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Toeplitz operator theory and Hilbert factorization applied to the Fredholmness of PDE

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1. Introduction

The object in this note is to show the validity of the Toeplitz operator method in the study of index theorems for both (systems of) ordinary and partial differential equations. Concerning the works on index theorems, we refer the works [5], [6], [7], [8], [9] [11] and [12]. As for the elementary properties of Toeplitz operators and general treatments of Toeplitz operators we refer [4] and [3], respectively.

In order to show clearly how the Toeplitz operator method is used we first state the results in the case of ordinary differential equations. Then we consider the case of partial differential equations. Although the results in the case of ordinary differential equations do not include essentially new results, our new proof of the index theorem is a very simple one based on the Fredholm property of Toeplitz operators, which is applicable to partial differential operators without essential changes. We believe that this shows the usefulness of the Toeplitz operator method.

2. Index theorems for ordinary differential equations

Let $\mathbf{C}[[x]]$ be the set of formal power series $u(x)=\sum_{n\in\mathbf{N}}u_nx^n/n!$ of a complex variable x, where $\mathbf{N}$ is the set of nonnegative integers. We say that u is in a (formal) Gevrey space $\mathcal{G}^s_w$ $(s\in\mathbf{R}, w>0)$ if

$$\sum_{n\in\mathbf{N}}u_nx^n/(n!)^s\in\mathcal{O}(|x|<w),$$

where $\mathcal{O}(|x|<w)$ denotes the set of holomorphic functions on the domain $\{|x|<w\}$. We denote by $W(\mathbf{C})$ the set of ordinary differential operators with polynomial coefficients, and we denote by $M_N(W(\mathbf{C}))$ the set of $N\times N$ matrices with entries in $W(\mathbf{C})$ for $N\geq 1$.

For $P\in M_N(W(\mathbf{C}))$ we want to give an index formula of the mapping

$$P(x,D):(\mathcal{G}^s_w)^N\longrightarrow(\mathcal{G}^s_w)^N, \tag{2.1}$$

where $D:=d/dx$.

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For $p(x,D)=\sum_{j,k}a_{jk}x^jD^k\in W(\mathbf{C})$ we define an s - Gevrey order $\mathrm{ord}_s p$ of $p(x,D)$ by the maximum of $sk+(1-s)j$ when j and k satisfy $a_{jk}\neq 0$. The s - Gevrey symbol $\sigma_s(p)(x,\xi)$ and s - Gevrey Toeplitz symbol $p_s(z)$ are defined, respectively, by

$$\sigma_s(p)(x,\xi)=\sum_{\mathrm{ord}_s p=sk+(1-s)j}a_{jk}x^j\xi^k,\quad p_s(z)=\sigma_s(P)(z,z^{-1}). \tag{2.2}$$

The s-Gevrey order induces a filtration on $W(\mathbf{C})$ in a natural manner, and we denote by $\mathrm{gr}^sW(\mathbf{C})$ the associated graded ring which is isomorphic to the set of s-Gevrey symbols. Because $\mathrm{gr}^sW(\mathbf{C})$ is a unique factorization ring, we can define $\det_s$ the determinant for matrices over $W(\mathbf{C})$ associated with the s-Gevrey filtration, which is a homomorphism of multiplicative groups from $M_N(W(\mathbf{C}))$ to $\mathrm{gr}^sW(\mathbf{C})$ by the well-known manner (see [1], [2], [8] and [9])

$$\det_s: M_N(W(\mathbf{C}))\longrightarrow \mathrm{gr}^sW(\mathbf{C}).$$

Then an s-Gevrey order $\mathrm{ord}_s P$ of a matrix $P(x,D)$ is defined in a natural manner from the determinant of $P(x,D)$. Now we define the s-Gevrey Toeplitz symbol $P_s(z)$ of $P(x,D)\in M_N(W(\mathbf{C}))$ by $P_s(z)=\det_s(P)(z,z^{-1})$. Then we have

Theorem 2.1. *Suppose that $P_s(z)\not\equiv 0$. Then the index $\chi(P;\mathcal{G}^s_w)$ of the mapping* (2.1) *is given by*

$$-\chi(P;\mathcal{G}^s_w)=\lim_{r\uparrow w}I_r(P_s):=\frac{1}{2\pi}\lim_{r\uparrow w}\oint_{|z|=r}d\arg P_s(z), \tag{2.2}$$

where $I_r(P_s)$ denotes the winding number of an oriented curve $\{P_s(z);|z|=r\}$ at the origin for r such that $P_s(z)\neq 0$ on $|z|=r$.

This theorem was proved by Ramis for single operators at the first time, but the expression in his paper seems to be somwhat complicated (cf. [11]). We also remark that the case of general systems was studied by Adjamagbo by use of determinant theory as above, but his treatment was purely algebraic based on the result of Ramis (cf. [1]).

The following theorem gives an estimate of the dimension of the kernel of the mapping.

Proposition 2.2. *We put $\mathcal{G}^\infty:=\mathbf{C}[[x]]$ and $\mathcal{G}^{-\infty}:=\mathbf{C}[x]$, the set of polynomials. Then we have*

$$\chi(P;\mathcal{G}^{-\infty})=\lim_{s\downarrow-\infty}\chi(P;\mathcal{G}^s_w)=k-j,\quad \chi(P;\mathcal{G}^{\infty})=\lim_{s\uparrow\infty}\chi(P;\mathcal{G}^s_w)=n-m, \tag{2.3}$$

where $\det_{-\infty}P:=\lim_{s\downarrow-\infty}\det_s P=ax^j\xi^k\neq 0$, and $\det_\infty P:=\lim_{s\uparrow\infty}\det_s P=bx^m\xi^n\neq 0$ are uniquely determined. And it holds that

$$\max\{0,k-j\}\leq \dim_{\mathbf{C}}\mathrm{Ker}(P;\mathcal{G}^s_w)\leq n\leq \mathrm{ord}_1 P.$$

Remarks. (a) When $s = 1$, the filtration is the usual one given by the order of differentiation, and the determinant is given in the form, $\det_1(P)(x,\xi) = a(x)\xi^m$ ($a(x) \in \mathbf{C}[x] \setminus 0$), and $P_1(z) = a(z)z^{-m}$. Therefore, under the assumption that $a(z) \not\equiv 0$ we have

$$\chi(P; \mathcal{O}(|x| < w)) = m - \sum_{z \in Z_w} \mathrm{ord}_z\, a,$$

where Z_w denotes the set of zero points of $a(z)$ in $|z| < w$ and $\mathrm{ord}_z\, a$ denotes the order of zeros of a at the point z.

(b) For a single operator, the s-Gevrey Toeplitz symbol is recognized in a visual way in terms of a Newton polygone as follows. Let $p(x,D) = \sum_{j,k} a_{jk}x^jD^k \in W(\mathbf{C})$. For $(j,k) \in \mathbf{N}^2$, we associate the left half line $Q(j,k) := \{(s, j-k) \in \mathbf{R}^2; s \leq k\}$. Then the Newton polygone $N(p)$ is defined by $N(p) := \mathrm{ch}\{Q(j,k); a_{jk} \neq 0\}$, where $\mathrm{ch}\{\cdot\}$ denotes the convex hull of points in $\{\cdot\}$. We draw a line L_s with slope $k = 1/(s-1)$ so that L_s contacts on a side or at a vertex of $N(p)$, and we put $N_s = L_s \cap N(p)$. Then the s-Gevrey Toeplitz symbol $p_s(z)$ is given by

$$p_s(z) = \sum_{(j,k) \in \mathring{N}_s} a_{jk} z^{j-k},$$

where $\mathring{N}_s = \{(j,k) \in \mathbf{N}^2; (k, j-k) \in N_s\} = \{(j,k); sk + (1-s)j = \mathrm{ord}_s p,\ a_{jk} \neq 0\}$.

Proof of Theorem 2.1. We shall give the sketch of the proof by restricting to the case $N = 1$ and $s \geq 0$. Let $\mu \in \mathbf{R}$ be fixed. For $u = \sum u_n x^n/n! \in \mathbf{C}[[x]]$, we set $U_n = s^{sn}u_n/(sn-\mu)!$, where $(sn-\mu)! = 1$ if $sn - \mu \leq 0$. Then we say that u is in the class $G^s_w(\mu)$ if $\|u\|_{w,\mu} := \sum_0^\infty |U_n| w^n < \infty$. By definition $G^s_w(\mu)$ is a Banach space. Moreover we can easily see that $\mathcal{G}^s_w = \mathrm{proj}\lim_{r \uparrow w} G^s_r(\mu)$ for any fixed μ. We first note

Lemma 2.3. *If s-Gevrey order of P is equal to μ, then P is a bounded operator from $G^s_w(\mu)$ to $G^s_w(0)$. If s-Gevrey order of Q is strictly smaller than μ, then, Q is a compact operator from $G^s_w(\mu)$ to $G^s_w(0)$.*

The proof of Lemma 2.3 is based on elementary calculations. For the detailed proof we refer [10].

In what follows, let μ be the s-Gevrey order of $P(x,D)$. Because $\mathcal{G}^s_w = \mathrm{proj}\lim_{r \uparrow w} G^s_r(\mu)$ and since the index is stable under compact perturbations, we may assume, by Lemma 2.3, that $P(x,D)$ consists of terms corresponding to $\mathring{N}_s$.

In case L_s contacts at a vertex of $N(P)$, that is $\mathring{N}_s$ is a single point, $P(x,D)$ is written in the form $P = bx^jD^k$ ($b \neq 0$) and it is easy to see that the index of the mapping $x^jD^k : G^s_w(\mu) \to G^s_w(0)$ is equal to $k - j (= -I_w(p_s))$, which implies the theorem.

Next let us consider the case where $\overset{\circ}{N}_s$ consists of more than two points. This condition implies that s is a rational number. Let $s = q/p$ be the irreducible fraction, where we put $(p, q) = (1, 0)$ if $s = 0$. Then $P(x, D)$ is written in the form

$$P(x, D) = x^j D^k \sum_{i=0}^{m} b_i x^{qi} D^{(q-p)i} =: x^j D^k L(x, D),$$

where $b_0 \neq 0$. Here we have to remark that in the case $0 \leq s < 1$ (i.e., $q < p$), $D^{(q-p)i} := (D^{-1})^{(p-q)i}$ and $D^{-1} := \int_0^x$ denotes the formal integration from 0 to x. The notion of s-Gevrey order is defined similarly for integro-differential operators, and we see that the s-Gevrey order of $L(x, D)$ is equal to 0. We put $p_s(z) = z^{j-k} \sum b_i z^{pi} =: z^{j-k} q_s(z)$. Since $x^j D^k : G_w^s(\mu) \to G_w^s(0)$ is a mapping with an index $k - j$ and $I_w(p_s) = j - k + I_w(q_s)$, it is sufficient to prove that $L : G_w^s(0) \to G_w^s(0)$ is a mapping with an index $-I_w(q_s)$ under the assumption that $q_s(z) \neq 0$ on $|z| = w$. In what follows, we always assume this condition, and we put $G_w^s := G_w^s(0)$ for simplicity.

We consider the equation $Lu = f \in G_w^s$. By substituting the expansion of u, $u = \sum u_n x^n/n!$ and f in the equation we have the infinite system of linear equations

$$\sum_{i=0}^{m} c_{n,i} b_i U_{n-pi} = F_n, \quad n \geq 0,$$

where $U_n = s^{sn} u_n/(sn)!$ and $F_n = s^{sn} f_n/(sn)!$. Here the coefficient $c_{n,i}$ satisfies that $c_{n,i} \to 1$ when $n \to \infty$ for each i. We recall that $u \in G_w^s$ if and only if $\sum_0^\infty |U_n| w^n < \infty$. For simplicity, we denote the set of $\{U\}_n$ satisfying $\sum_0^\infty |U_n| w^n < \infty$ by ℓ_w^1. We set $\mathcal{M} := \{U = \{U_n\} \in \ell_w^1, U_0 = 0\}$ the maximal ideal of ℓ_w^1 and we consider the following decomposition

$$0 \longrightarrow \mathcal{M}^N \longrightarrow \ell_w^1 \longrightarrow \ell_w^1/\mathcal{M}^N \longrightarrow 0 \quad \text{(exact)},$$

for a fixed $N \geq 1$. From the assumption we can easily see that the matrix representation of the above infinite system is a lower trianguler matrix of infinite order with a diagonal element $b_0 \neq 0$. Hence $L(x, D)$ is bijective on $\ell_w^1/\mathcal{M}^N$ for any $N \geq 1$. Hence it is sufficient to prove that the index of the mapping $L : \mathcal{M}^N \longrightarrow \mathcal{M}^N$ is equal to $-I_w(q_s)$ for sufficiently large N under the assumption that $q_s(z) \neq 0$ on $|z| = w$. This means that we consider the above infinite system of linear equations for $\{U_n\}_{n \geq N}$ and $\{F_n\}_{n \geq N}$. Now our result follows from the following proposition

Proposition 2.3. *For $q_s(z) = \sum_{i=0}^{pm} g_i z^i$, let the Toeplitz matrix $T(q_s)$ be defined by*

$$T(q_s) = \left(g_{j-k}; \begin{array}{l} j \downarrow 0, 1, 2, \cdots \\ k \to 0, 1, 2, \cdots \end{array} \right).$$

Suppose that $q_s(z) \neq 0$ on $|z| = w$. Then the operator $T(q_s)$ on ℓ_w^1 has an index $-I_w(q_s)$. In this case, the mapping is injective.

3. Partial differential equations

We first consider the case of two independent variables. Let $P \equiv P(t,x;D_t,D_x)$ be a partial differential operator of finite order with holomorphic coefficients in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x$, and write it in the form,

$$P(t,x;D_t,D_x) = \sum_{\sigma\in\mathbf{N}} \sum_{j,\alpha\in\mathbf{N}}^{\text{finite}} a_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha, \tag{3.1}$$

where, $\mathbf{N}$ denotes the set of non negative integers. For $(\sigma, j, \alpha) \in \mathbf{N}^3$, we associate a left half line $Q(\sigma, j, \alpha)$ in a (u,v)-plane defined by

$$Q(\sigma, j, \alpha) := \{(u, \sigma - j) \in \mathbf{R}^2 \,;\, u \le j + \alpha\}.$$

Then the Newton polygon $N(P)$ of the operator P is defined by,

$$N(P) := \mathrm{ch}\,\{\, Q(\sigma,j,\alpha)\,;\, a_{\sigma j\alpha}(x) \not\equiv 0\}, \tag{3.2}$$

where $\mathrm{ch}\{\cdot\}$ denotes the convex hull.

For a given $s > 0$, we draw a line L_s with slope $k := 1/(s-1) (\in \mathbf{R} \cup \{\infty\})$ which contacts to $N(P)$ at a vertex or on a side of $N(P)$. We put $N_s := N(P) \cap L_s$ and

$$\overset{\circ}{N}_s := \{(j,\alpha) \in \mathbf{N}^2 \,;\, a_{0j\alpha}(0) \neq 0,\ (j+\alpha, -j) \in N_s\}. \tag{3.3}$$

The principal part $P_s \equiv P_s(D_t, D_x)$ and the Toeplitz symbol $f_s(z)$ associated with the Gevrey index s are defined by,

$$P_s(D_t, D_x) := \sum_{(j,\alpha)\in \overset{\circ}{N}_s} a_{0j\alpha}(0) D_t^j D_x^\alpha \,, \tag{3.4}$$

$$f_s(z) := \sum_{(j,\alpha)\in \overset{\circ}{N}_s} a_{0j\alpha}(0) z^{-j}. \tag{3.5}$$

We define Gevrey space $\mathcal{G}_w^s(R)$ $(s, w, R > 0)$ as follows. Let $\mathbf{C}[[t,x]]$ denote the set of formal power series of variables $t, x \in \mathbf{C}$, and $\mathcal{O}(\Omega)$ the set of holomorphic functions on a domain $\Omega \subset \mathbf{C}_t \times \mathbf{C}_x$. Then the Gevrey space $\mathcal{G}_w^s(R)$ is defined by the following isomorphism,

$$\mathbf{C}[[t,x]] \supset \mathcal{G}_w^s(R) \underset{\sim}{\xrightarrow{\text{Borel transf.}}} \mathcal{O}((|t|/w)^{1/s} + |x| < R), \tag{3.6}$$

where the Borel trmansformation is defined by

$$\mathcal{G}_w^s(R) \ni \sum_{l,\eta\in\mathbf{N}} u_{l\eta} \frac{t^l x^\eta}{l!\,\eta!} \overset{\sim}{\mapsto} \sum_{l,\eta\in\mathbf{N}} u_{l\eta} \frac{t^l x^\eta}{(sl)!\,\eta!} \in \mathcal{O}((|t|/w)^s + |x| < R).$$

The factorial is defined by the gamma function, $r! := \Gamma(r+1)$ for $r \geq 0$.

We consider the following Goursat problem in $\mathcal{G}^s_w(R)$,

$$\begin{cases} P(t,x;D_t,D_x)\,u(t,x) = f(t,x) \in \mathcal{G}^s_w(R), \\ u(t,x) - v(t,x) = O(t^j x^\alpha), \quad v(t,x) \in \mathcal{G}^s_w(R), \end{cases} \tag{3.7}$$

where $w(t,x) = O(t^j x^\alpha)$ in $\mathcal{G}^s_w(R)$ means that $w(t,x)t^{-j}x^{-\alpha} \in \mathcal{G}^s_w(R)$. Then we can prove the following,

Theorem 3.1. *Assume $\overset{\circ}{N}_s \neq \phi$ and that $(j,\alpha) \in \mathbf{N}^2$ belongs to $\mathrm{ch}\{\overset{\circ}{N}\}$ the convex hull of points in $\overset{\circ}{N}_s$. Further we assume*

$$f_s(z) \neq 0 \text{ on } |z| = w, \quad \text{and} \quad \oint_{|z|=w} d(\arg f_s(z)z^j) = 0. \tag{3.8}$$

Then there exists $R_0 > 0$ such that every formal solution $u(t,x) \in \mathbf{C}[[t,x]]$ (if it exists) belongs to $\mathcal{G}^s_w(R)$ for $0 < R < R_0$. Precisely, the problem (3.7) has the Fredholm property, that is, the mapping $P : t^j x^\alpha \mathcal{G}^s_w(R) \longrightarrow \mathcal{G}^s_w(R)$ has the same finite dimensional kernel and cokernel for sufficiently small $R > 0$. Furthermore, if one of the following conditions is satisfied, then the problem (3.7) is uniquely solvable in $\mathcal{G}^s_w(R)$ for sufficiently small $R > 0$;

(i) (j,α) *is an end point of* $\overset{\circ}{N}_s$.

(ii) *There exists $c > 0$ such that $\{f_s(z)z^j; |z| = c\}$ is a segment.*

(iii) *There exists $c > 0$ such that $0 \notin \mathrm{ch}\{f_s(z)z^j; |z| = c\}$.*

We remark that whenever s is irrational, $\overset{\circ}{N}_s$ consists of a point, and the problem (3.7) is uniquely solvable in $\mathcal{G}^s_w(R)$ for every $w > 0$ for sufficiently small $R > 0$.

4. Equations of n independent variables

In this section we shall consider the Fredholm property of Goursat problems for n independent variables. We write $x = (x_1, \ldots, x_n)$ and, for a multi index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ we set $D^\alpha_x = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

Let $w_k > 0 (k = 1, \ldots, n)$ and $s > 0$. We set $w = (w_1, \ldots, w_n)$. Then we define the Gevrey space $\mathcal{G}^s_w$ by the following isomorphism

$$\mathbf{C}[[x]] \supset \mathcal{G}^s_w \underset{\sim}{\overset{\text{Borel transf.}}{\longrightarrow}} \mathcal{O}(\{|x_1| < w_1\} \times \cdots \times \{|x_n| < w_n\}), \tag{4.1}$$

where the Borel trmansformation is defined by

$$\mathcal{G}^s_w \ni \sum_{\eta\in\mathbf{N}^n} u_\eta \frac{x^\eta}{|\eta|!} \overset{\sim}{\mapsto} \sum_{\eta\in\mathbf{N}^n} u_\eta \frac{x^\eta}{|\eta|!^s} \in \mathcal{O}(\{|x_1| < w_1\} \times \cdots \times \{|x_n| < w_n\}).$$

Let $P \equiv P(x, D_x)$ be a partial differential operator of finite order with holomorphic coefficients in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x$, and write it in the form,

$$P(x, D_x) = \sum_{\alpha\in\mathbf{N}^n, |\alpha|\le m} a_\alpha(x) D_x^\alpha. \tag{4.2}$$

We study the following Goursat problem

$$\left\{\begin{array}{c} P(x, D_x)\, u(x) = f(x) \in \mathcal{G}^s_w, \\ u(x) - v(x) = O(x^\beta), \quad v(x) \in \mathcal{G}^s_w, \end{array}\right. \tag{4.3}$$

where $\beta \in \mathbf{N}^n$, $|\beta| = m$ is given and fixed and where $w(x) = O(x^\beta)$ in $\mathcal{G}^s_w$ means that $w(x)x^{-\beta} \in \mathcal{G}^s_w$.

We define the Toeplitz symbol associated with (4.3) as follows.

$$L(z) := \sum_{|\alpha|=m} a_\alpha(0) z^{\alpha-\beta}, \quad z = (z_1, \ldots, z_n). \tag{4.4}$$

We note that $L(z)$ is a homogeneous function of z with degree zero. For $|\zeta| = 1$ and $j, (1 \le j \le n)$ we set

$$L^j_\zeta(z) = L(z)\,|_{z_j=\zeta}\,. \tag{4.5}$$

Then we can prove the following,

Theorem 4.1. *Suppose that there exist $\zeta, |\zeta| = 1$ and $j, (1 \le j \le n)$ such that $L^j_\zeta(z)$ satisfies that the origin is not in the convex hull of the set*

$$\left\{L^j_\zeta(z)\,;\, |z_k| = w_k, \ \text{for all} \ k \neq j\right\}.$$

Then the problem (4.3) has the Fredholm property, that is, the mapping $P : x^\beta\mathcal{G}^s_w \longrightarrow \mathcal{G}^s_w$ has the same finite dimensional kernel and cokernel. Furthermore, if the following condition is satisfied, then the problem (4.3) is uniquely solvable in $\mathcal{G}^s_w$. ;

the origin is not in the convex hull of the set $\{L(z); |z_k| = w_k, \ \text{for all} \ 1 \le k \le n\}$.

We shall give the proof of this theorem in the forthcoming papers.

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