数理解析研究所講究録
第 835 巻 1993 年 237-243

A POTENTIAL OF FUZZY RELATIONS WITH A LINEAR STRUCTURE : THE UNBOUNDED CASE

千葉大学 教養部　吉田祐治 (Yuji YOSHIDA)
千葉大学 教養部　安田正実 (Masami YASUDA)
千葉大学 理学部　中神潤一 (Jun-ichi NAKAGAMI)
千葉大学 教育学部 蔵野正美 (Masami KURANO)

Abstract: This paper is a sequel of Yoshida, et al.[8], in which the potential theory for linear fuzzy relations on the positive orthant R_+^n is considered in the class of fuzzy sets with compact support under the contractive assumption. In this paper, potential treatment for unbounded fuzzy sets is developed without the assumption of contraction and compactness. The objective of this paper is to give the existence and the characterization of potential for linear fuzzy relations under some reasonable conditions.

Also, introducing the partial order in fuzzy sets, we prove Riesz decomposition theorem in the fuzzy potential theory. The proofs are shown by using only the linear structure and the monotonicity of fuzzy relations. In the one-dimensional case, the potential and its α-cuts are explicitly calculated. Numerical examples are given to comprehend further discussions.

Keyword: Fuzzy potential; superharmonic fuzzy set; partial order; linear structure; fuzzy relation; fuzzy relational equation.

1. Introduction.

A potential theory for linear fuzzy relations on the positive orthant R_+^n of an n-dimensional Euclidean space is developed. Yoshida, et al.[8] have introduced a linear structure for fuzzy relations and considered the potential theory in the class of fuzzy sets with a compact support. In this paper, the unbounded case is considered. We shall develop the relevant potential theory using only the linear structure and the monotonicity of fuzzy relations.

We adopt the notations in [8]. Let R^n be an n-dimensional Euclidean space with a basis $\{e_1, e_2, \cdots, e_n\}$. Let w_i be an orthogonal projection from R^n to the subspace $\{\lambda e_i \mid \lambda \in R^1\}$. Then, for $x \in R^n, x = \sum_{i=1}^n w_i(x)e_i$. We put a norm $\|\cdot\|$ and a metric d by $\|x\| = (\sum_{i=1}^n (w_i(x))^2)^{\frac{1}{2}}$ and $d(x,y) = \|x-y\|$ for $x, y \in R^n$. Let $R_+^n := \{x \in R^n \mid w_i(x) \geq 0 \text{ for all } i = 1, 2, \cdots, n\}$ be a positive orthant of R^n. (R_+^n, d) is a complete separable metric space. Let $\mathcal{C}(R_+^n)$ be a collection of all the closed convex subsets of R_+^n. We put $A + B := \{x + y \mid x \in A \text{ and } y \in B\}$ $(A, B \subset R_+^n)$ and $\lambda A := \{\lambda x \mid x \in A\}$ $(A \subset R_+^n, \lambda \geq 0)$. Especially, $\phi + A = A + \phi = \phi$ $(A \subset R_+^n)$ and $\lambda\phi = \phi$ $(\lambda \geq 0)$.

Definition. (Partial order) For $A, B \in \mathcal{C}(R_+^n)$,

$$A \succeq B \text{ means that there exists } C \in \mathcal{C}(R_+^n) \text{ such that } A = B + C.$$

We represent a fuzzy set on R^n_+ by its membership function $\tilde{s} : R^n_+ \to [0,1]$ (see Novák[6] and Zadeh[9]). For any fuzzy set $\tilde{s}$ on R^n_+ and $\alpha \in [0,1]$, the α-cut is defined by $\tilde{s}_\alpha := \{x \in R^n_+ \mid \tilde{s}(x) \geq \alpha\}$ $(\alpha > 0)$ $\tilde{s}_0 := c\ell\{x \in R^n_+ \mid \tilde{s}(x) > 0\}$, where $c\ell$ means the closure of a set. We call $\tilde{s}(\in \mathcal{F}(R^n_+))$ to be convex if its α-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(R^n_+)$ be a collection of all the convex fuzzy sets $\tilde{s}$ on R^n_+ which are upper semi-continuous.

The linear structure of $\tilde{s}$ on $\mathcal{F}(R^n_+)$ is introduced in [8]. For fuzzy sets $\tilde{s}, \tilde{r}$ and a scalar λ,

$$(\tilde{s} + \tilde{r})(x) := \sup_{y+z=x,\ y,z \in R^n_+} \{\tilde{s}(y) \wedge \tilde{r}(z)\},$$

$$(\lambda\tilde{s})(x) := \begin{cases} \tilde{s}(x/\lambda) & \text{if } \lambda > 0, \\ I_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases} \quad x \in R^n_+,$$

where $\lambda \wedge \mu := \min\{\lambda, \mu\}$ for scalars λ, μ, and $I_A(\cdot)$ is the classical characteristic function of $A(\subset R^n_+)$. We note that $(\tilde{s} + \tilde{r})_\alpha = \tilde{s}_\alpha + \tilde{r}_\alpha$ and $(\lambda\tilde{s})_\alpha = \lambda\tilde{s}_\alpha$ for fuzzy sets $\tilde{s}, \tilde{r}$, $\lambda \in R^1_+$ and $\alpha \in [0,1]$ (c.f. Madan, et al.[4]).

Lemma 1.1. (Linearity) Let $\tilde{s}, \tilde{r}, \tilde{p} \in \mathcal{F}(R^n_+)$ and $\lambda, \mu \in R^1_+$. Then :

(i) $\tilde{s} + \tilde{r} = \tilde{r} + \tilde{s}$, (ii) $(\lambda(\mu\tilde{s})) = (\mu(\lambda\tilde{s}))$, (iii) $(\tilde{s} + \tilde{r}) + \tilde{p} = \tilde{s} + (\tilde{r} + \tilde{p})$.

Assumption A. The fuzzy relation $\tilde{q}$ satisfies the following conditions $(A1)$ – $(A5)$:

$(A1)$ $\tilde{q}$ is continuous on $R^n_+ \times R^n_+ - \{(0,0)\}$. $(A2)$ $\tilde{q}(\cdot, y) \in \mathcal{F}(R^n_+)$ for all $y \in R^n_+$.

$(A3)$ $\sup_{x \in R^n_+} \tilde{q}(x,y) = 1$ for all $y \in R^n_+$. $(A4)$ $\tilde{q}(\cdot, 0) = I_{\{0\}}$ and $\tilde{q}(0, \cdot) = I_{\{0\}}$.

$(A5)$ $\tilde{q}(\cdot, \lambda y + \mu z) = \lambda\tilde{q}(\cdot, y) + \mu\tilde{q}(\cdot, z)$ for all $y, z \in R^n_+$ and $\lambda, \mu \in R^1_+$.

Assumption $(A5)$ of the linear structure is introduced by [8] firstly. When a fuzzy set $\tilde{q}(\cdot, e_i)(\in \mathcal{F}(R^n_+))$ is given for each $e_i (i = 1, 2, \cdots, n)$, we can construct a fuzzy relation $\tilde{q}$ on R^n_+ which satisfies Assumption A, by defining $\tilde{q}(\cdot, y) := \sum_{i=1}^n w_i(y)\tilde{q}(\cdot, e_i)$, $y \in R^n_+$ (see [8]).

We also introduce the transition : For $\tilde{p} \in \mathcal{F}(R^n_+)$,

$$\tilde{q}(\tilde{p})(x) := \sup_{y \in R^n_+} \{\tilde{q}(x,y) \wedge \tilde{p}(y)\} \ (x \in R^n_+).$$

Then we can inductively define the sequence of fuzzy states $\{\tilde{q}^k(\tilde{p})\}_{k=0}^\infty$ by $\tilde{q}^0(\tilde{p}) := \tilde{p}$ and $\tilde{q}^k(\tilde{p}) := \tilde{q}(\tilde{q}^{k-1}(\tilde{p}))$ $(k = 1, 2, \cdots)$. If *formal* infinite sums $Q(\tilde{p}) := \sum_{k=0}^\infty \tilde{q}^k(\tilde{p})$ can be defined, we call it a fuzzy potential or simply a potential.

For $\alpha \in [0,1]$, we also define a map $\tilde{q}_\alpha$ on $\mathcal{C}(R^n_+)$ by

$$\tilde{q}_\alpha(D) := \begin{cases} \{x \in R^n_+ \mid \tilde{q}(x,y) \geq \alpha \text{ for some } y \in D\} & \text{for } \alpha > 0,\ D \in \mathcal{C}(R^n_+)(D \neq \phi) \\ c\ell\{x \in R^n_+ \mid \tilde{q}(x,y) > 0 \text{ for some } y \in D\} & \text{for } \alpha = 0,\ D \in \mathcal{C}(R^n_+)(D \neq \phi) \\ \phi & \text{for } \alpha \in [0,1],\ D = \phi. \end{cases}$$

Then, Assumption $(A5)$ means that

$$\tilde{q}_\alpha(\lambda y + \mu z) = \lambda\tilde{q}_\alpha(y) + \mu\tilde{q}_\alpha(z) \quad \text{for all } y, z \in R^n_+ \text{ and } \lambda, \mu \in R^1_+ \tag{2.1}$$

where $\tilde{q}_\alpha(y) = \tilde{q}_\alpha(\{y\})$. Note that $\tilde{q}_\alpha(D) = \bigcup_{y \in D} \tilde{q}_\alpha(y)$ for all $D \in \mathcal{C}(R^n_+)$ holds.

For any $D \in \mathcal{C}(R^n_+)$, it follows from the continuity of $\tilde{q}$ and (2.1) that $\tilde{q}_\alpha(D) \in \mathcal{C}(R^n_+)$ which implies $\tilde{q}_\alpha : \mathcal{C}(R^n_+) \to \mathcal{C}(R^n_+)$. Inductively we define the map $\tilde{q}^k_\alpha : \mathcal{C}(R^n_+) \to \mathcal{C}(R^n_+) (k = 0, 1, 2, \cdots)$ by $\tilde{q}^0_\alpha$ is an identity map and $\tilde{q}^k_\alpha = \tilde{q}_\alpha(\tilde{q}^{k-1}_\alpha)$ $(k = 1, 2, \cdots)$.

Lemma 1.3. (See Kurano, et al.[3].) For $\tilde{p} \in \mathcal{F}(R^n_+)$, it holds that

$$(\tilde{q}^k(\tilde{p}))_\alpha = \tilde{q}^k_\alpha(\tilde{p}_\alpha), \qquad k = 0, 1, 2, \cdots, \ \alpha \in [0, 1].$$

Definition. (Partial order) For $\tilde{s}, \tilde{r} \in \mathcal{F}(R^n_+)$,

$$\tilde{s} \succeq \tilde{r} \text{ means that there exists } \tilde{p} \in \mathcal{F}(R^n_+) \text{ such that } \tilde{s} = \tilde{r} + \tilde{p}.$$

Lemma 1.4. (Partial order) Let $\tilde{s}, \tilde{r}, \tilde{p} \in \mathcal{F}(R^n_+)$. Then,
(i) $\tilde{s} \succeq \tilde{r}$ and $\tilde{r} \succeq \tilde{s}$, then $\tilde{s} = \tilde{r}$, (ii) $\tilde{s} \succeq \tilde{r}$ and $\tilde{r} \succeq \tilde{p}$, then $\tilde{s} \succeq \tilde{p}$.

Lemma 1.5. (Monotonicity) For $\tilde{s}, \tilde{r} \in \mathcal{F}(R^n_+)$, The following (i) and (ii) hold :
(i) If $\tilde{s} \succeq \tilde{r}$, then $\tilde{s}_\alpha \succeq \tilde{r}_\alpha$ for all $\alpha \in [0, 1]$. (ii) If $\tilde{s} \succeq \tilde{r}$, then $\tilde{q}(\tilde{s}) \succeq \tilde{q}(\tilde{r})$.

2. Preliminary lemmas.

Definition. (Convergence) For $\{A_k\}_{k=1}^\infty \subset \mathcal{C}(R^n_+)$ and $A \in \mathcal{C}(R^n_+)$,

$$\lim_{k\to\infty} A_k = A$$

means that $\overline{\lim}_{k\to\infty} A_k = \underline{\lim}_{k\to\infty} A_k = A$ where $\overline{\lim}_{k\to\infty} A_k := \{x \in R^n_+ \mid \underline{\lim}_{k\to\infty} d(x, A_k) = 0\}$, $\underline{\lim}_{k\to\infty} A_k := \{x \in R^n_+ \mid \overline{\lim}_{k\to\infty} d(x, A_k) = 0\}$ and $d(x, D) := \sup_{y\in D} d(x, y)$, $D \in \mathcal{C}(R^n_+)$.

Lemma 2.1. (Non-increasing case) Let $\{A_k\}_{k=1}^\infty \subset \mathcal{C}(R^n_+)$. If $A_k \succeq A_{k+1} (k = 1, 2, \cdots)$, then there exists $A \in \mathcal{C}(R^n_+)$ with $\lim_{k\to\infty} A_k = A$.

Lemma 2.2. (Non-decreasing case) Let $\{A_k\}_{k=1}^\infty \subset \mathcal{C}(R^n_+)$. If $A_k \preceq A_{k+1} (k = 1, 2, \cdots)$, then there exists $A \in \mathcal{C}(R^n_+)$ with $\lim_{k\to\infty} A_k = A$.

Lemma 2.3. (Linear structure of $\tilde{q}$) It holds that

$$\tilde{q}_\alpha(A + B) = \tilde{q}_\alpha(A) + \tilde{q}_\alpha(B) \quad \text{for } \alpha \in [0, 1]. \text{ and } A, B \in \mathcal{C}(R^n_+).$$

Theorem 2.1. Let $\{A_k\}_{k=1}^\infty \subset \mathcal{C}(R^n_+)$ and $A \in \mathcal{C}(R^n_+)$ such that $\{A_k\}_{k=1}^\infty$ is non-increasing (non-increasing) with respect to the order $\succeq$ on $\mathcal{C}(R^n_+)$ and $\lim_{k\to\infty} A_k = A$. Then it holds that

$$\lim_{k\to\infty} \tilde{q}_\alpha(A_k) = \tilde{q}_\alpha(A) \quad \text{for } \alpha \in (0, 1].$$

Lemma 2.4. Let $\{A_\alpha \mid \alpha \in [0, 1]\} \subset \mathcal{C}(R^n_+)$ such that $A_{\alpha'} \supset A_\alpha$ for $0 < \alpha' < \alpha \le 1$. Then it holds that

$$\tilde{q}_\alpha(\lim_{\alpha'\uparrow\alpha} A_{\alpha'}) = \lim_{\alpha'\uparrow\alpha} \tilde{q}_{\alpha'}(A_{\alpha'}) \quad \text{for } \alpha \in (0, 1].$$

Lemma 2.5. (c.f.[6]) We suppose that a family of subsets $\{A_\alpha \mid \alpha \in [0, 1]\} (\subset \mathcal{C}(R^n_+))$ satisfies the following conditions (i) and (ii):

(i) $A_\alpha \subset A_{\alpha'}$ for $0 \leq \alpha' < \alpha \leq 1$. (ii) $\lim_{\alpha' \uparrow \alpha} A_{\alpha'} = A_\alpha$ for $\alpha \in (0, 1]$. Then,

$$\tilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A_\alpha}(x)\}, \ x \in R^n_+$$

satisfies $\tilde{s} \in \mathcal{F}(R^n_+)$ and $\tilde{s}_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$.

3. General potential theorems.

In this section we shall show the existence of potentials in $\mathcal{F}(R^n_+)$, which are defined formally in Section 1. Further we shall develop a fuzzy potential theory.

Definition. For $\{\tilde{s}_k\}_{k=0}^\infty \subset \mathcal{F}(R^n_+)$ and $\tilde{r} \in \mathcal{F}(R^n_+)$,

$$\lim_{k \to \infty} \tilde{s}_k = \tilde{r} \text{ means that } \tilde{s}_{k,\alpha} \to \tilde{r}_\alpha (k \to \infty) \text{ for all } \alpha \in [0, 1].$$

Assumption B. $\tilde{q}_1(e_i) = \{e_i\}$ for $i = 1, 2, \cdots, n$.

Note that Assumption B implies $\tilde{q}_1(A) = A$ for all $A \in \mathcal{C}(R^n_+)$. It is more restrictive but the contractivity of $\tilde{q}$ is excluded(c.f. [8]).

Lemma 3.1. Suppose Assumption B holds. For non-empty $A \in \mathcal{C}(R^n_+)$,

(i) If $0 \notin A$, then $\sum_{k=0}^\infty \tilde{q}_1^k(A) = \phi$.

(ii) If A satisfies $A \cap E_i \neq \phi$ for all $i = 1, 2, \cdots, n$, then $\lim_{k \to \infty} \tilde{q}_\alpha^k(A) = R^n_+$ for $0 < \alpha < 1$, where $E_i := \{\lambda e_i \mid 0 < \lambda < \infty\}$ for $i = 1, 2, \cdots, n$.

(iii) If A satisfies $A \cap E_i \neq \phi$ for all $i = 1, 2, \cdots, n$, then $\sum_{k=1}^\infty \tilde{q}_\alpha^k(A) \cap E_i \neq \phi$ for all $0 < \alpha < 1$ and $i = 1, 2, \cdots, n$.

Let $\mathcal{F}^*(R^n_+)$ be the set of all $\tilde{p} \in \mathcal{F}(R^n_+)$ such that $0 \notin \tilde{p}_1$ and $\tilde{p}_\alpha \cap E_i \neq \phi$ for all $\alpha \in (0, 1)$ where E_i is defined in the above. The next theorem says that for any $\tilde{p} \in \mathcal{F}^*(R^n_+)$ its potential $Q(\tilde{p})$ is well-defined.

Theorem 3.1. Suppose Assumption B holds. For any $\tilde{p} \in \mathcal{F}^*(R^n_+)$, the potential $\tilde{u} := Q(\tilde{p})$ exists in $\mathcal{F}(R^n_+)$ and $\tilde{u}$ satisfies the following fuzzy relational equation:

$$\tilde{u} = \tilde{p} + \tilde{q}(\tilde{u}). \tag{3.2}$$

Definition. For $\tilde{s} \in \mathcal{F}(R^n_+)$, $\tilde{s}$ is called superharmonic (harmonic respectively) provided that

$$\tilde{s} \succeq \tilde{q}(\tilde{s}) \quad (\tilde{s} = \tilde{q}(\tilde{s})).$$

Theorem 3.2. (Decomposition) Suppose Assumption B holds. Let a superharmonic fuzzy set $\tilde{s} \in \mathcal{F}(R^n_+)$ satisfy $\tilde{s}_\alpha \cap E_i \neq \phi$ for all $\alpha \in (0, 1)$ and $i = 1, 2, \cdots, n$. Then,

(i) $\lim_{k \to \infty} \tilde{q}^k(\tilde{s})_\alpha = R^n_+$ for all $\alpha \in [0, 1)$.

(ii) There exist a potential $\tilde{u}$ and a harmonic $\tilde{h}$ such that $\tilde{s} = \tilde{u} + \tilde{h}$.

(iii) If $\tilde{s} \succeq \tilde{p} + \tilde{q}(\tilde{s})$ for some $\tilde{p} \in \mathcal{F}^*(R^n_+)$, then $\tilde{s} \succeq Q(\tilde{p})$.

4. One-dimensional case

In this section we consider a fuzzy potential of fuzzy number on $R_+ := R^1_+$. Using the results in Section 3 and [8], we could treat with the contractive and non-expansive examples simultaneously.

We will calculate several examples, which is related to the existence of the potential. Let us denote that $\tilde{u}_\alpha := \sum_{k=0}^{l} \tilde{q}^k(\tilde{p}) = Q_l(\tilde{p})$.

Example 4.1. First we consider the example which satisfies Assumption A and B. Let a fuzzy set $\tilde{q}(\cdot, 1)$ by

$$\tilde{q}(x,1) = \begin{cases} 1 - 2|x-1|, & 1/2 \le x \le 3/2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

$$\tilde{q}(x,y) = \begin{cases} \tilde{q}(x/y,1), & x \ge 0 \text{ and } y > 0, \\ I_{\{0\}}(x), & x \ge 0 \text{ and } y = 0. \end{cases} \tag{4.3}$$

$$\tilde{p}(x) = \begin{cases} 1 - |\frac{5}{2}x - 1|, & 0 \le x \le 4/5 \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

$$\tilde{u}(x) = 5x/(4+5x), \quad x \ge 0.$$

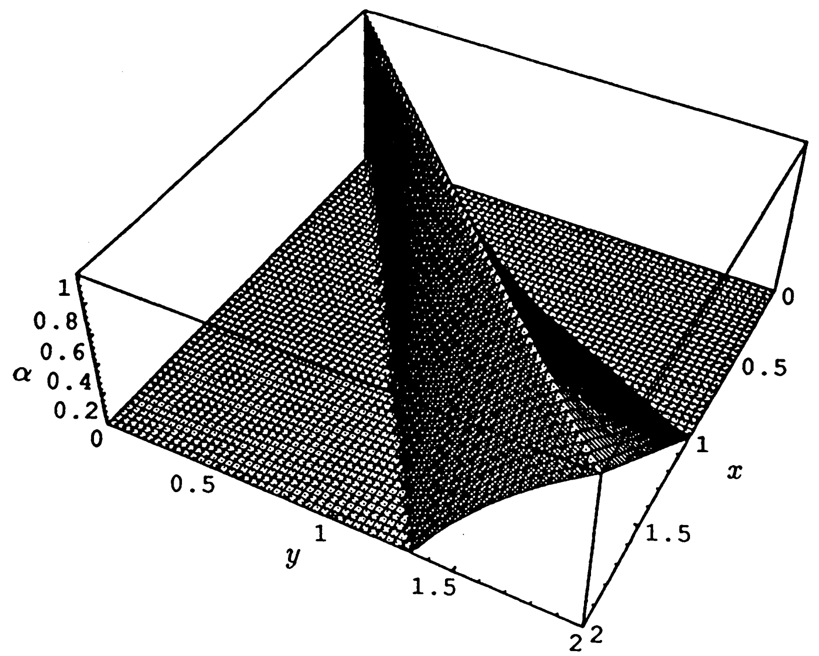

Figure 1. The fuzzy relation $\tilde{q}(x,y)$ of (4.2), (4.3).

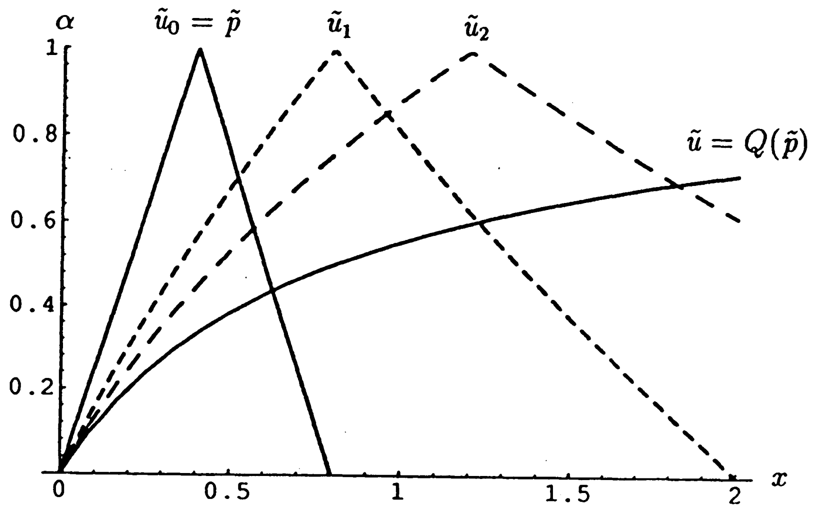

Figure 2. The potential $\tilde{u}$ for Example 4.1.

Example 4.2. Next we consider an example where $\tilde{q}_\alpha$ is contractive for $\alpha > 1/2$ (see [8]) and not contractive for $\alpha < 1/2$. The fuzzy relation $\tilde{q}$ does not satisfy Assumption B. Let a fuzzy set $\tilde{q}(\cdot, 1)$ by

$$\tilde{q}(x,1) = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 3/2 - x, & 1/2 < x \leq 3/2 \\ 0, & \text{otherwise.} \end{cases}$$

$$\tilde{u}(x) = \begin{cases} 10x/(4+5x), & 0 \leq x \leq 4/5 \\ (8+5x)/(4+10x), & 4/5 < x. \end{cases}$$

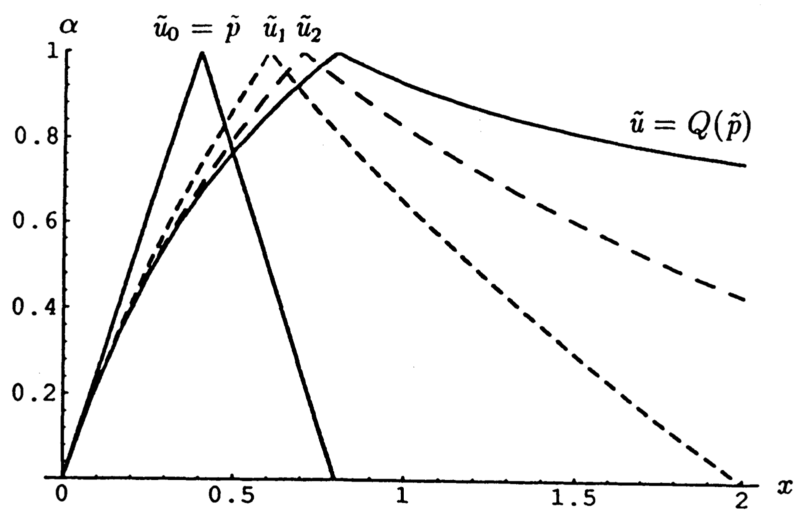

Figure 3. The potential $\tilde{u}$ for Example 4.2.

Example 4.3. Finally we consider an example which is not contractive. The fuzzy relation $\tilde{q}$ does not satisfy Assumption B.

$$\tilde{q}(x,1) = \begin{cases} x - 1/2, & 1/2 \le x \le 3/2 \\ 4 - 2x, & 3/2 < x \le 2 \\ 0, & \text{otherwise,} \end{cases}$$
$$\tilde{u}(x) = 5x/(4+10x), \quad x \ge 0.$$

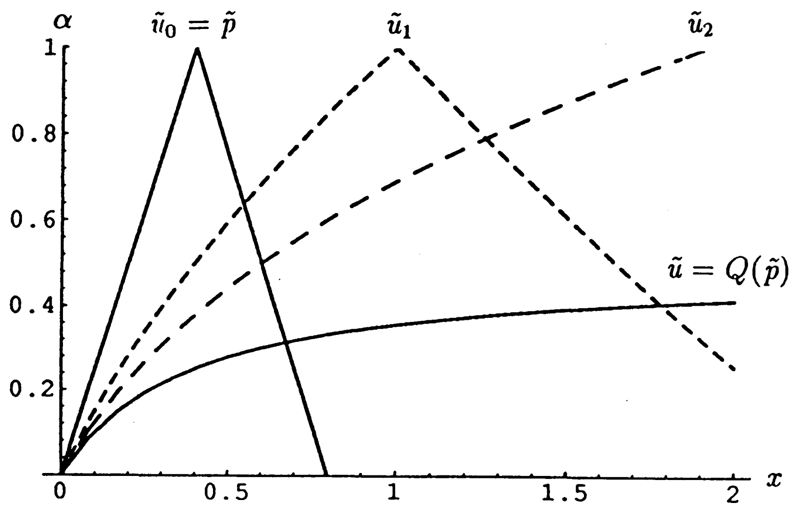

Figure 4. The potential $\tilde{u}$ for Example 4.3.

References

[1] D.P.Bertsekas and S.E.Shreve, *Stochastic Optimal Control, the discrete time case*, Academic Press, New York, 1978.

[2] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, *A Limit Theorem in Some Dynamic Fuzzy Systems*, to appear in *Fuzzy sets and systems.*

[3] K.Kuratowski, *Topology I*, Academic Press, New York, 1966.

[4] L.M.Puri and D.A.Palescu, *The concept of normality for fuzzy random variables, Annals of probability*, **13** 1985, pp. 1373-1379.

[5] S.Nanda, *On sequences of fuzzy numbers, Fuzzy Sets and Systems*, **33** 1989, pp. 123-126.

[6] V.Novàk, *Fuzzy sets and their applications* Adam Hilder, 1989.

[7] E.Nummelin, *General Irreducible Markov Chain and Non-negative Operators*, Cambridge University Press, London, 1984.

[8] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, *A Potential of Fuzzy Relations with a Linear Structure : The Contractive Case*, submitted.

[9] L.A.Zadeh, *Fuzzy sets, Information and Control*, **8** 1965, pp. 338-353.