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Kirillov-Kostant theory and Feynman path integrals on coadjoint orbits of a certain real semisimple Lie group

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0 Introduction

Alekseev, Faddeev and Shatashvili showed in [1] that any irreducible unitary representation of compact groups can be obtained by path integrals. They computed characters of the representations. We showed in [3] that path integrals give unitary operators of the representation which is constructed by Kirillov-Kostant theory for some Lie groups.

In [4] we found that, in order to compute the path integrals with nontrivial Hamiltonians for $SU(1,1)$ and $SU(2)$ to obtain unitary operators realized by Borel-Weil theory, we have to regularize the Hamiltonian functions, and in [5] we extended the results to the case that the maximal compact subgroup K of a connected semisimple Lie group G has equal rank to the complex rank of G.

In the rest of this section we shall show how the path integral reproduces the representation constructed by Kirillov-Kostant theory in the case of $SL(2,\mathbb{R})$ with real polarization. This was done in [3].

Let

$$G = SL(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ; \ ad - bc = 1 \right\}$$

$$\mathfrak{g} = sl(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ; \ a + d = 0 \right\}$$

Since the bilinear form $\langle , \rangle$ on $\mathfrak{g}$ given by $\langle X, Y \rangle = \mathrm{tr} XY$ is nondegenerate, the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is identified with $\mathfrak{g}$.

For a nonzero real number σ, we put $\lambda = \begin{pmatrix} \sigma/2 & 0 \\ 0 & -\sigma/2 \end{pmatrix} \in \mathfrak{g}^*$ and put $\mathcal{H}_\lambda = L_2(\mathbb{R})$. We define a representation $(U_\lambda, \mathcal{H}_\lambda)$ of G as follows:

$$U_\lambda(g)F(x) = |-cx + a|^{-(\sqrt{-1}\sigma + 1)} F\left(\frac{dx - b}{-cx + a} \right)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $F \in \mathcal{H}_\lambda$.

We can obtain this representation by path integrals as we shall show below.

We introduce local coordinates on G by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} \pm e^{v} & 0 \\ 0 & \pm e^{-v} \end{pmatrix}.$$

Note that such elements forms an open subset of G which is also dense.

Then define a 1-form φ by

$$\varphi = \langle \lambda, g^{-1}dg \rangle = \sigma(udx + dv).$$

Since dv is exact 1-form, we choose $\alpha = \sigma u dx$ and put $p = \sigma u$. The p is called momentum in quantum mechanics. Define a function $H(g : Y)$ for $Y \in \mathfrak{g}$, which we call *Hamiltonian function*, by

$$\begin{aligned} H(g:Y) &= \langle \mathrm{Ad}^*(g)\lambda, Y \rangle \\ &= \begin{cases} a(\sigma + 2px) & \text{if } \ Y = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \\ bp & \text{if } \ Y = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \\ -c(\sigma x + px^2) & \text{if } \ Y = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \end{cases} \end{aligned}$$

where Ad^* denotes the coadjoint action of G on $\mathfrak{g}^*$.

The path integral we should compute is given, symbolically, by

$$\int \mathcal{D}(x,p) \exp\left(\sqrt{-1} \int_0^T \gamma^* \alpha - H(g:Y)dt \right),$$

where γ denotes the paths in the phase space given below.

We divide the time interval $[0,T]$ into N small intervals $[\frac{k-1}{N}T, \frac{k}{N}T]$ $(k = 1, \cdots, N)$ and fix $x_0(= x'), x_1, \cdots, x_{N-1}, x_N(= x'')$ and $p_0, p_1, \cdots, p_{N-1}$ arbitrarily. Then we consider the following paths:

$$\begin{aligned} x(0) &= x', \quad x(T) = x'' \\ x(t) &= x_{k-1} + \left(t - \frac{k-1}{N}T \right) \left(\frac{x_k - x_{k-1}}{T/N} \right) \\ p(t) &= p_{k-1} \end{aligned}$$

for $t \in [\frac{k-1}{N}T, \frac{k}{N}T]$.

Furthermore, corresponding to a quantization of the Hamiltonian functions, we take the following ordering of the Hamiltonians: On each interval $[\frac{k-1}{N}T, \frac{k}{N}T]$, we replace $H(g:Y)$ by

$$H_k(g:Y) = \begin{cases} a(\sigma + p_{k-1}(x_k + x_{k-1})) & \text{if } \; Y = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \\ bp_{k-1} & \text{if } \; Y = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \\ -c(\sigma x_{k-1} + p_{k-1}x_{k-1}x_k) & \text{if } \; Y = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \end{cases}$$

For each generator $Y \in \mathfrak{g}$, we compute

$$K_Y(x'', x' : T) = \lim_{N\to\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^{N-1} dx_j \prod_{j=0}^{N-1} \frac{dp_j}{2\pi}$$
$$\exp\sqrt{-1}\left\{\sum_{k=1}^{N} p_{k-1}(x_k - x_{k-1}) - H_k(g:Y)\frac{T}{N}\right\}.$$

Then we have

$$\int_{\mathbb{R}} K_Y(x'', x' : T)F(x')dx' = (U_\lambda(\exp TY)F)(x'')$$

for each generator $Y \in \mathfrak{g}$ and $F \in \mathcal{H}_\lambda$.

Now we take another polarization and construct, following Kirillov-Kostant theory, another unitary representation which is known to be equivalent to the one given above.

Put $\mathcal{H}_{\tilde{\lambda}} = L^2(\mathbb{R})$. Then the representation $(U_{\tilde{\lambda}}, \mathcal{H}_{\tilde{\lambda}})$ is given by

$$U_{\tilde{\lambda}}(g)F(y) = |-by + d|^{\sqrt{-1}\sigma - 1} F\left(\frac{ay - c}{-by + d}\right)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $F \in \mathcal{H}_{\tilde{\lambda}}$.

Corresponding to the second polarization, we introduce local coordinates on G by

$$g = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pm e^s & 0 \\ 0 & \pm e^{-s} \end{pmatrix}.$$

Then the 1-form φ is, in this parametrization, given by

$$\varphi = \sigma(-wdy + ds).$$

Since ds is exact 1-form, we choose $\tilde{\alpha} = -\sigma w dy$ and put $p' = \sigma w$.

Then, proceeding analogously to the argument above, we can show that the path integrals give the kernel functions $\widetilde{K}_Y(y'', y' : T)$ of the unitary operators $U_{\tilde{\lambda}}(\exp TY)$ for each generator $Y \in \mathfrak{g}$.

Now consider the difference of the two 1-forms:

$$\tilde{\alpha} - \alpha = \sigma d \log |1 - xy|.$$

Therefore

$$\begin{aligned}&\int_0^T \tilde{\gamma}^* \tilde{\alpha} - H(g : Y)dt - \int_0^T \gamma^* \alpha - H(g : Y)dt \\ =&\sigma(\log |1 - x''y''| - \log |1 - x'y'|),\end{aligned}$$

which implies that

$$\begin{aligned}&\int_0^T \tilde{\gamma}^* \tilde{\alpha} - H(g : Y)dt + \sigma \log |1 - x'y'| \\ =&\sigma \log |1 - x''y''| + \int_0^T \gamma^* \alpha - H(g : Y)dt.\end{aligned}$$

Suggested by this, consider an integral operator with kernel function

$$e^{\sqrt{-1}\sigma \log |1-xy|} = |1 - xy|^{\sigma}.$$

But this operator does not commute with the unitary operators $U_\lambda(g)$ and $U_{\tilde{\lambda}}(g)$ ($g \in G$), so we modify the kernel function by multiplying $|1 - xy|^{-1}$. Then the following integral operator A gives a formal intertwining operator between $(U_\lambda, \mathcal{H}_\lambda)$ and $(U_{\tilde{\lambda}}, \mathcal{H}_{\tilde{\lambda}})$ [9][10]:

$$(AF)(y) = \int_{\mathbb{R}} |1 - xy|^{\sqrt{-1}\sigma - 1} F(x)dx$$

for $F \in \mathcal{H}_\lambda$.

We shall give a slight generalization of this in the following.

1 Kirillov-Kostant theory

Let G be a linear connected noncompact semisimple Lie group, $\mathfrak{g}$ its Lie algebra. We fix a Cartan involution θ of $\mathfrak{g}$ and denote the Cartan involution of G corresponding to that of $\mathfrak{g}$, also by θ. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the corresponding Cartan decomposition, B the Killing form on $\mathfrak{g}$. Since B is nondegenerate, the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is identified with $\mathfrak{g}$ by

$$\mathfrak{g}^* \ni \nu \leftrightarrow X_\nu \in \mathfrak{g}, \tag{1.1}$$

where

$$B(X_\nu, X) = \nu(X) \quad \text{for all } X \in \mathfrak{g}.$$

We also use the notation $\langle \nu,\, X\rangle$ for $\nu(X)$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, Σ the corresponding set of nonzero restricted roots, and $\mathfrak{m}$ the centralizer $Z_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$. Fix a Weyl chamber in $\mathfrak{a}$ and let Σ^+ denote the corresponding set of positive restricted roots. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g}\,;[H,X] = \alpha(H)X \text{ for } H \in \mathfrak{a}\}.$$

Define

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \quad \text{and} \quad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha,$$

where $m_\alpha = \dim \mathfrak{g}_\alpha$.

Let K, A, N be the analytic subgroups corresponding to $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$, respectively, and M the centralizer $Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K. Then $NMA\overline{N}$ is an open subset of G whose complement is of lower dimension and has Haar measure 0, where $\overline{N} = \theta N$.

For any element $\nu \in \mathfrak{a}^*$ we denote by H_ν the element of $\mathfrak{a}$ such that

$$B(H,\, H_\nu) = \nu(H) \quad \text{for all } H \in \mathfrak{a}. \tag{1.2}$$

We extend any linear form ν on $\mathfrak{a}$ to a linear form on $\mathfrak{g}$ by defining ν to vanish on the orthogonal complement of $\mathfrak{a}$ with respect to the Killing form.

Let λ be an element of $\mathfrak{a}^*$ which corresponds to a regular element of $\mathfrak{a}$ by (1.2). We denote the coadjoint action of G on $\mathfrak{g}^*$ by Ad^*. Then it is easy to see that the isotropy subgroup

$$G_\lambda = \{g \in G\,;\mathrm{Ad}^*(g)\lambda = \lambda\}$$

at λ equals MA, and its Lie algebra $\mathfrak{g}_\lambda$ equals $\mathfrak{m} \oplus \mathfrak{a}$. As a real polarization we take $\mathfrak{s}_- = \mathfrak{m} \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}$, where $\overline{\mathfrak{n}} = \theta\mathfrak{n}$. Correspondingly, we put $S_- = MA\overline{N}$.

Then the Lie algebra homomorphism

$$-\sqrt{-1}\lambda : \mathfrak{s}_- \longrightarrow \sqrt{-1}\mathbb{R}, \quad X_0 + H + X_- \longmapsto -\sqrt{-1}\lambda(H)$$

lifts to the unitary character of S_-:

$$S_- \longrightarrow U(1), \qquad m \exp H\, \overline{n} \longmapsto e^{-\sqrt{-1}\lambda(H)}.$$

We define a one-dimensional representation ξ_λ of S_- by

$$\xi_\lambda \; : S_- \longrightarrow \mathbb{C}^\times, \qquad m \exp H\, \overline{n} \longmapsto e^{-(\sqrt{-1}\lambda+\rho)(H)}.$$

Let L_λ be the C^∞-line bundle over G/S_- associated to the one-dimensional representation ξ_λ of S_-. Then we can identify the space of all C^∞-sections of L_λ with

$$C^\infty(L_\lambda) = \left\{f \in C^\infty(G); f(xb) = \xi_\lambda(b)^{-1} f(x), x \in G, b \in S_-\right\}.$$

For any $f \in C^\infty(L_\lambda)$ we put

$$\|f\|^2 = \int_K |f(k)|^2 dk,$$

where dk is a Haar measure on K. Let V_λ be the completion of $C^\infty(L_\lambda)$ with respect to the norm. For $g \in G, f \in C^\infty(L_\lambda)$ and $x \in G$, we define

$$\pi_\lambda(g) f(x) = f(g^{-1}x).$$

Then π_λ can be uniquely extended to a unitary operator on V_λ, which we also denote by π_λ.

For each $\alpha \in \Sigma^+$ we can find nonzero root vectors $E_{\alpha,i} \in \mathfrak{g}_\alpha$ $(i = 1, \cdots, m_\alpha)$ such that

$$B(E_{\alpha,i}, \theta E_{\alpha,j}) = -\delta_{ij},$$

where δ_{ij} is Kronecker's delta. Put $E_{-\alpha,i} = -\theta E_{\alpha,i}$ and introduce differentiable coordinates on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ as follows:

$$\mathbb{R}^m \longrightarrow \mathfrak{n}, \quad x = (x_{\alpha,i})_{\alpha\in\Sigma^+, i=1,\cdots,m_\alpha} \longmapsto \sum_{\alpha\in\Sigma^+} \sum_{i=1}^{m_\alpha} x_{\alpha,i} E_{\alpha,i}$$

$$\mathbb{R}^m \longrightarrow \overline{\mathfrak{n}}, \quad y = (y_{\alpha,i})_{\alpha\in\Sigma^+, i=1,\cdots,m_\alpha} \longmapsto \sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha} y_{\alpha,i}E_{-\alpha,i},$$

where $m = \dim \mathfrak{n}$. Put

$$n_x = \exp \sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha} x_{\alpha,i}E_{\alpha,i} \in N \tag{1.3}$$

$$\overline{n}_y = \exp \sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha} y_{\alpha,i}E_{-\alpha,i} \in \overline{N}. \tag{1.4}$$

We define a map L of $C^\infty(L_\lambda)$ into $C^\infty(N)$ by

$$Lf(n) = f(n) \quad \text{for} \quad f \in C^\infty(L_\lambda). \tag{1.5}$$

Then, defining a norm on $C^\infty(N)$ with respect to a Haar measure on N, one can show that

$$\|f\|^2 = \|Lf\|^2,$$

when the Haar measures are suitably normalized.

Let $\mathcal{H}_\lambda$ be the completion of the image of $C^\infty(L_\lambda)$ by L. Then one can show that L is extended to an isometry of V_λ onto $\mathcal{H}_\lambda$. Define a representation $(U_\lambda, \mathcal{H}_\lambda)$ of G such that the following diagram commutes for any $g \in G$:

$$\begin{array}{ccc} V_\lambda & \xrightarrow{L} & \mathcal{H}_\lambda \\ {\scriptstyle \pi_\lambda(g)}\downarrow & & \downarrow{\scriptstyle U_\lambda(g)} \\ V_\lambda & \xrightarrow{L} & \mathcal{H}_\lambda. \end{array}$$

For $g \in NMA\overline{N}$, we write as

$$g = n(g)m(g)a(g)\overline{n}(g). \tag{1.6}$$

Then

$$U_\lambda(g)F(x) = e^{(\sqrt{-1}\lambda+\rho)\log a(g^{-1}n_x)}F(n(g^{-1}n_x)) \tag{1.7}$$

for $F \in L(C^\infty(L_\lambda))$.

2 Quantization

We retain the notation of §1. Moreover, for $x = (x_{\alpha,i})_{\alpha\in\Sigma+, i=1\cdots m_\alpha}$, we put

$$X = \sum_{\alpha\in\Sigma+}\sum_{i=1}^{m_\alpha} x_{\alpha,i}E_{\alpha,i}. \tag{2.1}$$

In this section we compute the differential representation dU_λ of U_λ and quantize the Hamiltonian functions for $Y \in \mathfrak{m}\oplus\mathfrak{a}$ or $\mathfrak{n}$.

We decompose $\mathrm{Ad}(e^{-X})Y$ as

$$\mathrm{Ad}(e^{-X})Y = X_+ + X_0 + H + X_- \tag{2.2}$$

with $X_+ \in \mathfrak{n}, X_0 \in \mathfrak{m}, H \in \mathfrak{a}$ and $X_- \in \overline{\mathfrak{n}}$.

Then, for $Y \in \mathfrak{g}$ and $F \in C_c^\infty(N)$, $dU_\lambda(Y)$ is given by

$$\begin{aligned} dU_\lambda(Y)F(x) = &- \left(\sqrt{-1}\left\langle\lambda, \mathrm{Ad}(n_x)^{-1}Y\right\rangle + \left\langle\rho, \mathrm{Ad}(n_x)^{-1}Y\right\rangle\right) F(x) \\ &- \sum_{\alpha\in\Sigma+}\sum_{i=1}^{m_\alpha} c_{\alpha,i}\,\partial_{\alpha,i}F(x), \end{aligned} \tag{2.3}$$

where $x = (x_{\alpha,i}), n_x = \exp X, \partial_{\alpha,i} = \partial/\partial x_{\alpha,i}$ and

$$c_{\alpha,i} = B\left(\frac{\mathrm{ad}X}{1-e^{-\mathrm{ad}X}}X_+,\ E_{-\alpha,i}\right).$$

Define a 1-form φ by

$$\begin{aligned} \varphi &= \left\langle\lambda, g^{-1}dg\right\rangle \\ &= \left\langle\mathrm{Ad}^*(\overline{n}))\lambda, n(g)^{-1}dn(g)\right\rangle + \left\langle\lambda, a(g)^{-1}da(g)\right\rangle, \end{aligned}$$

where d is the exterior derivative on G and $\overline{n} = m(g)a(g)\overline{n}(g)(m(g)a(g))^{-1}$. Since the second term is an exact 1-form, we choose

$$\alpha_{\mathfrak{s}_-} = \left\langle\mathrm{Ad}^*(\overline{n})\lambda,\ n(g)^{-1}dn(g)\right\rangle.$$

and parametrize $n(g)$ as $n(g) = \exp X$, where X is of the form (2.1). Let

$$p_{\alpha,i} = \alpha_{\mathfrak{s}_-}(\partial_{\alpha,i})$$

i.e. $p_{\alpha,i}$ is the coefficient of $dx_{\alpha,i}$ in $\alpha_{\mathfrak{s}_-}$: $\alpha_{\mathfrak{s}_-} = \sum_{\alpha,i} p_{\alpha,i}dx_{\alpha,i}$. Then $p_{\alpha,i}$ is given by

$$p_{\alpha,i} = B\left(\frac{e^{\mathrm{ad}X}-1}{\mathrm{ad}\,X}\mathrm{Ad}(\overline{n})H_\lambda, E_{\alpha,i}\right).$$

Using $c_{\alpha,i}$ and $p_{\alpha,i}$, we have, for $Y \in \mathfrak{g}$,

$$H(g:Y) = \langle \lambda, \mathrm{Ad}(n_x)^{-1}Y \rangle + \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} c_{\alpha,i} p_{\alpha,i}, \tag{2.4}$$

where $g \in NMA\overline{N}$ and $n(g) = n_x = \exp X$.

Now, using (2.4), we quantize the Hamiltonian function for $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$, replacing $x_{\alpha,i}$ and $\sqrt{-1}p_{\alpha,i}$ in $H(g:Y)$ by $x_{\alpha,i}\times$ (multiplication operator) and $\partial_{\alpha,i}$, respectively, (canonical quantization !) and choosing an operator ordering between $x_{\alpha,i}\times$'s and $\partial_{\alpha,i}$'s.

PROPOSITION 2.1. *For $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$, we define quantized Hamiltonians $\mathbf{H}(Y)$ as follows :*

(i) For $Y \in \mathfrak{m} \oplus \mathfrak{a}$,

$$\mathbf{H}(Y) = \langle \lambda, Y \rangle - \frac{\sqrt{-1}}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} \{c_{\alpha,i}\partial_{\alpha,i} + \partial_{\alpha,i} \circ c_{\alpha,i}\};$$

(ii) For $Y \in \mathfrak{n}$,

$$\mathbf{H}(Y) = -\sqrt{-1} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} \partial_{\alpha,i} \circ c_{\alpha,i},$$

where $\circ$ denotes the composition of operators. Then the quantized Hamiltonian coincides with $\sqrt{-1}dU_\lambda(Y)$.

Remark. If $Y \in \mathfrak{n}$, since $\partial_{\alpha,i} c_{\alpha,i} = 0$, we obtain

$$\begin{aligned} \mathbf{H}(Y) &= -\sqrt{-1} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} c_{\alpha,i}\partial_{\alpha,i} \\ &= -\frac{\sqrt{-1}}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} \{c_{\alpha,i}\partial_{\alpha,i} + \partial_{\alpha,i} \circ c_{\alpha,i}\}. \end{aligned}$$

But we do not adopt these quantizations in the present paper.

3 Path integrals

In this section we show that the path integrals with Hamiltonian functions with $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$ give the kernel function of the unitary operator constructed in §1. For detail, we refer the reader to [6].

The path integral is, symbolically, given by

$$\int \mathcal{D}(x,p) \exp\left(\sqrt{-1}\int_0^T \gamma^* \alpha_{s_-} - H(g:Y)dt\right)$$

for $Y \in \mathfrak{g}$, where γ denotes certain paths in the phase space [3].

Here we divide the time interval $[0,T]$ into N small intervals

$$\left[\frac{k-1}{N}T, \frac{k}{N}T\right] \quad (k = 1, \cdots, N).$$

On each small interval $[\frac{k-1}{N}T, \frac{k}{N}T]$, Proposition 2.1 indicates that we should take the following ordering of Hamiltonian functions $H_k(g:Y)$ with $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$.

(i) For $Y \in \mathfrak{m} \oplus \mathfrak{a}$,

$$H_k(g:Y) = \langle \lambda, Y \rangle + \frac{1}{2} \sum_{\alpha \in \Sigma+} \sum_{i=1}^{m_\alpha} (c^k_{\alpha,i} p^{k-1}_{\alpha,i} + p^{k-1}_{\alpha,i} c^{k-1}_{\alpha,i}),$$

where $c^k_{\alpha,i} = \alpha(Y) x^k_{\alpha,i}$.

(ii) For $Y \in \mathfrak{n}$,

$$H_k(g:Y) = \sum_{\alpha \in \Sigma+} \sum_{i=1}^{m_\alpha} p^{k-1}_{\alpha,i} c^{k-1}_{\alpha,i},$$

where

$$c^k_{\alpha,i} = B\left(\frac{\mathrm{ad}X^k}{e^{\mathrm{ad}X^k} - 1} Y, E_{-\alpha,i}\right)$$

and $X^k = \sum_{\alpha,i} x^k_{\alpha,i} E_{\alpha,i}$.

Now the computation of the path integral.

For $x = (x_{\alpha,i}), x' = (x'_{\alpha,i})$ given, let $x^0_{\alpha,i} = x_{\alpha,i}$, $x^N_{\alpha,i} = x'_{\alpha,i}$. We put

$$dx^j = \prod_{\alpha \in \Sigma+} \prod_{i=1}^{m_\alpha} dx^j_{\alpha,i} \text{ and } dp^j = \frac{1}{(2\pi)^m} \prod_{\alpha \in \Sigma+} \prod_{i=1}^{m_\alpha} dp^j_{\alpha,i}$$

for brevity, where $m = \dim \mathfrak{n}$. Remark that the Haar measure dx on N equals the Haar measure dn given in §1, up to constant multiple.

A. Path integral for $Y \in \mathfrak{m} \oplus \mathfrak{a}$

Recall that if $Y \in \mathfrak{m} \oplus \mathfrak{a}$, then $H_k(g : Y)$ is given by

$$H_k(g : Y) = \langle \lambda, Y \rangle + \frac{1}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} p_{\alpha,i}^{k-1}(c_{\alpha,i}^k + c_{\alpha,i}^{k-1}),$$

where $c_{\alpha,i}^k = \alpha(Y) x_{\alpha,i}^k$.

B. Path integral for $Y \in \mathfrak{n}$

Recall that if $Y \in \mathfrak{n}$, then $H_k(g : Y)$ is given by

$$H_k(g : Y) = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} p_{\alpha,i}^{k-1} c_{\alpha,i}^{k-1},$$

where

$$c_{\alpha,i}^k = B\left(\frac{\mathrm{ad} X^k}{e^{\mathrm{ad} X^k} - 1} Y,\ E_{-\alpha,i}\right)$$

and $X^k = \sum_{\alpha,i} x_{\alpha,i}^k E_{\alpha,i}$. Now we assume that

$$\mathcal{C}^0\mathfrak{n} \supset \mathcal{C}^1\mathfrak{n} \supset \mathcal{C}^2\mathfrak{n} \supset \mathcal{C}^3\mathfrak{n} = \{0\}, \tag{3.1}$$

where $\mathcal{C}^0\mathfrak{n} = \mathfrak{n}$ and $\mathcal{C}^{i+1}\mathfrak{n} = [\mathfrak{n}, \mathcal{C}^i\mathfrak{n}]$.

Then, computing the path integrals as in §0, we obtain

THEOREM 3.1. *(i) For* $Y \in \mathfrak{m} \oplus \mathfrak{a}$*, taking the ordering of the Hamiltonian function* $H(g : Y)$ $(g \in NMA\overline{N})$ *described in this section, the path integral with the Hamiltonian gives the kernel function of the operator* $U_\lambda(\exp TY)$.

(ii) Assume that the length of the central descending series of $\mathfrak{n}$ *is* ≤ 3 *(see (3.1)). Then for* $Y \in \mathfrak{n}$*, taking the ordering of the Hamiltonian function* $H(g : Y)$ $(g \in NMA\overline{N})$ *described in this section, the path integral with the Hamiltonian gives the kernel function of the operator* $U_\lambda(\exp TY)$.

4 Intertwining Operator

In this section we take another real polarization and show that the formal intertwining operator between the two representations can be obtained from the path integral.

Let λ be the same element of $\mathfrak{a}^*$ as in §1. We take another real polarization $\mathfrak{s}_+ = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Correspondingly, we put $S_+ = MAN$. Then the Lie algebra homomorphism

$$-\sqrt{-1}\lambda : \mathfrak{s}_+ \longrightarrow \sqrt{-1}\mathbf{R}, \qquad X_0 + H + X_+ \longmapsto -\sqrt{-1}\lambda(H)$$

lifts to the unitary character of S_+:

$$S_+ \longrightarrow U(1), \qquad m \exp H\, n \longmapsto e^{-\sqrt{-1}\lambda(H)}.$$

We define a one-dimensional representation $\tilde{\xi}_\lambda$ of S_+ by

$$\tilde{\xi}_\lambda \; : S_+ \longrightarrow \mathbf{C}^\times, \qquad m \exp H\, n \longmapsto e^{(-\sqrt{-1}\lambda+\rho)(H)}.$$

Let $(\mathcal{H}_{\tilde{\lambda}}, U_{\tilde{\lambda}})$ be the unitary representation of G which is constructed from $\tilde{\xi}_\lambda$ as in §1, instead of ξ_λ. Note that $\tilde{F} \in \mathcal{H}_{\tilde{\lambda}}$ is a function on $\overline{N}$, on which we introduced coordinates by (1.6).

For $g \in \overline{N}MAN$, we write as

$$g = \overline{n}'(g) m'(g) a'(g) n'(g) \tag{4.1}$$

and parametrize $\overline{n}'(g)$ as $\overline{n}'(g) = \overline{n}_y = \exp Y$, where Y is of the form

$$Y = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} y_{\alpha,i} E_{-\alpha,i}. \tag{4.2}$$

Then for $g \in G$ and $\tilde{F} \in \mathcal{H}_{\tilde{\lambda}}$ the action is

$$U_{\tilde{\lambda}}(g)\tilde{F}(y) = e^{(\sqrt{-1}\lambda-\rho)\log a'(g^{-1}\overline{n}_y)} \tilde{F}(\overline{n}'(g^{-1}\overline{n}_y)), \tag{4.3}$$

where $y = (y_{\alpha,i})$ and $\overline{n}_y = \exp \sum_{\alpha\in\Sigma^+} y_{\alpha,i} E_{-\alpha,i}$. If we use the parametrization (4.1), then φ is given by

$$\begin{aligned}\varphi &= \left\langle \lambda, g^{-1}dg \right\rangle \\ &= \left\langle \mathrm{Ad}^*(n')\lambda, \overline{n}'(g)^{-1} d\overline{n}'(g) \right\rangle + \left\langle \lambda, a'(g)^{-1} da'(g) \right\rangle,\end{aligned}$$

where $n' = m'(g)a'(g)n'(g)(m'(g)a'(g))^{-1}$. Since the second term is an exact 1-form, we choose

$$\alpha_{\mathfrak{s}_+} = \left\langle \mathrm{Ad}^*(n')\lambda, \; \overline{n}'(g)^{-1} d\overline{n}'(g) \right\rangle.$$

Fixing $y' = (y'_{\alpha,i})$ and $y = (y_{\alpha,i})$, we can explicitly compute the path integral with Hamiltonian function for $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\bar{\mathfrak{n}}$, in the same way as in §3.

For $g \in NMA\overline{N} \cap \overline{N}MAN$, write g in two ways :

$$\begin{aligned} g &= n(g)\overline{n}m(g)a(g) \\ &= \overline{n}'(g)n'm'(g)a'(g). \end{aligned}$$

Then we have

$$\alpha_{s_-} - \alpha_{s_+} = \left\langle \lambda, \, a^{-1}da \right\rangle , \tag{4.4}$$

where $a = a(\overline{n}'(g)^{-1}n(g))$.

We parametrize $n(g) = n_x = \exp X$ and $\overline{n}'(g) = \overline{n}_y = \exp Y$, where X (or Y) is of the form (2.1) (or (4.2), respectively), and fix $x' = (x'_{\alpha,i})$, $x = (x_{\alpha,i})$, $y' = (y'_{\alpha,i})$ and $y = (y_{\alpha,i})$.

Then using (4.4) and proceeding analogously to the argument in §0, we can show that an integral operator with kernel function

$$\exp((-\sqrt{-1} + \rho) \log a(\bar{n}_y^{-1} n_x)) \tag{4.5}$$

coincides with the formal intertwining operator $A(S_+ : S_- : 1 : \sqrt{-1}\lambda)$ given in [9][10]. The integral operator with kernel function (4.5) is not well-defined in the sense that the integral

$$\int_N e^{(-\sqrt{-1}\lambda + \rho) \log a(\bar{n}_y^{-1} n_x)} F(x) dx$$

need not converge for $F \in \mathcal{H}_\lambda$. Knapp and Stein showed in [9][10] that if one regularizes the integral suitably, then the regularized operator, $\mathcal{A}(S_+ : S_- : 1 : \sqrt{-1}\lambda)$ in their notation, is a well-defined intertwining operator and is invertible, i.e., the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} \mathcal{H}_\lambda & \xrightarrow{\mathcal{A}(S_+:S_-:1:\sqrt{-1}\lambda)} & \mathcal{H}_{\bar{\lambda}} \\ {\scriptstyle U_\lambda(g)}\big\downarrow & & \big\downarrow{\scriptstyle U_{\bar{\lambda}}(g)} \\ \mathcal{H}_\lambda & \xrightarrow{\mathcal{A}(S_+:S_-:1:\sqrt{-1}\lambda)} & \mathcal{H}_{\bar{\lambda}} \end{array}$$

THEOREM 4.1. *The path integral with the action defined by (4.5) provides the formal intertwining operator* $A(S_+ : S_- : 1 : \sqrt{-1}\lambda)$, *where* $A(S_+ : S_- : 1 : \sqrt{-1}\lambda)$ *is given by*

$$A(S_+ : S_- : 1 : \sqrt{-1}\lambda) f(\bar{n}_y) = \int_N f(\bar{n}_y n_x) dx \quad \textit{for } f \in V_\lambda$$

when the indicated integrals are convergent.

We can compute the path integral for $Y \in \overline{\mathfrak{n}}$ using the polarization given in this section in the same way as in §3.

Thus, considering the composition

$$\mathcal{A}(S_+ : S_- : 1 : \sqrt{-1}\lambda)^{-1} \circ U_{\tilde{\lambda}}(\exp TY) \circ \mathcal{A}(S_+ : S_- : 1 : \sqrt{-1}\lambda),$$

we can obtain the unitary operators $U_\lambda(\exp TY)$ for $Y \in \overline{\mathfrak{n}}$ by the path integrals.

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