

# グラフの辺取りゲーム

(ニムとケイレスの一般化)

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## 1.Introduction.

We first define a new game played on graphs. Let  $G$  be a finite graph without loops or multiple edges, and  $\mathcal{H}$  a set of graphs. This game is played by two players on the graph  $G$ . Each player in turn removes a set of edges which induces a graph isomorphic to a graph in  $\mathcal{H}$ . The winner is a player who removes edges such that the resulting graph contains no graph of  $\mathcal{H}$ , that is, the player who cannot move loses. We call this game an *edge-removing game of normal  $\mathcal{H}$  type*. If we change the rule to one where the player who removes the last edges loses, then the game is called an *edge-removing game of misere  $\mathcal{H}$  type*. In this paper, we shall discuss only games of normal type.

We call the complete bipartite graph  $K_{1,n} = K(1,n)$  the *star* of order  $n+1$ , and denote by  $P_n$  the path of order  $n$ . If  $\mathcal{H}$  is the set of all stars, then we call this game the *edge-removing game of normal star type*, or simply *ER-game of star type*. If we play ER-game of star type on a graph consisting of some stars, then this game is nothing but the game of Nim. Similarly ER-game of star type played on a graph consisting of some paths is equivalent to the game called Kayles. So ER-game of star type is a generalization of these

two games. In this paper we give some results on ER-game of star type played on double stars, forks and trees.

## 2. ER-game of star type played on doubles stars

In order to solve ER-game of normal  $\mathcal{H}$  type played on a graph  $G$ , it suffices to determine the Sprague-Grundy number  $g(G)$  of  $G$ , which is often called the Grundy number [1,2,3]. The Grundy number is defined inductively as follows: If a graph  $G_1$  contains no graph of  $\mathcal{H}$ , then  $g(G_1) = 0$ . Let  $H_1, H_2, \dots, H_m$  be the set of all graphs which can be obtained from a graph  $G$  by one move. Then

$$g(G) = \min\{\{0,1,2,3,\dots\} - \{g(H_i) \mid 1 \leq i \leq m\}\}$$

By this definition, we can easily show that  $g(G) \leq |E(G)|$  by induction on  $|E(G)|$ . It is well-known that if a graph  $G$  consists of the components  $D_1, \dots, D_r$ , then

$$\begin{aligned} g(G) &= \text{the nim-sum of } g(D_1), g(D_2), \dots, g(D_r). \\ &= g(D_1) \dot{+} g(D_2) \dot{+} \dots \dot{+} g(D_r). \end{aligned}$$

Namely if

$$g(D_k) = \sum_{i \geq 0} x_k(i) 2^i, \quad x_k(i) \in \{0,1\}$$

then

$$g(G) = \sum_{i \geq 0} y(i) 2^i, \quad y(i) \equiv \sum_{k=1}^r x_k(i) \pmod{2} \quad \text{and} \quad y(i) \in \{0,1\}.$$

Moreover, it is easy to see that the player going second can win if and only if  $g(G) = 0$ .

We denote by  $(\dots)$  an order set, that is,  $(x_1, x_2, \dots, x_k) = (y_1, y_2, \dots, y_k)$  means that  $x_i = y_i$  for all  $i, 1 \leq i \leq k$ .

**Theorem A.** [1,2,3] The Grundy numbers of stars  $K_{1,n}$  and paths  $P_n$  of ER-game of star type are given by the following statements.

- (i)  $g(K_{1,n}) = n$ .
- (ii)  $g(P_{n+12}) = g(P_n)$  if  $n \geq 72$ , and  
 $(g(P_k) \mid 72 \leq k < 84) = (7, 4, 1, 2, 8, 1, 4, 7, 2, 1, 8, 4)$ .

For convenience, we denote by  $K_{1,0}$  a graph with one vertex and no edge. The double star  $DS(n,m)$  is a graph obtained from two stars  $K_{1,n}$  and  $K_{1,m}$  by joining their two centers by a new edge. Then the order of  $DS(n,m)$  is  $n+m+2$  and its size is  $n+m+1$ . We now give a conjecture on the Grundy numbers of double stars.

**Conjecture B.** Suppose that ER-game of star type is played on a double star  $DS(n,m)$ . Then

- (i) For every positive odd integer  $n$ , there exists an integer  $M = M(n)$  for which  $g(DS(n,m)) = n+m+1$  if  $m \geq M$ .
- (ii) For every positive even integer  $n$ , there exists integers  $p = p(n)$  and  $M = M(n)$  for which  $g(DS(n,m+p)) = g(DS(n,m)) + p$  if  $m \geq M$ .

We shall show that the conjecture is true if  $n = 2^k - 1$ ,  $n = 2^k$  or  $1 \leq n \leq 10$ . Moreover, by making use of computer, we observe that if  $n < 50$  and  $m \leq 5000$  then the conjecture holds and that  $M(n) < 800$  except  $n = 33$  ( $M(n) = 1953$ ),  $n = 34$  ( $M(n) = 2141$ ) and  $n = 48$  ( $M(n) = 2157$ ), furthermore, we may give a conjecture on  $p(n)$  that  $p(n) = 2^{k+1}$  if  $2^k \leq n < 2^{k+1}$  except  $n = 24$  ( $p(n) = 64$ ).

**Theorem 1.** Suppose that ER-game of star type is played on a double star. Then

- (i) For every integers  $k \geq 1$  and  $m \geq 0$ , we have

$$g(DS(2^k - 1, m)) = 2^k + m.$$

(ii) For every integers  $k \geq 1$  and  $h \geq 0$ , we have

$$g(DS(2^k, h \cdot 2^{k+1} + s)) = h \cdot 2^{k+1} + 2^k + s + 1,$$

where  $-1 \leq s \leq 2^k - 1$  and

$$g(DS(2^k, h \cdot 2^{k+1} + 2^k + s)) = h \cdot 2^{k+1} + s + 1,$$

where  $0 \leq s \leq 2^k - 2$ . In particular,

$$g(DS(2^k, m + 2^{k+1})) = g(DS(2^k, m)) + 2^{k+1} \text{ for all } m \geq 0.$$

**Proof** We first prove Statement (i). For convenience, let  $n = 2^k - 1$ . We shall prove that  $g(DS(n, m)) = n + m + 1$  by induction on  $m$ . Since a double star  $DS(n, 0)$  is a star  $K_{1, n+1}$ ,  $g(DS(n, 0)) = n + 1$  by Theorem A.

Suppose that  $1 \leq m \leq n$ . For every integer  $j$ ,  $0 \leq j \leq n$ , let  $r = m + j$ . Then  $0 \leq r \leq n$  and  $r + m = j$ . We can remove a star from the double star  $DS(n, m)$  such that the resulting graph is  $K_{1, r} \cup K_{1, m}$ , whose Grundy number is  $r + m = j$ . By the induction hypothesis, we have that  $g(DS(n, r)) = n + r + 1$  for every  $0 \leq r < m$ . Therefore  $g(DS(n, m)) \geq n + m + 1$ . Since  $g(DS(n, m)) \leq |E(DS(n, m))| = n + m + 1$ , we can conclude that  $g(DS(n, m)) = n + m + 1$ .

Next assume that  $n < m$ . For every integer  $j$ ,  $0 \leq j \leq n$ , let  $r = n + j$ . Then  $0 \leq r \leq n$  and  $n + r = j$ . We can remove a star from the double star  $DS(n, m)$  such that the resulting graph is  $K_{1, n} \cup K_{1, r}$ , whose Grundy number is  $n + r = j$ . By the same argument as above, we can also show that  $g(DS(n, m)) = n + m + 1$ .

For convenience, we denote the star  $K_{1, l}$  by  $K(1, l)$ . In order to prove Statement (ii) we need to show that the following equation holds.

$$g(DS(h 2^{k+1} - 1, m)) = h 2^{k+1} + m$$

for every integers  $0 \leq m \leq 2^k$  and  $h \geq 1$ . We prove the above equation by induction on  $m$ . Let  $0 \leq j < 2^k$  and  $0 \leq t \leq h - 1$ . If  $m < 2^k$ , then  $r = j < 2^k$ , and we can remove stars from  $DS(h 2^{k+1} - 1, m)$  such that the resulting graphs are  $K(1, t 2^{k+1} + r) \cup K(1, m)$  and  $K(1, t 2^{k+1} + 2^k + r) \cup K(1, m)$ , whose grundy numbers are  $t 2^{k+1} + j$  and  $t 2^{k+1} + 2^k + j$ , respectively. If  $m = 2^k$  then we can remove stars from  $DS(h 2^{k+1} - 1, m)$  such that the resulting graphs are  $K(1, t 2^{k+1} + j) \cup K(1, 2^k)$  and  $K(1, t 2^{k+1} + 2^k + j) \cup K(1, 2^k)$ , whose grundy numbers are  $t 2^{k+1} + 2^k + j$  and  $t 2^{k+1} + j$ , respectively. By the induction hypothesis, we have that  $g(DS(h 2^{k+1} - 1, y)) = h 2^{k+1} + y$  for every  $0 \leq y < m$ . Thus  $g(DS(h 2^{k+1} - 1, m)) \geq h 2^{k+1} + m$ , and so  $g(DS(h 2^{k+1} - 1, m)) = h 2^{k+1} + m$ .

We now prove Statement (ii) by induction on  $h 2^{k+1} + s$  or  $h 2^{k+1} + 2^k + s$ . By Theorem A and the above statement,  $g(DS(2^k, 0)) = 2^k + 1$  and  $g(DS(2^k, h 2^{k+1} - 1)) = h 2^{k+1} + 2^k$ . Consider a double star  $DS(2^k, h 2^{k+1} - 1)$ ,  $0 \leq h$ ,  $0 \leq s \leq 2^k - 1$ . For every integers  $0 \leq t \leq h - 1$ , and  $0 \leq x < 2^k$ , we have that

$$g(K(1, 2^k)) + g(K(1, t 2^{k+1} + 2^k + x)) = t 2^{k+1} + x$$

and

$$g(K(1, 2^k)) + g(K(1, t 2^{k+1} + x)) = t 2^{k+1} + 2^k + x.$$

Moreover,

$$g(K(1, x')) + g(K(1, h 2^{k+1})) = h 2^{k+1} + x' \text{ for } 0 \leq x' \leq 2^k,$$

and for every integer  $y$ ,  $0 \leq y < s$ , it follows from the induction hypothesis that

$$g(DS(2^k, h 2^{k+1} + y)) = h 2^{k+1} + 2^k + y + 1.$$

Therefore  $g(DS(2^k, h 2^{k+1} + s)) \geq h 2^{k+1} + 2^k + s + 1$ , and thus  $g(DS(2^k, h 2^{k+1} + s)) = h 2^{k+1} + 2^k + s + 1$ .

We next consider a double star  $DS(2^k, h 2^{k+1} + 2^k + s)$ ,  $0 \leq s \leq 2^k - 2$ .

By the same argument as above, we can easily show that

$$\begin{aligned} & \{g(K(1, 2^k)) \dot{+} g(K(1, y)) \mid 0 \leq y \leq h 2^{k+1} + 2^k + s\} \\ &= \{0, 1, 2, \dots, h 2^{k+1} + s\} \cup \{h 2^{k+1} + 2^k, \dots, (h+1)2^{k+1} - 1\}. \end{aligned}$$

Thus  $g(DS(2^k, h 2^{k+1} + 2^k + s)) \geq h 2^{k+1} + s + 1$ . It is obvious that for every  $0 < t < 2^k$ ,  $DS(t, h 2^{k+1} + 2^k + s)$  contains  $K(1, t) \cup K(1, h 2^{k+1} + r)$ ,  $r = t + (s + 1)$ , whose Grundy number is  $h 2^{k+1} + s + 1$ . Therefore  $g(DS(t, h 2^{k+1} + 2^k + s)) \neq h 2^{k+1} + s + 1$ . Consequently we can conclude that  $g(DS(2^k, h 2^{k+1} + 2^k + s)) = h 2^{k+1} + s + 1$ .

**Theorem 2.** The Grundy numbers of double stars  $DS(n, m)$ ,  $n \leq 10$ , of ER-game of star type are given by the following statements.

(i) If  $n = 0$ ,  $n = 1$ ,  $n = 3$ ,  $n = 5$  and  $m \geq 15$ ,  $n = 7$  or  $n = 9$  and  $m \geq 95$  then

$$g(DS(n, m)) = n + m + 1.$$

(ii) Let  $p = 4, 8$  or  $16$  according as  $n = 2$ ,  $n = 4, 6$ , or  $n = 8, 10$ . Suppose that  $m \geq 15$  if  $n = 6$ , and  $m \geq 110$  if  $n = 10$ . Then

$$g(DS(n, m + p)) = g(DS(n, m)) + p.$$

Note that Theorem 2 holds for  $n = 0, 1, 2, 3, 4, 7, 8$  by Theorem A and Theorem 1. We shall prove the following proposition instead of remaining Statement (ii) of Theorem 2.

**Proposition 3.** Consider  $g(DS(n, m))$  with  $n = 2, 4, 6, 8$  or  $10$ . Let  $t$  and  $s$  be integers such that  $0 \leq t$ , and  $0 \leq s < 4$ ,  $0 \leq s < 8$ , or  $0 \leq s < 16$  according as  $n = 2$ ,  $n = 4, 6$  or  $n = 8, 10$ . Then the following statements hold.

(i)  $g( DS(2,4t+s) ) = 4t+3, 4t+4, 4t+1, 4t+6$  if  $s = 0, 1, 2, 3$ , respectively.

(ii)  $g( DS(4,8t+s) ) = 8t+5, 8t+6, 8t+7, 8t+8, 8t+1, 8t+2, 8t+3, 8t+12$  if  $s = 0, 1, 2, \dots, 7$ , respectively.

(iii)  $g( DS(6,15+8t+s) ) = 8t+22, 8t+23, 8t+24, 8t+21, 8t+26, 8t+25, 8t+28, 8t+27$  if  $s = 0, 1, 2, \dots, 7$ , respectively.

(iv)  $g( DS(8,16t+s) ) = 8t+s+9, 8t+s-7$  or  $8t+24$  if  $0 \leq s \leq 7, 8 \leq s \leq 14$  or  $s = 15$ , respectively.

(v)  $g( DS(10,110+8t+s) ) = 8t+117, 8t+122, 8t+123, 8t+124, 8t+121, 8t+126, 8t+127, 8t+128, 8t+125, 8t+130, 8t+119, 8t+132, 8t+129$  if  $s = 0, 1, \dots, 15$ , respectively.

## 2. Grundy Numbers of Forks and trees

A fork  $F(n,m)$  is defined to be a graph which is obtained from a star  $K_{1,n}$  and a path  $P_m$  by joining the center of the star to one of the end vertices of the path by a new edge. Then the order of  $F(n,m)$  is  $n+m+1$  and its size is  $n+m$ . Note that  $F(0,m) = P_{m+1}$ ,  $F(1,m) = P_{m+2}$ ,  $F(n,0) = K_{1,n}$  and  $F(n,1) = K_{1,n+1}$  and these Grundy numbers are given by Theorem A.

**Theorem 2.** The Grundy numbers of forks  $F(n,m)$  with  $n \leq 10$  or  $m \leq 10$  of ER-game of star type are given by the following statements.

(i) If  $n = 2$  and  $m \geq 152$ ,  $n = 3$  and  $m \geq 141$ ,  $n = 4$  and  $m \geq 142$ ,  $n = 5$  and  $m \geq 286$ ,  $n = 6$  and  $m \geq 286$ ,  $n = 7$  and  $m \geq 215$ ,  $n = 8$  and  $m \geq 112$ ,  $n = 9$  and  $m \geq 141$ , or  $n = 10$  and  $m \geq 190$  then

$$g( F(n,m+12) ) = g( F(n,m) ).$$

(ii) If  $m = 2, m = 3$  and  $n \geq 2$ ,  $m = 4$  and  $n \geq 5$ ,  $m = 6$  and  $n \geq 15$ ,

or  $m = 10$  and  $n \geq 30$  then

$$g(F(n, m)) = n + m.$$

(iii) Let  $p = 4, 16$  or  $8$  according as  $m = 5, m = 7, 8$  or  $m = 9$ . Suppose that  $n \geq 8, 9, 10$  or  $15$  if  $m = 5, 7, 8,$  or  $9,$  respectively. Then

$$g(F(n+p, m)) = g(F(n, m)) + p.$$

**Conjecture C.** (i) For every positive integer  $n$ , there exists an integer  $M = M(n)$  for which  $g(F(n, m+12)) = g(F(n, m))$  if  $m \geq M$ .

(ii) For every positive even integer  $m$ , there exists integers  $p = p(m)$  and  $M = M(m)$  for which  $g(F(n+p, m)) = g(F(n, m)) + p$  if  $n \geq M$ .

We finally give some remarks on ER-game of star type on trees and propose a related problem. The Grundy number of every tree with order less than 10 is non-zero, and there exist 16 trees of order 10 and seven trees of order 11 whose Grundy numbers are equal to 0. These trees are given below. Let  $T_i$  denote a tree of order 10 or 11 whose Grundy number is 0, and let  $V(T_i) = \{1, 2, \dots, 9, a\}$  or  $\{1, 2, \dots, 9, a, b\}$ . If the order of  $T_i$  is 10, then  $T_i$  contains a set of edge  $F = \{12, 23, 34, 45, 56\}$ , and so we denote only  $F_i = E(T) - F$ .

$F_1 = \{67, 78, 89, 2a\}, F_2 = \{67, 78, 89, 3a\}, F_3 = \{67, 78, 79, 4a\}, F_4 = \{67, 58, 89, 2a\},$   
 $F_5 = \{67, 78, 79, 6a\}, F_6 = \{67, 78, 69, 4a\}, F_7 = \{67, 58, 89, 3a\}, F_8 = \{37, 78, 89, 1a\},$   
 $F_9 = \{37, 78, 89, 2a\}, F_{10} = \{67, 48, 29, 3a\}, F_{11} = \{37, 78, 29, 7a\}, F_{12} = \{47, 78, 29, 3a\},$   
 $F_{13} = \{47, 78, 79, 5a\}, F_{14} = \{67, 78, 79, 7a\}, F_{15} = \{67, 68, 69, 4a\}$  and  $F_{16} = \{47, 78, 29, 2a\}.$

If the order of  $T_i$  is 11, then the edge set  $E_i$  of  $T_i$  are given as follows:



$$\begin{aligned}
E_1 &= \{12,23,34,45,56,67,58,89,4a,ab\}, \quad E_2 = \{12,23,34,45,56,37,78,29,4a,4b\}, \\
E_3 &= \{12,23,34,45,56,57,78,59,4a,4b\}, \quad E_4 = \{12,23,34,45,46,67,48,89,3a,3b\}, \\
E_5 &= \{12,23,34,45,56,57,58,59,4a,4b\}, \quad E_6 = \{12,13,14,15,16,67,68,89,7a,7b\} \text{ and} \\
E_7 &= \{12,23,35,56,34,37,78,29,2a,2b\}.
\end{aligned}$$

**Problem** Characterize trees whose Grundy numbers are equal to 0.

ER-game of path type (i.e.  $\mathcal{K}$  is the set of all paths ) will be deal with other paper. Is it possible to solve ER-games of the following  $\mathcal{K}$  type on certain class of graphs:  $\mathcal{K}$  is the set of all cycles,  $\mathcal{K}$  is the set of all trees,  $\mathcal{K}$  is the set of all matchings,  $\mathcal{K}$  is the set of all forests, and so on.

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