p, r) with $p \neq 0$, let $\Pi_{p,r} : S^1 \times S^1 \times D^2 \to D^2$ be a map defined by $\Pi_{p,r}(x, z, w) = z^p \cdot w^r$ where the second factor S^1 is regarded as $\{z \in \mathbf{C} |\ |z| = 1\}$, and the third factor D^2 is regarded as $\{w \in \mathbf{C} |\ |w| \leq 1\}$. Note that $\Pi_{p,r}$ has a multiple torus fibre over $0 \in D^2$ with multiplicity $|p|$.

Let $T = h^{-1}(a)$ be a regular torus fibre of h and $a \in D$ a closed 2-disc in $S^2 \backslash h(S(h))$. Take an essential closed curve d in T, co-prime integers p, q, and a matrix

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbf{Z}).$$

We define a pair (M', h') as,

$$(M', h') = (\overline{M \backslash h^{-1}(D)}, h|\ ) \cup_\varphi (S^1 \times S^1 \times D^2, \Pi_{p,r})$$

where $\varphi : \partial \overline{M \backslash h^{-1}(D)} \to \partial(S^1 \times S^1 \times D^2)$ is a diffeomorphism given in (2.3).

Definition. The *log-transformation along* T, *of type* (p, q), *with direction* d is the operation that changes (M, h) to (M', h'). We call p the *multiplicity*, and q the *sub-multiplicity*.

(2.3) To describe the glueing φ, we fix essential simple closed curves c', d' and e in $h^{-1}(\partial D)$. Take a path λ connecting a point $b \in \partial D$ and a. Let d' be a curve in $h^{-1}(b)$ which is isotopic to d in $h^{-1}(\lambda)$. Let c' be a curve in $h^{-1}(b)$ with which d' spuns $H_1(h^{-1}(b), \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$. Take a cross section $\tilde{D}$ of h over D which passes through $c' \cap d'$. Let $e = \partial\tilde{D}$.

Then φ is a diffeomorphism which induces an isomorphism between the first homology groups with $\mathbf{Z}$ coefficients, of the form $\varphi_* = 1 \oplus A$ where $H_1(h^{-1}(\partial D))$ and $H_1(\partial(S^1 \times S^1 \times D^2))$ are identified with $\mathbf{Z}^3$ with respect to the basis $< c', d', e >$ and the canonical basis, respectively.

Note that (M', h') is independent of the choice of r and c', that is, for another pair (M'', h'') derived from another r and c', one can show that M' and M'' are diffeomorphic, and h' and h'' are right-equivalent via the diffeomorphism.

(2.4) Let T_a and T_b be two torus fibres of h, and let $M(p_1, q_1; p_2, q_2)$ be the manifold obtained from M by a log-transformation of type (p_1, q_1) along T_a, and another one of type (p_2, q_2) along T_b. The directions d_1, d_2 of the log-transformations were taken as follows.
(1) d_1 and d_2 are homotopic in $h^{-1}(\lambda)$ where λ is a path in R connecting $h(T_a)$ and $h(T_b)$.
(2) d_1 is not a meridian with respect to some (C, J), namely, does not bound a disc in $h^{-1}(J) = S^1 \times D^2 \backslash Int D^3$ where C is a curve in $h(S(h))$ and J a path starting $h(T_a)$ and meeting C transversely at a single point.

THEOREM C. *$M(p_1, q_1; p_2, q_2)$ is diffeomorphic to either (a) $L(a)\sharp k(S^2 \times S^2)$, (a)' $L'(a)\sharp k(S^2 \times S^2)$, a is even, or (b) $L(a)\sharp k(S^2 \tilde{\times} S^2)$. Here k is a non-negative integer and $L'(a)$ is the other Pao's manifold defined in [Pa].*

Remark ([Pa]). $L'(a)\sharp S^2 \tilde{\times} S^2 \cong L(a)\sharp S^2 \tilde{\times} S^2, L(0) = S^3 \times S^1 \sharp S^2 \times S^2$, and $L'(0) = S^3 \times S^1 \sharp S^2 \tilde{\times} S^2$.

Note that for the manifolds listed above, homeo-types and diffeo-types coincide, which follows from the facts that $\pi_1(L(a)) = \pi_1(L'(a)) = \mathbf{Z}_a$, that (a) and (c) are spin and others are not spin, and that the intersection form of (a)' is even ([Iw]). Note also that M is diffeomorphic to one of the manifolds listed above, since $M = M(1, 0; 1, 0)$. It turns out that the log-transformations are inactive.

(2.5) It is shown that $\chi(M) = 2$, our M is S^4. This implies that M with any Euler numbre is simply connected. On S^4, one can construct h directly from a simple map $g : S^4 = L(1) \to \mathbf{R}^2$ (which is the procedure converse to the one mentioned in (4.3)). It is shown that $k(S^2 \times S^2)$ and $k(S^2 \tilde{\times} S^2)$ admit our h.

(2.6) With respect to Viro's inactive log-transformations, (which is performed for certain two tori in $S^2 \times S^2$ and which he defines without maps), we can show that the tori are regular fibres of our h. It is checked that his direction coincides with ours. Theorem C together with this gives another proof to Theorem 3 in [Vi].

3. Stable change of "twin" fibres.

(3.1) The map h has a deep connection with good torus fibrations. Let $C \subset S(h)$ be a connected component, D a closed 2-disc in $S^2 \backslash h(S(h))$ which contains C in its interior. Let N' be a fibred tubular neighbourhood of a twin fibre of multiplicities $(1,1)$, and $\varphi : N' \to D^2$ be the torus fibration (see [Iw]).

LEMMA. *(1) There is a diffeomorphism $\phi : h^{-1}(D) \to N'$.*
(2) $(\varphi \circ \phi)|\partial h^{-1}(D)$ and $h|\partial h^{-1}(D)$ are right-equivalent.

By this, in a neighbourhood of a twin fibre of this type, we can replace the torus fibration with a stable map, preserving the map of the outside.

(3.2) Let $\phi : M \to S^2$ be a good torus fibration with at least one singular fibre. Assume that the singular fibres are of type I_1^+, I_1^- and that the signature $\sigma(M) = 0$. Then one can deform φ to $\varphi' : M \to S^2$, a good torus fibration with twin singular fibres of multiplicities $(1,1)$ [Mt]. Therefore we get the corollaries below.

COROLLARY. *Let $\varphi : M^4 \to S^2$ be a good torus fibration mentioned above. Assume also that $\chi(M) > 2$. Then the log-transformations performed along whose two distinct regular fibres are inactive.*

COROLLARY. *S^4 admits a good torus fibration mentioned above such that the log-transformations performed along whose two distinct regular fibres are inactive.*

Note that $L(a)$ admit the map h exept for the additional condition (6). Hence they admit the torus fibrations mentioned above. It is an open problem whether such torus fibrations on $L(a)$ have an active log-transformation or not.

Torus fibrations with twin fibres are studied in [Iw]. One can see the complete list of the domain manifolds of such fibrations there.

4. Proofs of Theorem B,C, outlines.

(4.1) Here we state a result which is the base of previous two theorems. Let $f : M^4 \to P^2$ be a stable map into any oriented, connected surface, possibly non-compact, possibly with boundary. Let $C \subset S(f)$ be a connected component with the four conditions.

(1) $q_f|C$ is an embedding.

(2) $q_f(C)$ separates one 1-region and one 0-region.

(3) The 0-region bounded by $q_f(C)$ is an open 2-disc.

(4) The monodromy over $q_f(C)$ is trivial.

THEOREM A. *There is an embedded 2-sphere S in M containing C, and a stable map $f' : M' \to P^2$ such that (M', f') is obtained from (M, f) by a surgery detaching S, namely, M' is obtained from M by the surgery and f' on $M'\backslash\nu(s)$ and f on $M\backslash\nu(S)$ coincide via the natural identification $M'\backslash\nu(s) = M\backslash\nu(S)$, and that $q_f(S(f')) = q'_f(S(f))\backslash C$, where s is the attaching circle and where $\nu(s), \nu(S)$ are tubular neighbourhoods.*

(4.2) Applying this to the pair (M, f) of Theorem B, one obtains a sequence of surgeries

$$(M, f) = (M_k, f_k) \to (M_{k-1}, f_{k-1}) \to \cdots \to (M_2, f_2).$$

Here the index is taken to indicate the numbre of connected components of $S(f_j)$. The terminal domain M_2 is seen to be $S^3 \times S^1$. Thus (M, f) is obtained from $(S^3 \times S^1, f_2)$ by surgeries detaching $(k-2)$ simple closed curves in $S^3 \times S^1$. This, with some detailed discussion, implies the theorem.

(4.3) Let (M, h) be the pair of Theorem C. It is obtained from (M_1, h_1) by a sequence of surgeries, in the same way as (4.2). Since the log-transformations are compatible with the surgeries, (M', h'), the pair after performing the log-transformations to (M, h), is obtained from (M_1', h_1') by a sequence of surgeries. We construct a simple map $g' : M_1' \to \mathbf{R}^2$ with $g_{g'} \leq 1$ and with trivial monodromies, which is the crucial point of the proof. Applying Theorem B, we get $M_1' = L(a)$. The effect of log-transformations appears as the change of the glueings of the bundle decomposition mentioned in $(1, 2)$. Theorem C follows in the same way as Theorem B.

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