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A relation between the conformal factor in the Einstein's vacuum equations and the central extension of a formal loop group

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In this notes we shall briefly show that the space of all the solutions of the Einstein's vacuum equations in 2-dimensional reduction has an infinite dimensional homogeneous space structure of the centrally extended Hauser group by the usage of the formal loop group techniques. Moreover the conformal factor coming from metrics on our space-time manifolds is related to a central extension of the formal loop group, into which the potential space being all of the solutions of our linearized equation is naturally embedded. For more details discussion, see [1][3][4][5].

0. Preliminaries

Let G be $PSL(2,\mathbb{R}) \equiv SL(2,\mathbb{R})/\{\pm I_2\}$ and θ be the Cartan involution defined by $\theta\,(g) = {}^t g^{-1}$ for $g \in G$. Let $G = KAN$ be an Iwasawa decomposition, where a maximal compact subgroup K of G is given by $K = \{g \in G\,;\, \theta\,(g) = g\}$.

Let $F = \mathbb{R}[[z,\rho]]$ be an associative filtered algebra over $\mathbb{R}$ with a filtration $\{F_l\}_{l\in\mathbb{Z}} = \{\rho^{|l|}\mathbb{R}[[z,\rho]]\}_{l\in\mathbb{Z}}$.

And let $\mathcal{FG}$ be the formal loop group as follows:

$$\mathcal{FG} = \left\{ g = \sum_{l\in\mathbb{Z}} g_l t^l \,;\, g_l \in \mathfrak{gl}(2,F_l), \det g = 1 \right\} / \{\pm I_2\}\,.$$

G is naturally embedded into $\mathcal{FG}$.

We introduce an involutive automorphism $\theta^{(\infty)}$ of $\mathcal{FG}$, which is also called the Cartan involution, by

$$\theta^{(\infty)} : \mathcal{FG} \ni g(t) \longmapsto \theta\left(g(-\frac{1}{t})\right) \in \mathcal{FG}\,.$$

By use of the Cartan involution we define the subgroup of $\mathcal{FG}$ such that

$$\mathcal{FK} = \left\{ k \in \mathcal{FG}\,;\, \theta^{(\infty)}\,(k) = k \right\}.$$

Let $AN(\mathbb{R}[[z,\rho]])$ be the set of the formal power series with values in AN of the Iwasawa decomposition and let

$$\mathcal{FP} = \left\{ \mathcal{P}(t) = \sum_{l=0}^{\infty} P_l t^l \in \mathcal{FG}\,;\, P_0 \in AN(\mathbb{R}[[z,\rho]]) \right\}.$$

Then from the theory of Takasaki's formal loop group it is easily obtained that the following proposition holds.

PROPOSITION. *The formal loop group $\mathcal{FG}$ is uniquely decomposed as*

$$\mathcal{FG} = \mathcal{FK}\,\mathcal{FP}\,.$$

Let α be the map : $\mathcal{FG} \longrightarrow \mathcal{FP}$ through the above decomposition. We denote by $\overline{\alpha}$ the map from $\mathcal{FK}\backslash\mathcal{FG}$ to $\mathcal{FP}$ induced from α.

Then for any $g \in \mathcal{FG}$ we define an action on $\mathcal{FP}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{FK}\backslash\mathcal{FG} \arrow[r, "g"] \arrow[d, "\overline{\alpha}"'] & \mathcal{FK}\backslash\mathcal{FG} \arrow[d, "\overline{\alpha}"] \\
\mathcal{FP} \arrow[r, "g"] & \mathcal{FP}\,.
\end{tikzcd}$$

1. Basic equations in 2-dimensional reduction

Let $ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu$ be a space-time metric on $\mathbf{R}^{1+3}$. Then the Einstein's vacuum equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \quad (\mu,\nu = 0,1,2,3)\,,$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature given by:

$$\begin{aligned}
\Gamma^\beta_{\mu\nu} &= \frac{1}{2}g^{\beta\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})\,,\\
R_{\mu\nu} &= \partial_\beta\Gamma^\beta_{\mu\nu} - \partial_\nu\Gamma^\beta_{\mu\beta} + \Gamma^\beta_{\mu\nu}\Gamma^\kappa_{\mu\beta} - \Gamma^\kappa_{\mu\beta}\Gamma^\beta_{\nu\kappa}\,,\\
R &= g^{\mu\nu}R_{\mu\nu}\,.
\end{aligned}$$

As for 2-dimensional reduction, we assume that the stationary and axially symmetric space-times have the following metric form in cylindrical polar coordinates

$$ds^2 = \sum_{p,q=0}^{1} h_{pq}dx^p \otimes dx^q - \lambda^2(dz\otimes dz + d\rho \otimes d\rho)\,,$$

where λ is a positive function, $h = (h_{pq})$ is symmetric, and h and λ depend only on the variables z, ρ, and $\det h = -\rho^2$.

Then the Einstein's vacuum equations become as follows:

(1.a) $\qquad d\,(\rho^{-1}h\epsilon * dh) = 0,$

(1.b) $\qquad \tau^{-1}\partial_z\tau = -\dfrac{\partial_z f}{2f} + \dfrac{\rho}{4}\mathrm{tr}(\partial_z h^{-1}\partial_\rho h),$

$$\tau^{-1}\partial_\rho\tau = -\frac{\partial_\rho f}{2f} + \frac{1}{2\rho} - \frac{\rho}{8}\mathrm{tr}\big(\partial_z h^{-1}\partial_z h - \partial_\rho h^{-1}\partial_\rho h\big)\,, \tag{1.c}$$

where $f(>0) =$ the (1,1) component of h, $\tau = 1/\sqrt{f}\lambda$, $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $* =$ Hodge operator for the metric $dz^2 + d\rho^2$. τ is called the conformal factor in this notes.

We parametrize h by introducing a new function γ as

$$h = \begin{pmatrix} f\gamma^2 - \rho^2/f & f\gamma \\ f\gamma & f \end{pmatrix}.$$

Introducing the Ernst potential ψ defined by

$$d\psi = \rho^{-1} f^2 * d\gamma\,,$$

we have the following equations equivalent to the equations $(1.a)$.

$$d\,(\rho f^{-2} * d\psi) = 0\,, \tag{2.a}$$

$$d\,(\rho f^{-1} * df + \rho f^{-2}\psi * d\psi) = 0\,. \tag{2.b}$$

Let $M(\mathbf{R}[[z,\rho]])$ be as follows:

$$\{m \in \mathfrak{gl}(2, \mathbf{R}[[z,\rho]])\,;\ {}^t m = m,\ \det m = 1,\ \text{the (2,2) component of m } > 0\}\,.$$

Then, we fix the parametrization of $m \in M(\mathbf{R}[[z,\rho]])$ by

$$m = \begin{pmatrix} f + \dfrac{\psi^2}{f} & \dfrac{\psi}{f} \\ \dfrac{\psi}{f} & \dfrac{1}{f} \end{pmatrix}.$$

DEFINITION. *Let $M(\mathbf{R}[[z,\rho]])$ be as above.*

Then we define SM to be the set of all elements $m \in M(\mathbf{R}[[z,\rho]])$ satisfying the equation $d\,(\rho m^{-1} * dm) = 0$.

For the conformal factor defined by the equations $(1.b)$ and $(1.c)$, using the matrix m, we have a more elegant expression as follows:

$$\tau^{-1}\partial_z\tau = \frac{\rho}{4}\mathrm{tr}(\partial_z m^{-1}\partial_\rho m)\,, \tag{3.a}$$

$$\tau^{-1}\partial_\rho\tau = \frac{\rho}{8}\mathrm{tr}(\partial_\rho m^{-1}\partial_\rho m - \partial_z m^{-1}\partial_z m)\,. \tag{3.b}$$

LEMMA. *For any element m of the solution space SM there exists a unique conformal factor τ up to a multiplicative positive constant, which satisfies the equations $(3.a)$ and $(3.b)$.*

From the lemma we define the mapping

$$\eta : SM \longrightarrow F\,,$$

where for any given $m \in SM$ $\tau = \eta(m)$ is given by solving the equations $(3.a)$ and $(3.b)$, and by adjusting a multiplicative positive constant so that $\tau(0,0) = 1$.

REMARK. *The Minkowski space-time, which has the metric in the cylindrical polar coordinate*

$$ds^2 = dt \otimes dt - \rho^2 d\varphi \otimes d\varphi - dz \otimes dz - d\rho \otimes d\rho \ ,$$

is explicitly expressed by

$$\begin{aligned} h_e &= \begin{pmatrix} 1 & 0 \\ 0 & -\rho^2 \end{pmatrix} \in \mathcal{S}E \ , \\ m_e &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \in \mathcal{S}M \ , \\ \tau_e &= \eta(m) = 1 \quad \in F \ . \end{aligned}$$

2. Linearization and Total space construction

Let notations be as in Section 0. Let $G = KAN$ be an Iwasawa decomposition, where we employ the following parametrization

$$\begin{aligned} A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} ; a > 0 \right\} \ , \\ N &= \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} ; x \in \mathbf{R} \right\} \ . \end{aligned}$$

Corresponding to the above parametrization we parametrize the element P in $AN(\mathbf{R}[[z,\rho]])$ as follows:

$$P = \begin{pmatrix} \sqrt{f} & 0 \\ \dfrac{\psi}{\sqrt{f}} & \dfrac{1}{\sqrt{f}} \end{pmatrix} \ . \tag{4}$$

Fix the above parametrization of $AN(\mathbf{R}[[z,\rho]])$. Then we define the solution space $\mathcal{S}P$, which is equivalent to $\mathcal{S}M$, by

$$\mathcal{S}P = \left\{ P \in AN(\mathbf{R}[[z,\rho]]) \ ; \ d\Big(\rho\, P^{-1}\theta\,(P) * d(\theta\,(P^{-1})\,P)\Big) = 0 \ \right\} \ .$$

The map

$$\overline{\theta} : \mathcal{S}P \longrightarrow \mathcal{S}M.$$

is defined by defining $\overline{\theta}(P^{-1})P$ for $P \in \mathcal{S}P$.

Next we introduce a 1-form with the spectral parameter, the exterior derivative on which is defined as follows:

$$dt = \frac{t}{\rho(1+t^2)}\Big((1-t^2)d\rho + 2t dz\Big).$$

Definition. *For $P \in AN(\mathbf{R}[[z,\rho]])$, let $\mathcal{A}$ and $\mathcal{I}$ be the $sl(2,\mathbf{R}[[z,\rho]])$-valued 1-forms defined by*

$$\mathcal{A} = \frac{1}{2}(dPP^{-1} + \theta(dPP^{-1}))\,,$$
$$\mathcal{I} = \frac{1}{2}(dPP^{-1} - \theta(dPP^{-1}))\,.$$

A $sl(2,\mathbf{R}[[z,\rho]])$-valued 1-form Ω_P for P is defined by

$$\Omega_P = \mathcal{A} + \frac{1-t^2}{1+t^2}\mathcal{I} - \frac{2t}{1+t^2} * \mathcal{I}.$$

The map $proj$ is defined on $\mathcal{SP}$ as follows:

$$proj : \mathcal{SP} \ni \mathcal{P}(t) = \sum_{l=0}^{\infty} P_l t^l \longmapsto P_0 \in AN(\mathbf{R}[[z,\rho]]).$$

Definition. *Let $\mathcal{FP}$ be the formal loop group defined in Section 0. We define $\mathcal{SP}$ to be the set of all elements $\mathcal{P}(t) = \sum_{m=0}^{\infty} P_m t^m$ of $\mathcal{FP}$ satisfying the equation $d\mathcal{P}(t) = \Omega_P \mathcal{P}(t)$, where we put $P = proj(\mathcal{P}(t))$.*

Proposition. *Let $\mathcal{P}(t)$ be any element of the potential space $\mathcal{SP}$. Then $proj(\mathcal{P}(t))$ is an element of SP.*

That is to say, the map

$$proj : \mathcal{SP} \longrightarrow SP$$

is well-defined.

In summary, from the proposition and the discussions so far we have the following well-defined diagram:

$$\mathcal{SP} \xrightarrow{proj} SP \xrightarrow{\overline{\theta}} SM.$$

Now we consider the following total solution space $E(\mathcal{SP})$:

$$\begin{aligned} E(\mathcal{SP}) = \Big\{ (\mathcal{P}(t), e^{\mu}) \in \mathcal{SP} \times F^+ \,;\ & \\ & \mathcal{P}(t) \in \mathcal{SP},\ (\text{ put } m = \overline{\theta}(proj(\mathcal{P}(t)))\ ) \\ & \partial_z \mu = -\frac{\rho}{4}\mathrm{tr}(\partial_z m^{-1} \partial_\rho m)\,, \\ & \partial_\rho \mu = -\frac{\rho}{8}\mathrm{tr}(\partial_\rho m^{-1} \partial_\rho m - \partial_z m^{-1} \partial_z m) \quad \Big\} \end{aligned}$$

and denote by $\pi : E(\mathcal{SP}) \longrightarrow \mathcal{SP}$ the surjective map defined by

$$\pi((\mathcal{P}(t), e^{\mu})) = \mathcal{P}(t) \quad \text{for} \quad (\mathcal{P}(t), e^{\mu}) \in E(\mathcal{SP}).$$

Then a triplet $(E(\mathcal{SP}), \pi, \mathcal{SP})$ is considered to be a fiber space with fiber $\mathbb{R}^+$, in fact a principal bundle.

By the lemma in Section 1, we can define the following global section *sect* of the fiber space

$$sect : \mathcal{SP} \in \mathcal{P}(t) \longmapsto (\mathcal{P}(t), \overline{\eta}(\mathcal{P}(t))) \ni E(\mathcal{SP}),$$

where $\overline{\eta} : \mathcal{SP} \longrightarrow F$ is given by the following diagram:

$$\begin{tikzcd} \mathcal{SP} \arrow[r, "proj"] & SP \arrow[r, "\overline{\theta}"] & SM \arrow[d, "\eta"'] \\ & & F \end{tikzcd}$$

We put $\Gamma(\mathcal{SP}) = \text{Im}(sect)$. The map *sect* is a global section of the fiber space, the fiber space is trivial.

REMARK. *The Minkowski space-time has in the potential space $\mathcal{SP}$ is*

$$\begin{aligned} &\mathcal{P}_e &&= I_2 &&\in \mathcal{SP} \quad \text{and} \\ &(\mathcal{P}_e, \tau_e) &&= (I_2, 1) &&\in \Gamma(\mathcal{SP}). \end{aligned}$$

3. Central extensions and transformations

Let $\mathcal{G}^{(\infty)} = PSL(2, \mathbb{R}[[s]])$ be an infinite dimensional group

$$\{g(s) \in \mathfrak{gl}(2, \mathbb{R}[[s]]) \ ; \ \det g(s) = 1\}/\{\pm I_2\} \ ,$$

where $\mathbb{R}[[s]]$ is the associative algebra of formal power series in s over $\mathbb{R}$. We call $\mathcal{G}^{(\infty)}$ the Hauser group.

Let $\mathcal{FG}$ be the formal loop group with values in $PSL(2, \mathbb{R})$. Then we define an injective homomorphism j such that

$$j : \mathcal{G}^{(\infty)} \ni g(s) \longmapsto g(\rho(\frac{1}{t} - t) + 2z) \in \mathcal{FG} \ .$$

$\mathrm{Im}(j)$ is denoted by $\mathcal{FH}$.

First we define the central extension of Hauser group by the additive group $\mathbb{R}\,(\cong \mathbb{R}^+)$ as follows.

DEFINITION. *Let* $\mathcal{G}^{(\infty)}_{ce} = \mathcal{G}^{(\infty)} \times \mathbb{R}^+$ *with the group multiplication such that*

$$(g_1, e^v) \cdot (g_2, e^u) = (g_1 g_2, e^{v+u}) \quad \textit{for} \ (g_1, e^v), (g_2, e^u) \in \mathcal{G}^{(\infty)} \times \mathbb{R}^+ \ .$$

Next we consider the central extension of the formal loop group $\mathcal{FG}$ by the additive formal group $F = \mathbb{R}[[z, \rho]]$, which is diagramatically expressed as

$$0 \longrightarrow F \longrightarrow \mathcal{FG}_{ce} \longrightarrow \mathcal{FG} \longrightarrow 0 \ .$$

Since the cohomology group $H^2(\mathcal{FG}, F)$ is not trivial, we take the nontrivial central extension by the choice of the representative Ξ in the nontrivial cohomology class (see [1][5] for the definition of Ξ).

DEFINITION. *We define the centrally extended formal loop group* $\mathcal{FG}_{ce}$ *to be the direct product* $\mathcal{FG} \times F^+$ *with the group multiplication:*

$$(g_1, e^\mu)(g_2, e^\nu) = (g_1 g_2, e^{\mu+\nu+\Xi(g_1,g_2)}) \quad \textit{for} \ (g_1, e^\mu), (g_2, e^\nu) \in \mathcal{FG}_{ce} \ .$$

Define a map j_{ce} from $\mathcal{G}^{(\infty)}_{ce}$ to $\mathcal{FG}_{ce}$ by the mapping product $j \times i$, where i is the inclusion map into F. Then j_{ce} is an injective homomorphism. And the images of j_{ce} is denoted by $\mathcal{FH}_{ce}$.

We introduce an involutive automorphism $\theta^{(\infty)}_{ce}$ of $\mathcal{FG}_{ce}$ defined by

$$\theta^{(\infty)}_{ce} : \mathcal{FG}_{ce} \ni (g, e^\mu) \longmapsto (\theta^{(\infty)}(g), e^{-\mu}) \in \mathcal{FG}_{ce} \ ,$$

which is also called the Cartan involution. Then we define the subgroup of $\mathcal{FG}_{ce}$ by

$$\mathcal{FK}_{ce} = \left\{ k_{ce} \in \mathcal{FG}_{ce}\,;\, \theta_{ce}^{(\infty)}(k_{ce}) = k_{ce} \right\},$$

which turns out to be

$$\mathcal{FK}_{ce} = \mathcal{FK} \times \{1\}\,.$$

Let $\mathcal{FP}$ denote the subgroup of $\mathcal{FG}$ defined in Preliminaries. We define the subgroup of $\mathcal{FG}_{ce}$ as follows:

$$\mathcal{FP}_{ce} = \{\mathcal{P}_{ce}(t) = (\mathcal{P}(t), e^{\mu}) \in \mathcal{FG}_{ce}\,;\, \mathcal{P}(t) \in \mathcal{FP},\ \mu \in F\}.$$

Then the following Proposition holds.

PROPOSITION. *Let $\mathcal{FG}_{ce}$ be the centrally extended formal loop group of $\mathcal{FG}$. Then $\mathcal{FG}_{ce}$ is uniquely decomposed as*

$$\mathcal{FG}_{ce} = \mathcal{FK}_{ce}\,\mathcal{FP}_{ce}.$$

Let α_{ce} be the map : $\mathcal{FG}_{ce} \longrightarrow \mathcal{FP}_{ce}$ through the decomposition (5.11). We denote by $\overline{\alpha}_{ce}$ the map from $\mathcal{FK}_{ce}\backslash\mathcal{FG}_{ce}$ to $\mathcal{FP}_{ce}$ induced from α_{ce}. Then for any $g_{ce} \in \mathcal{FG}_{ce}$ we define the action on $\mathcal{FP}_{ce}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{FK}_{ce}\backslash\mathcal{FG}_{ce} & \xrightarrow{g_{ce}} & \mathcal{FK}_{ce}\backslash\mathcal{FG}_{ce} \\ \overline{\alpha}_{ce}\downarrow & & \downarrow\overline{\alpha}_{ce} \\ \mathcal{FP}_{ce} & \longrightarrow & \mathcal{FP}_{ce}\,. \end{array}$$

For the action of $g_{ce} \in \mathcal{FG}_{ce}$ on $\mathcal{FP}_{ce}$ we use g_{ce} as a notation, that is,

$$g_{ce} : \mathcal{FP}_{ce} \longrightarrow \mathcal{FP}_{ce}.$$

It is noticed that $\Gamma(\mathcal{SP}) \subset E(\mathcal{SP}) \subset \mathcal{FP}_{ce}$.

Then we have the main theorem below.

THEOREM.

Let $\mathcal{FH}_{ce}$ and $E(\mathcal{SP})$ be the Hauser group and the total space defined in Section 2.

For any $g_{ce} \in \mathcal{FH}_{ce}$, the following diagram is well-defined:

$$\begin{array}{ccc} \mathcal{FK}_{ce}\backslash\mathcal{FK}_{ce}\,E(\mathcal{SP}) & \xrightarrow{g_{ce}} & \mathcal{FK}_{ce}\backslash\mathcal{FK}_{ce}\,E(\mathcal{SP}) \\ \overline{\alpha}\downarrow & & \downarrow\overline{\alpha} \\ E(\mathcal{SP}) & \xrightarrow{g_{ce}} & E(\mathcal{SP})\,. \end{array}$$

The centrally extended Hauser group $\mathcal{FH}_{ce}$ ($\cong \mathcal{G}^{(\infty)}$) acts transitively on the potential space $E(\mathcal{SP})$; $E(\mathcal{SP})$ is an infinite dimensional homogeneous space.

Let $g_{ce} = (g, e^a)$ be any element of $\mathcal{FH}_{ce}$.

Then for g in g_{ce} we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{FK}\backslash\mathcal{FK}\,\mathcal{SP} & \xrightarrow{g} & \mathcal{FK}\backslash\mathcal{FK}\,\mathcal{SP} \\ {\scriptstyle\overline{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\alpha}} \\ \mathcal{SP} & \longrightarrow & \mathcal{SP}. \end{array}$$

Furthermore we can prove that the Hauser group $\mathcal{FH}(\cong \mathcal{G}^{(\infty)})$ acts transitively on the potential space $\mathcal{SP}$; $\mathcal{SP}$ is an infinite dimensional homogeneous space.

Since g in (g, e^a) is an element of $\mathcal{FH}$ ($\subset \mathcal{FG}$), we have the decomposition of g such that $g^{-1} = k\,\mathcal{P}(t)$ ($\mathcal{P}(t) \in \mathcal{SP}$, $k \in \mathcal{FK}$). Let P denote $proj(\mathcal{P}(t))$, that is, $\mathcal{P}(0)$. We parametrize P as in Section 2.

Then for the derivative of the group 2-cocycle Ξ with respect to z and ρ we have

$$\partial_z \Xi(\mathcal{P}(t), g) = -\frac{\rho}{2f^2}(\partial_z f \partial_\rho f + \partial_z \psi \partial_\rho \psi),$$
$$\partial_\rho \Xi(\mathcal{P}(t), g) = -\frac{\rho}{4f^2}\left((\partial_\rho f)^2 - (\partial_z f)^2 + (\partial_\rho \psi)^2 - (\partial_z \psi)^2\right),$$

where f, ψ are given by the parametrization (4) of P. So we can complete the proof. For the details of the proof of the theorem, we refer to [5].

As for the conformal factor τ we have the following relation.

COROLLARY 5.12. *For any element $\mathcal{P}_{ce}(t) = (\mathcal{P}(t), \tau) \in \Gamma(\mathcal{SP})$, we have the following relation:*

$$\tau = \exp\left\{-\frac{1}{2}\Xi\left(\theta^{(\infty)}\left(\mathcal{P}(t)^{-1}\right), \mathcal{P}(t)\right)\right\}.$$

Let $E(\mathcal{S}P)$ and $E(\mathcal{S}M)$ be subspaces of $\mathcal{S}P \times F$ and $\mathcal{S}M \times F$ and defined by the same way in $E(\mathcal{SP})$. And, let $i : F \longrightarrow F$ be the identity map. Then from the discussions so far we have the following diagram for $g_{ce} \in \mathcal{FH}_{ce}$:

$$\begin{array}{ccccc} E(\mathcal{SP}) & \xrightarrow{proj \times i} & E(\mathcal{S}P) & \xrightarrow{\overline{\theta} \times i} & E(\mathcal{S}M) \\ {\scriptstyle g_{ce}}\big\downarrow & & \big\downarrow & & \big\downarrow \\ E(\mathcal{SP}) & \xrightarrow{proj \times i} & E(\mathcal{S}P) & \xrightarrow{\overline{\theta} \times i} & E(\mathcal{S}M). \end{array}$$

Therefore for the fiber space description we have the following commutative diagram for $g_{ce} = (g, e^a) \in \mathcal{G}_{ce}^{(\infty)}$:

$$\begin{array}{ccc} E(\mathcal{SP}) & \xrightarrow{g_{ce}} & E(\mathcal{SP}) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathcal{SP} & \xrightarrow{g} & \mathcal{SP}. \end{array}$$

It is clear that the center $\mathbb{R}^+$ of $\mathcal{G}_{ce}^{(\infty)}$ corresponds to the fiber $\mathbb{R}^+$ of $E(\mathcal{SP})$.

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