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Riemann-Hilbert factorization and Fredholm property of differential operators

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1. Introduction

In 1974 Kashiwara-Kawai-Sjöstrand showed the sufficient condition for the convergence of all formal power series solutions for the following linear partial differential operators of regular singular type with analytic coefficients in some neighborhood of the origin of $\mathbf{C}^n$ $(n \geq 2)$

$$\mathcal{L}(x, D) = \sum_{|\alpha|=|\beta|\leq m} x^{\alpha} a_{\alpha\beta} D^{\beta}, \tag{1.1}$$

where m is a positive integer and $a_{\alpha\beta}$'s are complex constants. Here we use the standard notations of multi indices, $D^{\beta} = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$ and $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \cdots x_n) \in \mathbf{C}^n$. They proved the following result.

Theorem 1.1. (cf. [3]) *Suppose that the following condition*

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} z^{\alpha} \bar{z}^{\beta} \neq 0, \tag{1.2}$$

is satisfied for any $z \in \mathbf{C}^n \setminus \{0\}$, where $z^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\bar{z}^{\beta} = \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$. Then, for any $f(x)$ analytic at the origin all formal power series solutions $u(x)$ of the equation $\mathcal{L}(x, D)u(x) = f(x)$ converges in some neighborhood of the origin.

They proved the result for somewhat more general operators than (1.1) admitting perturbations. In the following, we call (1.2) as a K-K-S condition.

Inspired from this theorem, we shall study in this paper the Fredholm property in the space of (formal) Gevrey classes for linear partial differential operators with analytic coefficients in a neighborhood of the origin of $\mathbf{C}^2$. We consider regular and irregular singular type operators including (1.1). We show that such a property is characterized by the Riemann-Hilbert factorization condition for the Toeplitz symbol on the two dimensional torus in $\mathbf{C}^2$. Here the Toeplitz symbols are introduced in a nautural way in connection with the filtration with respect to the Gevrey order and it coincides with the symbol given by Kashiwara-Kawai-Sjöstrand in some special cases. (cf. (1.2) and (2.7)). Moreover, we can give an alternative proof of Theorem 1.1 in case $n = 2$.

As to the geometrical relations between formal Gevrey spaces and the operators we refer the reader to [9] and [10].

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2. Statement of the results

Let $w_j > 0$ $(j = 1, 2)$ and $s > 0$. We set $w = (w_1, w_2)$. Let $\mathbf{C}[[x]]$ be the set of all formal power series

$$\mathbf{C}[[x]] := \left\{ u(x)\, ;\, u(x) = \sum_{\eta \in \mathbf{N}^2} u_\eta x^\eta / \eta! \right\}.$$

If we denote by $\mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\})$ the set of holomorphic functions on a domain $\{|x_1| < w_1\} \times \{|x_2| < w_2\} \subset \mathbf{C}^2$, then we define the class of entire and Gevrey spaces $\mathcal{G}^s_w$ by

$$\mathcal{G}^s_w = \left\{ u(x) = \sum u_\eta \frac{x^\eta}{\eta!} \in \mathbf{C}[[x]]\, ;\, \sum_\eta u_\eta \frac{x^\eta}{|\eta|!^s} \in \mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\}) \right\}, \quad (2.1)$$

where factorial is understood as the gamma function, $r! := \Gamma(r+1)$ for $r \geq 0$. $\mathcal{G}^s_w$ can be seen as a Frechét space by the following isomorphism of Frechét spaces

$$\mathbf{C}[[x]] \supset \mathcal{G}^s_w \underset{\sim}{\xrightarrow{\text{Borel transf.}}} \mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\}), \quad (2.2)$$

where the Borel transformation is defined by

$$\mathcal{G}^s_w \ni \sum_{\eta \in \mathbf{N}} u_\eta \frac{x^\eta}{\eta!} \overset{\sim}{\mapsto} \sum_{\eta \in \mathbf{N}} u_\eta \frac{x^\eta}{|\eta|!^s} \in \mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\}). \quad (2.3)$$

We note that $\mathcal{G}^s_w$ is equal to formal Gevrey space, the class of locally analytic functions and entirely analytic functions with finite order in case $s > 1$, $s = 1$ and $s < 1$, respectively. (cf. Lemma 3.1 which follows).

Let $P \equiv P(x, D_x)$ be a partial differential operator of finite order with holomorphic coefficients in a neighbourhood of the origin of $\mathbf{C}^2$ and write it in the form,

$$P(x, D_x) = \sum_{\beta \in \mathbf{N}^2, |\beta| \leq m} a_\beta(x) D^\beta, \quad (2.4)$$

where $a_\beta(x)$ is an analytic function of x in some neighborhood of the origin.

By substituting the Taylor expansion of $a_\beta(x)$, $a_\beta(x) = \sum_\alpha a_{\alpha\beta} x^\alpha$ in (2.4) we have the expression

$$P(x, D) = \sum_{\alpha, \beta} a_{\alpha\beta} x^\alpha D^\beta. \quad (2.5)$$

For $x^\alpha D^\beta$ we define the s-Gevrey order of $x^\alpha D^\beta$ by

$$\operatorname{ord}_s x^\alpha D^\beta := |\beta| + (1-s)(|\alpha| - |\beta|). \quad (2.6)$$

Then the s-Gevrey order of P in (2.5) is defined by

$$\operatorname{ord}_s P := \sup_{\alpha, \beta} \{ |\beta| + (1-s)(|\alpha| - |\beta|)\ ;\ a_{\alpha\beta} \neq 0 \}.$$

Here and in what follows we always assume that the s-Gevrey order of $P(x, D)$ is finite. This implies that P is of polynomial coefficients in case $s < 1$.

We shall define the Toeplitz symbol associated with $P(x, D)$ by

$$L_s(z;\xi) := \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=\mathrm{ord}_s P} a_{\alpha\beta} z^{\alpha-\beta} w^{\alpha-\beta} \xi^{\alpha}, \qquad \xi \in \mathbf{R}^2. \tag{2.7}$$

We define the two dimensional torus $\mathbf{T}^2$ by $\mathbf{T}^2 = \{(z_1, z_2) \in \mathbf{C}^2 ; |z_1| = 1, |z_2| = 1\}$. Then we can prove the following

Theorem 2.1. *The operator $P : \mathcal{G}^s_w \longrightarrow \mathcal{G}^s_w$ is Fredholm of index zero in the sense that the mapping has the same finite dimensional kernel and cokernel if the following conditions are satisfied.*

$$L_s(z, \xi) \neq 0 \quad \forall (z_1, z_2) \in \mathbf{T}^2, \ \forall \xi \in \mathbf{R}^2, \ |\xi| = 1, \ \xi \geq 0. \tag{2.8}$$

$$\mathrm{ind}_1 L_s = \mathrm{ind}_2 L_s = 0. \tag{2.9}$$

Here $\mathrm{ind}_1\, L_s$ (resp. $\mathrm{ind}_2\, L_s$) is defined by

$$\mathrm{ind}_1 L_s = \frac{1}{2\pi i} \oint_{|\zeta_1|=1} d \log L_s(\zeta_1, z_2, \eta). \tag{2.10}$$

Remarks. (a) The conditions (2.8) and (2.9) are equivalent to a Riemann-Hilbert factorization condition with respect to a certain closed subspace $G^s_{w,\xi}(\mu)$ of $G^s_w(\mu)$ which is isomorphic to the Hardy space and will be defined in the proof of Theorem 2.1. We have to note that these conditions are necessary and sufficient condition for the Fredholmness of a Toeplitz operator T_ξ on $G^s_{w,\xi}(\mu)$ which will be reduced from the mapping $P : \mathcal{G}^s_w \to \mathcal{G}^s_w$. (cf. Theorem 4.1.)

(b) We note that the right-hand side of (2.10) is an integer-valued continuous function of z_2 and η. Because the sets $|z_2| = 1$ and $|\eta| = 1$ are connected the integral (2.10) is constant. Hence the right-hand side is independent of z_2 and η. We write this quantity by $\mathrm{ind}_1 L_s$. We similarly define $\mathrm{ind}_2 L_s$.

Corollary 2.2. *Suppose that the K-K-S condition (1.2) is satisfied. Assume that $n = 2$. Then the operator $\mathcal{L} : \mathcal{G}^1_w \longrightarrow \mathcal{G}^1_w$ is a Fredholm operator of index zero. Especially, for $f(x)$ analytic in a neighbourhood of the origin, if formal power series $f(x)$ satisfies $P(x, D)u(x) = f(x)$ then $u(x)$ is also analytic in a neighbourhood of the origin.*

Proof. We shall show that (2.8) and (2.9) with $s = 1$. Let $p_K := \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} z^\alpha \bar{z}^\beta$. Suppose that $\xi_1\xi_2 \neq 0$. If we make the change of variables $x \mapsto wx = (w_1\xi_1^{1/2} x_1, \xi_2^{1/2} w_2 x_2)$ in the original equation (1.1), then the symbol p_K in (1.2) can be replaced by

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} z^\alpha \bar{z}^\beta w^{\alpha-\beta} \xi^{\alpha/2-\beta/2}. \tag{2.11}$$

If we set $z \mapsto \xi^{1/2}z$ in (2.11) it follows that

$$L_s(z,\xi) = p_K, \tag{2.12}$$

if $z = (z_1, z_2) \in \mathbf{T}^2$, and $\xi \in \mathbf{R}^2$ satisfies that $|\xi| = 1, \xi \geq 0, \xi \neq (1,0)$ and $(0,1)$. Because both sides of (2.12) are analytic functions of z in some neighborhood of $\mathbf{T}^2$ and ξ in some neighborhood of $\xi \in \mathbf{R}^2, |\xi| = 1, \xi \geq 0$ (2.12) holds in case $\xi = (1,0)$ or $\xi = (0,1)$. Therefore, (2.8) is direct consequence of a K-K-S condition. On the other hand, if we make the deformation of a path $z_1 \mapsto \varepsilon z_1, z_2 \neq 0$ $(\varepsilon > 0)$ or $z_2 \mapsto \varepsilon z_2, z_1 \neq 0$ $(\varepsilon > 0)$ we have (2.9). Hence, by Theorem 2.1, $\mathcal{L}$ is a Fredholm operator (with an index zero) on $\mathcal{G}^1_w$ for every $w > 0$.

Next, let assume for a formal power series $u(x) \in \mathbf{C}[[x]]$ it holds that $\mathcal{L}u(x) = f(x)$ is analytic in a neighbourhood of the origin. We may assume $f(x) \in \mathcal{G}^1_w$ for some $w > 0$. Let $u_n(x)$ be a homogeneous polynomial of degree n. Then $\mathcal{L}u_n(x)$ is also homogeneous of degree n by the definition of operator $\mathcal{L}$. Hence, the basis of kernel of the mapping $\mathcal{L} : \mathcal{G}^1_w \to \mathcal{G}^1_w$ consists of finite numbers of homogeneous polynomials. Therefore for homogeneous polynomial $f_n(x)$ of degree n of sufficiently large n, there exists a unique homogeneous polynomial $u_n(x)$ of degree n satisfying $\mathcal{L}u_n(x) = f_n(x)$, since the uniquness of solutions implies the solvability. This implies that the mapping $\mathcal{L} : \mathbf{C}[[x]] \to \mathbf{C}[[x]]$ is also a Fredholm operator (with an index zero) and has the same dimensional kernel and cokernel with the operator $\mathcal{L}$ on $\mathcal{G}^s_w$. Therefore, the mapping

$$\mathcal{L} : \mathbf{C}[[x]]/\mathcal{G}^1_w \longrightarrow \mathbf{C}[[x]]/\mathcal{G}^1_w$$

is a Fredholm operator with an index zero. The above reasoning shows that this mapping is surjective, and therefore is injective. This proves the latter half of the corollary.

3. Preliminary lemmas

We define the class $G^s_w(\mu)$ $(\mu \in \mathbf{R})$ by

$$G^s_w(\mu) := \left\{ u = \sum_\eta u_\eta \frac{x^\eta}{\eta!} ; \sum_\eta \left(|u_\eta| \frac{w^\eta}{(|\eta| - \frac{\mu}{s})!^s} \right)^2 < \infty \right\}, \tag{3.1}$$

where factorial is understood as the gamma function, $r! := \Gamma(r+1)$ for $r \geq 0$ and where we set $(|\eta| - (\mu/s))! = 1$ if $|\eta| - (\mu/s) \leq 0$. $G^s_w(\mu)$ is a Banach space with the norm

$$\|u\| := \left(\sum_\eta \left(|u_\eta| \frac{w^\eta}{(|\eta| - \frac{\mu}{s})!^s} \right)^2 \right)^{1/2}.$$

Lemma 3.1. *Let the class $\mathcal{G}^s_w$ be defined by* (2.1). *Then we have*

$$\mathcal{G}^s_w = \text{proj} \lim_{r \uparrow w} G^s_r(\mu) \tag{3.2}$$

for every $\mu \in \mathbf{R}$.

Proof. Suppose that $u(x) \in G^s_r(\mu)$ for any $r < w$. Then we have $|u_\eta| \le Mr^{-\eta}(|\eta| - (\mu/s))!^s$ for some $M > 0$ independent of η. Therefore we have, for $|x_j| < r_j$,

$$\sum |u_\eta| \frac{|x|^\eta}{|\eta|!^s} \le M \sum r^{-\eta}|x^\eta| \frac{(|\eta| - \frac{\mu}{s})!^s}{|\eta|!^s}.$$

Clearly, the right-hand side converges for $|x| < r$. Because $r < w$ is arbitray we have $u \in \mathcal{G}^s_w$.

Conversely, suppose that $u = \sum u_\eta x^\eta/\eta! \in \mathcal{G}^s_w$. Then we have $U(x) := \sum_\eta u_\eta x^\eta/|\eta|!^s \in \mathcal{O}(\{|x_1| \le \rho_1\} \times \{|x_2| \le \rho_2\})$ for any $\rho < w$. By Caucy's formula we have

$$v_\eta := \frac{u_\eta}{|\eta|!^s} = \frac{1}{(2\pi i)^2} \oint_{|\zeta_1|=\rho_1} \oint_{|\zeta_2|=\rho_2} \frac{U(\xi)}{\xi^{\eta+1}} d\xi.$$

Hence we have the estimate $|v_\eta| \le M\rho^{-\eta}$ for some $M > 0$. Because $\rho < w$ is arbitrary we have $u \in G^s_r(\mu)$ for any $r < w$. □

Let X_j $(j = 1, 2)$ be a positive number and set $X = (X_1, X_2)$. We denote by $\mathcal{O}(|x| \le X)$ the set of holomorphic functions on $\{x \in \mathbf{C}^2\,;\, |x_j| < X_j, j = 1, 2\}$ and continuous on its closure. For $a(x) \in \mathcal{O}(|x| \le X)$, we put $\|a\|_X := \max_{|x_j| \le X_j} |a(x)|$. Then we have

Lemma 3.2. *Let* $s \ge 1$. *Assume that* $a(x) \in \mathcal{O}(|x| \le \rho w)$ $(\rho > 1)$, *then for any* $U(x) \in G^s_w(\mu)$, *we have* $a(x)U(x) \in G^s_w(\mu)$ *and there exists a constant* C *depending only on* μ *such that*

$$\|aU\| \le C \left(\frac{\rho}{\rho - 1}\right)^2 \|a\|_{\rho w} \|U\|. \tag{3.3}$$

Proof. We put $a(x) = \sum a_\gamma x^\gamma/\gamma! \in \mathcal{O}(|x| \le \rho w)$. Then by Cauchy's integral formula, we have $|a_\gamma| \le \|a\|_{w\rho}\gamma!/(\rho w)^\gamma (\gamma \in \mathbf{N}^2)$. We put $a(x)U(x) = \sum V_\beta x^\beta/\beta!$. Then we have

$$V_\beta = \sum_{0 \le \gamma \le \beta} a_\gamma U_{\beta-\gamma} \frac{\beta!}{(\beta - \gamma)!\gamma!}.$$

Hence we have, for $C_1 > 0$

$$\begin{aligned}
\sum_\beta \left(|V_\beta| \frac{w^\beta}{(|\beta| - \frac{\mu}{s})!^s}\right)^2 &\le \|a\|^2_{\rho w} \sum_\beta \left(\sum_{0 \le \gamma \le \beta} |U_{\beta-\gamma}| \frac{1}{(\rho w)^\gamma} \frac{\beta!}{(\beta - \gamma)!} \frac{w^\beta}{(|\beta| - \frac{\mu}{s})!^s}\right)^2 \\
&\le C_1 \|a\|^2_{\rho w} \sum_\beta \left(\sum_{0 \le \gamma \le \beta} |U_{\beta-\gamma}| \frac{1}{\rho^{|\gamma|}} \frac{w^{\beta-\gamma}}{(|\beta| - \frac{\mu}{s} - |\gamma|)!^s}\right)^2 \\
&\le C_1 \|a\|^2_{\rho w} \sum_\beta \left(\sum_\gamma \frac{1}{\rho^{|\gamma|}}\right) \left(\sum_\gamma \frac{1}{\rho^{|\gamma|}} \left(|U_{\beta-\gamma}| \frac{w^{\beta-\gamma}}{(|\beta| - \frac{\mu}{s} - |\gamma|)!^s}\right)^2\right)
\end{aligned}$$

$$\leq C_1\left(\frac{\rho}{\rho-1}\right)^2 \|a\|^2_{\rho w}\sum_\gamma \frac{1}{\rho^{|\gamma|}}\sum_{\beta\geq\gamma}\left(|U_{\beta-\gamma}|\frac{w^{\beta-\gamma}}{(|\beta-\gamma|-\frac{\mu}{s})!^s}\right)^2$$
$$\leq \left(\frac{\rho}{\rho-1}\right)^4 C_1\|a\|^2_{\rho w}\|U\|^2. \qquad \square$$

Lemma 3.3. *Let $\mu = |\beta| + (1-s)(|\alpha| - |\beta|)$ be the s-Gevrey order of $x^\alpha D^\beta$. Then the map $x^\alpha D^\beta : G^s_w(\mu) \longrightarrow G^s_w(0)$ is continuous. Moreover, for every $\varepsilon > 0$ the map $x^\alpha D^\beta : G^s_w(\mu+\varepsilon) \longrightarrow G^s_w(0)$ is a compact operator.*

Proof. We first show that for every $\kappa < \mu$ the injection $i; G^s_w(\mu) \to G^s_w(\kappa)$ is compact. Let $B \subset G^s_w(\mu)$ be a bounded set of $G^s_w(\mu)$. If we write $u = \sum_\eta u_\eta x^\eta/\eta! \in B$, then for each fixed η the set $\{u_\eta\,;\, u \in B\}$ is bounded. Hence, by the diagonal argument, we can choose a sequence $\{u^{(k)}\} \subset B$, $u^{(k)}(x) = \sum_\eta u^{(k)}_\eta x^\eta/\eta!$ such that for each η, $u^{(k)}_\eta \to u_\eta$ when $k \to \infty$. Moreover we have that

$$\sum_{|\eta|\geq N}\left(|u^{(k)}_\eta|\frac{w^\eta}{(|\eta|-\frac{\kappa}{s})!^s}\right)^2 \leq \max_{|\eta|\geq N}\frac{(|\eta|-\frac{\mu}{s})!^{2s}}{(|\eta|-\frac{\kappa}{s})!^{2s}}\sum_{|\eta|\geq N}\left(|u^{(k)}_\eta|\frac{w^\eta}{(|\eta|-\frac{\mu}{s})!^s}\right)^2$$
$$\leq K\max_{|\eta|\geq N}\frac{(|\eta|-\frac{\mu}{s})!^{2s}}{(|\eta|-\frac{\kappa}{s})!^{2s}} \longrightarrow 0 \quad (n\to\infty),$$

where $K > 0$ is independent of k and N. This proves that the sequence $\{u^{(k)}\}$ converges in $G^s_w(\mu)$.

In order to complete the proof we shall show that the map $x^\alpha D^\beta : G^s_w(\mu) \longrightarrow G^s_w(0)$ is continuous. By simple calculations

$$x^\alpha D^\beta\sum_\eta u_\eta\frac{x^\eta}{\eta!} = \sum u_\eta\frac{x^{\eta+\alpha-\beta}}{(\eta-\beta)!} = \sum u_{\eta+\beta-\alpha}\frac{x^\eta}{(\eta-\alpha)!}. \tag{3.4}$$

Hence we have

$$\sum_\eta\left(|u_{\eta+\beta-\alpha}|\frac{w^\eta}{|\eta|!^s}\frac{\eta!}{(\eta-\alpha)!}\right)^2 = \sum_\eta\left(|u_\eta|w^{\eta-\beta+\alpha}\frac{1}{(|\eta|-|\beta|+|\alpha|)!^s}\frac{(\eta-\beta+\alpha)!}{(\eta-\beta)!}\right)^2. \tag{3.5}$$

If η is sufficiently large the term $(\eta-\beta+\alpha)!/(\eta-\beta)!$ can be estimated from the above and from the below by η^α. Therefore we have

$$\frac{(|\eta|-\frac{\mu}{s})!^s}{(|\eta|-|\beta|+|\alpha|)!^s}\frac{(\eta-\beta+\alpha)!}{(\eta-\beta)!} \leq C|\eta|^{s(|\beta|-|\alpha|)-\mu}|\eta|^{|\alpha|} = C|\eta|^{s(|\beta|-|\alpha|)+|\alpha|-\mu} \tag{3.6}$$

for some constant C is independent of η. Because $s(|\beta|-|\alpha|)+|\alpha|-\mu = 0$ the right-hand side of (3.6) is bounded when $|\eta|$ tends to infinity. By (3.4), (3.5) and (3.6) we see that the map $x^\alpha D^\beta : G^s_w(\mu) \longrightarrow G^s_w(0)$ is continuous. $\square$

Let $p(\eta)$ be a function on N^2 such that

$$|p(\eta)| \leq C|\eta|^m, \forall \eta \in \mathrm{N}^2 \tag{3.7}$$

for some $C > 0$ and $m \geq 0$ independent of η. Then we define the Euler type pseudodifferential operator $p(\partial)$ on $G^s_w(\mu)$ by

$$p(\partial)u := \sum_\eta u_\eta p(\eta) x^\eta/\eta!, \quad u = \sum_\eta u_\eta x^\eta/\eta! \in G^s_w(\mu), \tag{3.8}$$

where we set $\partial = (\partial_1, \partial_2)$, $\partial_j = x_j(\partial/\partial x_j), j = 1, 2$. We note that if $p(\eta) = \eta_1 + \eta_2$, then $p(\partial) = \partial_1 + \partial_2$ is a so-called Euler type differential operator. Then we have

Lemma 3.4. *Let $p(\eta)$ be a function on N^2 such that $\sup_{|\eta|\geq N} |p(\eta)| \to 0$ when $N \to \infty$. Then the map $p(\partial) : G^s_w(\mu) \longrightarrow G^s_w(\mu)$ is a compact operator for every μ.*

The proof of this lemma follows exactly the same arguments of the former half of the proof of Lemma 3.3. Therefore we omit the proof.

4. Proof of Theorem 2.1.

Let m be an s-Gevrey oder of P. In view of Lemma 3.1 it is sufficient to prove that for any $r < w$ the map $P : G^s_r(m) \to G^s_r(0)$ is Fredholm of index 0. For every β we collect $\alpha \in \mathrm{N}^2$ such that $|\beta| + (1-s)(|\alpha| - |\beta|) < m$ and we denote the set by C_β. Since C_β is a subset of N^2, we can choose finite $\alpha^{(j)}$'s ($j = 1, \ldots, k$) from C_β such that C_β is contained in the union of sets $\alpha^{(j)} + \mathrm{N}^2$ for $j = 1, \ldots, k$. We choose the set of $\alpha^{(j)}$'s ($j = 1, \ldots, k$) for each β. By using this grouping of α we can write P in (2.5) in the following form

$$\begin{aligned} P(x, D_x) = &\sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} x^\alpha D^\beta + \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)<m,\alpha,\text{finite}} a_{\alpha\beta} x^\alpha D^\beta \\ &+ \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)<m,\alpha,\text{finite}} b_{\alpha\beta}(x) x^\alpha D^\beta =: P_0(x, D) + P_1(x, D) + P_2(x, D), \end{aligned} \tag{4.1}$$

where $a_{\alpha\beta}$ are complex constants and $b_{\alpha\beta}(x)$ are analytic functions of x. We note that $P_2(x, D) \equiv 0$ if $s < 1$ by assumption.

Because the s-Gevrey order of terms in P_1 is small than m, it follows from Lemma 3.3 that the map $P_1 : G^s_w(m) \longrightarrow G^s_w(0)$ is compact. On the other hand, since $s \geq 1$, it follows from Lemmas 3.2 and 3.3 that the map $P_2 : G^s_w(m) \longrightarrow G^s_w(0)$ is compact. Therefore we shall consider the Fredholmness of $P_0 : G^s_w(m) \longrightarrow G^s_w(0)$.

By using the identity of ordinary differential equations

$$t^k \frac{d^k}{dt^k} = t\frac{d}{dt}\left(t\frac{d}{dt} - 1\right) \cdots \left(t\frac{d}{dt} - k + 1\right), \quad k = 1, 2, \ldots,$$

we have

$$x^\beta \left(\frac{\partial}{\partial x}\right)^\beta = \prod_{j=1}^2 x_j \frac{\partial}{\partial x_j}\left(x_j\frac{\partial}{\partial x_j} - 1\right) \cdots \left(x_j\frac{\partial}{\partial x_j} - \beta_j + 1\right) =: p_\beta(\partial). \tag{4.2}$$

By substituting (4.2) into (4.1) we have

$$P_0(x,D) = \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta}x^\alpha D^\beta = \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta}x^{\alpha-\beta}p_\beta(\partial). \tag{4.3}$$

We set $<\eta>:= (1+|\eta|^2)^{1/2}$ and we denote by $<\partial>$ the Euler type pseudodifferential operator with symbol $<\eta>$ given by (3.8).

Let $\gamma \in \mathbf{Z}^2$ and let $u = \sum u_\eta x^\eta/\eta! \in G^s_w(\mu)$. We set $v_\eta := u_\eta w^\eta/(|\eta|-(\mu/s))!^s$. Then we have

$$x^\gamma <\partial>^{(s-1)|\gamma|} \sum u_\eta \frac{x^\eta}{\eta!} = x^\gamma <\partial>^{(s-1)|\gamma|} \sum v_\eta w^{-\eta}(|\eta| - \frac{\mu}{s})!^s \frac{x^\eta}{\eta!}$$

$$= \sum v_\eta w^{-\eta}(|\eta| - \frac{\mu}{s})!^s <\eta>^{(s-1)|\gamma|} \frac{x^{\eta+\gamma}}{\eta!}$$

$$= \sum v_{\eta-\gamma} w^{-\eta+\gamma}(|\eta| - |\gamma| - \frac{\mu}{s})!^s <\eta-\gamma>^{(1-s)|\gamma|} \frac{\eta!}{(\eta-\gamma)!}\frac{x^\eta}{\eta!}.$$

On the other hand, we see that $u \in G^s_w(\mu)$ if and only if the sequence $\{v_\eta\}$ is in $\ell_2 := \ell_2(\mathbf{Z}^2)$, the set of square summable sequences on $\mathbf{Z}^2$, where we set $v_\eta = 0$ if $\eta \notin \mathbf{N}^2$. By this identification, $x^\gamma <\partial>^{(s-1)|\gamma|}$ induces the map

$$x^\gamma <\partial>^{(s-1)|\gamma|}: \{v_\eta\} \in \ell_2 \longrightarrow$$

$$\left\{ v_{\eta-\gamma} w^\gamma \frac{(|\eta|-|\gamma|-(\mu/s))!^s}{(|\eta|-(\mu/s))!^s} <\eta-\gamma>^{(s-1)|\gamma|} \frac{\eta!}{(\eta-\gamma)!} \right\} \in \ell_2. \tag{4.4}$$

If we set $\xi = \eta/|\eta|$ and if we let $|\eta|$ tends to infinity we see that

$$\frac{(|\eta|-|\gamma|-(\mu/s))!^s}{(|\eta|-(\mu/s))!^s} <\eta-\gamma>^{(s-1)|\gamma|} \frac{\eta!}{(\eta-\gamma)!} = \lambda_\gamma(\eta) + r_\gamma(\eta), \tag{4.5}$$

where $\lambda_\gamma(\eta) = \xi^\gamma, \xi = \eta/|\eta|$ and where $r_\gamma(\eta)$ consists of term such that $r_\gamma(\eta) \to 0$ when η tends to infinity. We define the shift operator S_γ by

$$S_\gamma : \{v_\eta\}_\eta \in \ell_2 \longrightarrow \{v_{\eta-\gamma}\}_\eta \in \ell_2. \tag{4.6}$$

It follows from (4.4), (4.5) and (4.6) that

$$x^\gamma <\partial>^{(s-1)|\gamma|} = S_\gamma \lambda_\gamma(\partial) w^\gamma + R_\gamma(\partial) \tag{4.7}$$

where $R_\gamma(\partial) := S_\gamma r_\gamma(\partial) w^\gamma$.

Next we consider the operator $<\partial>^{-|\beta|} p_\beta(\partial)$. Because we have

$$<\partial>^{-|\beta|} p_\beta(\partial) \sum u_\eta \frac{x^\eta}{\eta!} = \sum v_\eta w^{-\eta}(|\eta| - \frac{\mu}{s})!^s <\eta>^{-|\beta|} p_\beta(\eta)\frac{x^\eta}{\eta!},$$

$<\partial>^{-|\beta|}\, p_\beta(\partial)$ induces the map

$$<\partial>^{-|\beta|}\, p_\beta(\partial) : \{v_\eta\} \in \ell_2 \longrightarrow \left\{v_\eta <\eta>^{-|\beta|}\, p_\beta(\eta)\right\} \in \ell_2. \tag{4.8}$$

We note that

$$<\eta>^{-|\beta|}\, p_\beta(\eta) = \xi^\beta + \tilde{r}(\eta), \tag{4.9}$$

where $\tilde{r}(\eta)$ satisfies that $\sup_{|\eta|\geq n} |\tilde{r}(\eta)| \to 0$ when η tends to infinity. Therefore we have

$$<\partial>^{-|\beta|}\, p_\beta(\partial) = \lambda_\beta(\partial) + \tilde{r}(\partial). \tag{4.10}$$

We set

$$Q_0 := P_0 <\partial>^{-m}. \tag{4.11}$$

Clearly, $Q_0 : G^s_w(\mu) \longrightarrow G^s_w(\mu)$ is bounded for every μ. We want to show that Q_0 is Fredholm if and only if $P_0 : G^s_w(\mu) \longrightarrow G^s_w(\mu + m)$ is Fredholm. Indeed, let R_j and the compact operator $K_j (j = 1, 2)$ satisfy that

$$R_1 Q_0 = I + K_1, \quad Q_0 R_2 = I + K_2, \tag{4.12}$$

where I denotes the identity operator. Let us for the moment suppose that

$$Q_0 = <\partial>^{-m} P_0 + K \tag{4.13}$$

for some compact operator K. Because the map $<\partial>^{-m}$: $G^s_w(\mu) \longrightarrow G^s_w(\mu + m)$ is bijective it follows from (4.12) and (4.13) that P_0 is a Fredholm operator. We can similarly prove the converse.

It remains to prove (4.13). In view of (4.3), we shall consider the commutator

$$[x^\gamma, <\partial>^{-m}] := x^\gamma <\partial>^{-m} - <\partial>^{-m} x^\gamma,$$

where $\gamma = \alpha - \beta$. We have, for $u = \sum_\eta u_\eta x^\eta/\eta! \in G^s_w(\mu)$

$$[x^\gamma, <\partial>^{-m}]u = x^\gamma \sum_\eta u_\eta x^\eta \left(<\eta>^{-m} - <\gamma + \eta>^{-m}\right)/\eta!. \tag{4.14}$$

By Taylor's formula we have

$$<\eta>^{-m} - <\gamma + \eta>^{-m} = m \int_0^1 \gamma \cdot (\eta + s\gamma) <\eta + s\gamma>^{-m-2}\, ds =: C_\gamma(\eta). \tag{4.15}$$

It follows that $\Lambda_\gamma(\eta) := <\eta>^m C_\gamma(\eta)$ satisfies

$$\sup_{|\eta|\geq N}\, <\gamma>^{-1} |\Lambda_\gamma(\eta)| \to 0, \tag{4.16}$$

when $N \to \infty$. Here the limit is uniform with respect to γ when $|\gamma| \to \infty$. Therefore we have

$$[x^\gamma, <\partial>^{-m}] = x^\gamma <\partial>^{-m} \Lambda_\gamma(\partial) \tag{4.17}$$

where $\Lambda_\gamma(\partial)$ is the Euler type operator with symbol given by $\Lambda_\gamma(\eta)$. It follows from (4.16), (4.17) and Lemma 3.4 that $[x^\gamma, <\partial>^{-m}]$ is a compact operator. In order to show (4.13) we note that the summation in P_0 with respect to α and β is a finite sum in case $s \neq 1$. Hence the assertion is trivial. If $s = 1$, one may assume that $|a_{\alpha\beta}| \leq C\rho^{|\alpha|}$ for some $\rho < 1$ and $C > 0$ independent of α and β. Because $a_{\alpha\beta}$ is the Taylor coefficients of $a_\beta(x)$, which is, by a scaling of x, analytic in a larger domain. This implies that we can assume $\rho < 1$. Therefore it follows from (4.16) and lemma 3.4 that

$$\sum_{\alpha,\beta} a_{\alpha\beta}[x^{\alpha-\beta}, <\partial>^{-m}]p_\beta(\partial) = \sum_{\alpha,\beta} a_{\alpha\beta}x^{\alpha-\beta} <\alpha-\beta><\partial>^{-m} p_\beta(\partial) <\alpha-\beta>^{-1} \Lambda_{\alpha-\beta}(\partial)$$

is a compact operator.

Next we want to show that the Fredholmness of the operator Q_0 given by (4.10) is equivalent to that of a certain Toeplitz operator. To this end, we define the projection π by

$$\pi u := \sum_{\eta\in\mathbf{N}^2} u_\eta x^\eta/\eta! \qquad \text{for} \quad u = \sum_{\eta\in\mathbf{Z}^2} u_\eta x^\eta/\eta!. \tag{4.18}$$

It follows from the definition of $p_\beta(\partial)$ in (4.2) that $x^{\alpha-\beta}p_\beta(\partial) <\partial>^{-m}$ maps $G^s_w(\mu)$ into itself. Hence we have

$$Q_0 = \pi \sum a_{\alpha\beta}x^{\alpha-\beta}p_\beta(\partial) <\partial>^{-m} = \pi \sum a_{\alpha\beta}x^{\alpha-\beta} <\partial>^{(s-1)(|\alpha|-|\beta|)} <\partial>^{-|\beta|}p_\beta(\partial), \tag{4.19}$$

where the summation is taken for α and β such that $|\beta| + (1-s)(|\alpha| - |\beta|) = m$. By substituting (4.7) with $\gamma = \alpha - \beta$ and (4.10) into (4.19) we have

$$Q_0 = \pi \sum a_{\alpha\beta} \left(S_{\alpha-\beta}\lambda_{\alpha-\beta}w^{\alpha-\beta} + R_{\alpha-\beta}\right)(\lambda_\beta + \tilde{r}) = \pi \sum a_{\alpha\beta}S_{\alpha-\beta}w^{\alpha-\beta}\lambda_{\alpha-\beta}\lambda_\beta$$

$$+ \pi \sum K_{\alpha\beta} = \pi \sum a_{\alpha\beta}S_{\alpha-\beta}\lambda_\alpha w^{\alpha-\beta} + \pi \sum a_{\alpha\beta}K_{\alpha\beta}, \tag{4.20}$$

where

$$K_{\alpha\beta} = S_{\alpha-\beta}\lambda_{\alpha-\beta}w^{\alpha-\beta}\tilde{r} + R_{\alpha-\beta}\lambda_\beta + R_{\alpha-\beta}\tilde{r}. \tag{4.21}$$

We want to show that the second term in the right-hand side of (4.20) is a compact operator. To this end, we note that for each α and β, $K_{\alpha\beta}$ is a compact operator by the definition of the symbols $R_{\alpha-\beta}$ and $\tilde{r}$ and Lemma 3.4. Because the sum in the second term in the right-hand side of (4.19) is a finite sum except for the case $s = 1$, the desired compactness follows if $s \neq 1$. In case $s = 1$ we may assume that $|a_{\alpha\beta}| \leq C\rho^{|\alpha|}$ for some $C > 0$ and $0 < \rho < 1$ without loss of generality. In view of (4.21), it is sufficient to show that the sum $\sum_{\alpha,\beta} a_{\alpha,\beta}R_{\alpha-\beta}$ is a compact operator. In view of the definition of $R_{\alpha-\beta}$ in (4.7) ($\gamma = \alpha - \beta$), the inequality, $|(|\eta| - |\gamma|)!\eta!/(|\eta|!(\eta-\gamma)!)| \leq 1$ and (4.4) the symbol $a_{\alpha,\beta}r_{\alpha-\beta}(\eta)$ tends to zero uniformly with respect to α and β when $\eta \to \infty$. Hence we have the assertion by Lemma 3.4.

The operator

$$T := \pi \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta}w^{\alpha-\beta}S_{\alpha-\beta}\lambda_\alpha(\partial) : G^s_w(\mu) \longrightarrow G^s_w(\mu) \tag{4.22}$$

is called a Toeplitz operator. Here $S_{\alpha-\beta}$ is considered as an operator on $G^s_w(\mu)$ by the isomorphism between ℓ^2 and $G^s_w(\mu)$. The function (2.7) is called the symbol of a Toeplitz operator T.

It follows from (4.19) that Q_0 is a Fredholm operator if and only if the Toeplitz operator T is a Fredholm operator. By the obvious identification between $G^s_w(\mu)$ and $\ell_2(\mathbf{N}^2)$ used in (4.4) one may think T as an operator on $\ell_2(\mathbf{N}^2)$. Moreover, by the isomorphism between $\ell_2(\mathbf{N}^2)$ and the Hardy space $H^2(\mathbf{T}^2)$ one can also think T as an operator on $H^2(\mathbf{T}^2)$. If we take the cordinate $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbf{T}^2$ T is given by

$$T = \pi \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} w^{\alpha-\beta} e^{i(\alpha-\beta)\theta} \lambda_\alpha(D) : H^2(\mathbf{T}^2) \longrightarrow H^2(\mathbf{T}^2), \tag{4.23}$$

where $\lambda_\alpha(D)$ is a pseudo-differential operator with the symbol $\lambda_\alpha(\xi)$ where ξ is a covariable of $\theta = (\theta_1, \theta_2)$.

In view of the arguments above, one may assume that T is a Fredholm operator. We shall microlocalize T in the following way. Let $\varepsilon > 0$ and let $\xi \in \mathbf{R}^2, |\xi| = 1$. We define the convex cone $C(\xi, \varepsilon)$ and the projection π_ξ respectively, by

$$C(\xi,\varepsilon) := \left\{ \eta \in \mathbf{R}^2; \left| \frac{\eta}{|\eta|} - \xi \right| < \varepsilon \right\}, \qquad \pi_\xi \sum_\eta u_\eta x^\eta/\eta! := \sum_{\eta \in C(\xi,\varepsilon)\cap \mathbf{Z}^2} u_\eta x^\eta/\eta!. \tag{4.24}$$

We define the closed subspace $G^s_{w,\xi}(\mu)$ by

$$G^s_{w,\xi}(\mu) := \{\pi_\xi u; u \in G^s_w(\mu)\}. \tag{4.25}$$

$G^s_{w,\xi}(\mu)$ is a Banach space with the norm of $G^s_w(\mu)$. We define the microlocalized Toeplitz operator T_ξ of T in the direction ξ by

$$T_\xi := \pi_\xi T \pi_\xi : G^s_{w,\xi}(\mu) \longrightarrow G^s_{w,\xi}(\mu). \tag{4.26}$$

We want to show that $T : G^s_w(\mu) \to G^s_w(\mu)$ is a Fredholm operator if and only if, for every $\xi \in \mathbf{R}^2, \xi \geq 0$ T_ξ is a Fredholm operator. We first show the necessity. In order to see this, we take a finite number of ζ's ($|\zeta| = 1$, $\zeta \in \mathbf{R}^2, \zeta \geq 0$) and $\varepsilon > 0$ in such a way that

$$\mathbf{N}^2 = \cup_\zeta C(\zeta, \varepsilon) \cap \mathbf{N}^2 \tag{4.27}$$

and $C(\zeta,\varepsilon) \cap \mathbf{N}^2$ are distinct. It follows that we can decompose $u \in G^s_w(\mu)$ into a direct sum $u = \sum_\zeta \pi_\zeta u$. Let η be one of ζ's in the decomposition above. Then the equation $Tu = h$ can be written in the form

$$\pi_\eta T \pi_\eta u + \sum_{\zeta \neq \eta} \pi_\eta T \pi_\zeta u = \pi_\eta h, \qquad \forall \eta. \tag{4.28}$$

We decompose the space $G^s_w(\mu)$ into the direct sum $G^s_w(\mu) = \sum_\zeta G^s_{w,\zeta}(\mu)$. Similarly we decompose $u = \sum_\zeta u^\zeta$ and $h = \sum_\zeta h^\zeta$, where $u^\zeta := \pi_\zeta u$ and $h^\zeta := \pi_\zeta h$. If we define the vector $\mathcal{U}$ and $\mathcal{H}$ by $\mathcal{U} := {}^t(u^\zeta)_\zeta$ and $\mathcal{H} := {}^t(h^\zeta)_\zeta$ then (4.28) can be written in the form

$$\mathcal{A}\mathcal{U} = \mathcal{H}. \tag{4.29}$$

Here the (η,ζ) component of $\mathcal{A}$ is given by $\pi_\eta T\pi_\zeta$.

Because T is a Fredholm operator it follows that

$$\mathcal{A} : \bigoplus_\zeta G^s_{w,\zeta}(\mu) \longrightarrow \bigoplus_\zeta G^s_{w,\zeta}(\mu)$$

is a Fredholm operator. By setting $u^\zeta = 0$ if $\zeta \neq \eta$ and $u^\eta = \pi_\eta u$ in (4.29) we have, for some compact operator K and $c > 0$

$$\|\pi_\eta T\pi_\eta u^\eta\| + \sum_{\zeta\neq\eta} \|\pi_\zeta T\pi_\eta u^\eta\| + \|Ku^\eta\| \geq c\|u^\eta\|. \tag{4.30}$$

Let e be a vector in the interior of $C(\eta,\varepsilon)$ with integral components and let us define the shift operator U_e in the direction e by $U_e \sum_\eta u_\eta x^\eta/\eta! := \sum_\eta u_{\eta+e}x^\eta/\eta!$. By replacing u^η by $U_e^n u^\eta$ for sufficiently large n in (4.30) we have

$$\|\pi_\eta T\pi_\eta U_e^n u^\eta\| + \sum_{\zeta\neq\eta} \|\pi_\zeta T\pi_\eta U_e^n u^\eta\| + \|KU_e^n u^\eta\| \geq c\|U_e^n u^\eta\|. \tag{4.31}$$

We note that $\|U_e^n u^\eta\| = \|u^\eta\|$ because U_e is an isometry.

On the other hand, recalling that π_η and U commute each other we have

$$\pi_\eta T\pi_\eta U_e^n u^\eta = U_e^n \pi_\eta T\pi_\eta u^\eta + \pi_\eta[T, U_e^n]\pi_\eta u^\eta. \tag{4.32}$$

We easily see that $[T, U_e^n]$ is a compact operator by exactly the same arguments as in the proof of (4.17). Next we shall estimate the term $\|\pi_\zeta T\pi_\eta U_e^n u^\eta\|$ for $\zeta \neq \eta$. By the definition of U_e we see that the distance between the support of $\pi_\eta U_e^n u^\eta$ and the cone $C(\zeta,\varepsilon)$ is bounded from the below by $c_1 n$ for some $c_1 > 0$ independent of n. It follows from (4.22) that if $\pi_\zeta T\pi_\eta U_e^n u^\eta$ does not vanish, $\alpha - \beta$ in (4.22) should satisfy that $|\alpha - \beta| \geq c_2 n$ for some $c_2 > 0$. Because β moves on a finite set and $a_{\alpha\beta}$ tends to zero when α tends to zero, it follows that the norm $\|\pi_\zeta T\pi_\eta U_e^n u^\eta\|$ can be absorbed in $\|u^\eta\|$ when n is sufficiently large. We fix such an integer n. Then, we have

$$\|\pi_\eta T\pi_\eta u^\eta\| + \|K'u^\eta\| \geq \delta'\|u^\eta\| \tag{4.33}$$

for some constant $\delta' > 0$ and a compact operator K'. Because η is arbitrary we see that the localized operator $\pi_\eta T\pi_\eta : G^s_{w,\eta}(\mu) \to G^s_{w,\eta}(\mu)$ is a Fredholm operator for every η.

We shall show the sufficiency. Let us suppose that (4.33) is valid for every η. We want to show that, for every ζ

$$\|\pi_\zeta T\pi_\zeta u^\zeta\| + \sum_{\eta\neq\zeta} \|\pi_\zeta T\pi_\eta u^\eta\| + \sum_\eta \|Ku^\eta\| \geq c\sum_\eta \|u^\eta\|. \tag{4.34}$$

To this end, let us define the shift operators U_ξ by $U_\xi \sum_\eta u_\eta x^\eta/\eta! := \sum_\eta u_{\eta+e(\xi)}x^\eta/\eta!$, where $e(\xi)$ is a vector in the interior of the cone $C(\xi,\varepsilon)$ with integral components. Let n be an integer. We first prove the following, for some compact operator K and $c > 0$

$$\|\pi_\zeta T\pi_\zeta U_\zeta^n u^\zeta\| - \sum_{\eta\neq\zeta} \|\pi_\zeta T\pi_\eta U_\eta^n u^\eta\| + \sum_\eta \|KU_\eta^n u^\eta\| \geq c\sum_\eta \|u^\eta\|. \tag{4.35}$$

We recall that U_ζ is an isometry. We have $\pi_\zeta T\pi_\zeta U_\zeta^n = U_\zeta^n \pi_\zeta T\pi_\zeta + K_i$ for some compact operator K_1. Hence it follows from (4.33) that

$$\|\pi_\zeta T\pi_\zeta U_\zeta^n u^\zeta\| \geq \|U_\zeta^n \pi_\zeta T\pi_\zeta u^\zeta\| - \|K_1 u^\zeta\|$$

$$\geq \|\pi_\zeta T\pi_\zeta u^\zeta\| - \|K_1 u^\zeta\| \geq \delta'\|u^\zeta\| - \|K' u^\zeta\| - \|K_1 u^\zeta\|. \tag{4.36}$$

On the other hand, we can easily see by the argument in the proof of the necessity that $\|\pi_\zeta T\pi_\eta U_\eta^n u^\eta\|$ for $\zeta \neq \eta$ can be absorbed in $\|u^\eta\|$ if we take n sufficiently large. We fix such an integer n. In view of (4.33) we have (4.35).

We can remove U_η^n and U_ζ^n in (4.35) by replacing K if necessary because the commutators with T and U_ζ^n's are compact and U_ζ^n's are isometries. If we take the summation with respect to ζ in (4.35) we see that the operator $\mathcal{A}$ is a Fredholm operator.

We define the freezed operator T_η by

$$T_\eta := \pi_\eta \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} w^{\alpha-\beta} S_{\alpha-\beta} \lambda_\alpha(\eta) : G^s_{w,\eta}(\mu) \longrightarrow G^s_{w,\eta}(\mu). \tag{4.37}$$

We want to show that $T_\eta : G^s_{w,\eta}(\mu) \to G^s_{w,\eta}(\mu)$ is a Fredholm operator if and only if $\pi_\eta T\pi_\eta : G^s_{w,\eta}(\mu) \to G^s_{w,\eta}(\mu)$ is a Fredholm operator. Indeed, suppose that $\pi_\eta T\pi_\eta$ is a Fredholm operator. It follows from (4.33) that

$$\|\pi_\eta T_\eta u^\eta\| + \|K' u^\eta\| \geq \delta'\|u^\eta\| - \|(T_\eta - \pi_\eta T\pi_\eta)u^\eta\|$$

for some constant $\delta > 0$. In view of the definition of $\lambda_\alpha(\partial)$ in (4.22) and (4.37) we see that the operator norm of $T_\eta - \pi_\eta T\pi_\eta$ tends to zero if $\varepsilon \to 0$. This implies that the term $\|(T_\eta - \pi_\eta T\pi_\eta)u^\eta\|$ can be absorbed in $\delta'\|u^\eta\|$. This implies that T_η is a Fredholm operator. The converse part will be proved similarly.

Now our theorem is the consequence of the following theorem.

Theorem 4.1. *For every ξ the map $T_\xi : G^s_{w,\xi}(\mu) \to G^s_{w,\xi}(\mu)$ is a Fredholm operator if and only if the conditions (2.8) and (2.9) are satisfied.*

The proof of this theorem is purely based on the arguments of the theory of Toeplitz operators. In oder to make the presentation selfcontained, it is necessary to prepare many concepts, which would make this note considerably long. Hence we would like to omit the proof and we refer the readers to [2] for the necessary tools in proving this type of theorems.

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