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Extension algebras of C^*-algebras via canonical *-endomorphisms

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§1. Introduction. Let A be a unital C^*-algebra and γ a unit preserving *-endomorphism of A. When γ satisfies a certain condition which are naturally obtained through the index theory, we call the γ canonical. Let ϕ be a faithful state on A with $\phi(\gamma(x)) = \phi(x)$ for all $x \in A$. Using the canonical γ, we define a *-endomorphism ρ of A for which there exists a $\lambda (0 < \lambda < 1)$ so that $\phi(\rho(x)) = \lambda\phi(x)$ for all $x \in A$. From such a *-endomorphism ρ, we have a representation π of A and a non unitary isometry W which satisfy the following conditions :

(1) $W\pi(a)W^* = \pi(\rho(a))$ for all $a \in A$,

and

(2) $W^*\pi(A)W \subset \pi(A)$.

Let $< A, \gamma >$ be the C^*-algebra generated by $\pi(A)$ and W. We call $< A, \gamma >$ the extension algebra of A by the canonical *-endomorphism γ. We give a condition for $< A, \gamma >$ to be simple. We show that the canonical *-endomorphism γ of A is always extended to a canonical *-endomorphism $\hat{\gamma}$ of $< A, \gamma >$.

The terminology *canonical* is used by Ocneanu for *-endomorphism on some important *-algebras in the classification theory of subfactors of the hyperhinite II_1 factor. For *-endomorphism on the Cuntz algebra O_n, same terminology *canonical* is used by Cuntz. We show that the extension algebra $< A, \gamma >$ is always simple for pairs $\{A, \gamma\}$ due to Ocneanu and that the canonical *-

endomorphism of Cuntz is the extension $\hat{\gamma}$ of a special shift γ

Our extension algebra $< A, \gamma >$ is generated by A and an isometry W with the relation (1) and (2). Paschke([Pa]) proved that if A is strongly amenable and simple, then the C^*−algebra $C^*(A, W)$ generated by A and W with relations (1) and (2) is always simple. We show that th amenable simple C^*-algebra O_n (which is not strongly amenable) has two kind of isometries W_1 and W_2 with relations (1) and (2) which give simple $C^*(O_n, W_1)$ and non simple $C^*(O_n, W_2)$.

§**2. Crossed product by *-endomorphisms.** In this section, we define a crossed product of a C*-algebra A by a *-endomorphism ρ and investigate some properties of the crossed products which we need in the next section. Our method is somewhat similar to the method for von Neumann algebras in [A] and [T]. Related topics are investigated in [D], [Pa] and [S].

Let A be a unitel C^*-algebra with a faithful state ϕ and ρ a *-endomorphisms of A which satisfies for some $\lambda (0 < \lambda < 1)$ that

$$\phi(\rho(x)) = \lambda\phi(x), \qquad (x \in A).$$

Let ξ_0 be the image of the unit 1 of A in the Hilbert space $L^2(A, \phi)$. We denote by p_n the projection $\rho^n(1)$. Put

$$H_k = \begin{cases} L^2(A,\phi), & (k \leq 0) \\ p_k L^2(A,\phi), & (k \geq 1). \end{cases}$$

and

$$H = \sum_{k \in Z} \oplus H_k.$$

Let π be the representation of A on H defined by

$$(\pi(x)\eta)_k = \begin{cases} \rho(x)\eta_k, & (k \leq 0) \\ \rho^k(x)\eta_k, & (k \geq 1). \end{cases},$$

where $\eta = (\eta_k)_{k\in\mathbf{Z}}$ for $\eta_k \in H_k$. Let W be the isometry defined by

$$(W(\eta))_k = \begin{cases} \frac{1}{\sqrt{\lambda}}\rho(x_{k-1})\xi_0, & (k \leq 0) \\ \frac{1}{\lambda}\rho^2(x_{k-1})\xi_0, & (k \geq 1), \end{cases}$$

where $\eta = (\eta_k)_{k\in\mathbf{Z}}$ and $\eta_k = x_k\xi_0$, for some $x_k \in A$. Then W and π satisfies

$$W\pi(x) = \pi(\rho(x))W \qquad \text{for all} \qquad x \in A$$

and $W^kW^{*k} = \pi(p_k)$ for all k. We assume that $W^*\pi(A)W \subset \pi(A)$. In the next ection, we show that the *-endomorphism ρ induced by a canonical *-endomorphism γ satisfies this assumption.

Proposition 1. Let $C^*(A,W)$ be the C^*−algebra generated by $\pi(A)$ and the above isometry W. Then there exists a faithful conditional expectation E of $C^*(A,W)$ onto $\pi(A)$ with $E(aW_k) = 0$, for all $a \in \pi(A), k \geq 1$. The state ϕ is extended to a faithful state ψ of $C^*(A,W)$ which satisfies $\psi(aW^k) = 0, (a \in \pi(A), k \geq 1)$.

Let us consider the following condition (*) for a *-endomorphism γ of A due to Kishimoto[K] in the case of automorphisms.

Condition (*). For an $a \in A$, a finite set S in A, a finite set F in integers $\mathbf{N}$ and $\epsilon > 0$, there exists a positive $x \in A$ with $\|x\| = 1$ such that

$$\|xax\| \geq \|a\| - \epsilon, \qquad \|xs\rho^k(x)\| \leq \epsilon \ (s \in S, k \in F).$$

Prpposition 2. If ρ satisfies the condition (*), then $C^*(A,W)$ is simple when the only proper ideal J of A for which $WJW^* \subset J$ is the zero ideal.

Remark 3. If the C*-algebra A is strongly amenable, then we dont need the condition (*) for simplicity of $C^*(A,W)$ by [Pa]. However, in the case of A which is not strongly amenable, $< A, W >$ is not always simple without any condition like (*). we show in §3 an example of a pair of an amenable C^*−algebra A and a canonical *- endomorphism γ which inplies non simple $C^*(A, W)$.

§3. Canonical *-endomorphisms. In this section we define a canonical *-endomorphism γ on C^*-algebras, and applying the method of crossed products in §2 to the *-endomorphism ρ induced from γ, we investigate relations between extension algebras and canonical *-endomorphisms.

Definition 4. Let A be a unital C^* algebra with a faithful state ϕ. Let γ be a *-endomorphism of A with $\phi \cdot \gamma = \phi$. If there exist two projections $e \in \gamma(A)' \cap A$ and $f \in \gamma^2(A)' \cap A$ such that :

(1) $$eAe = \gamma(A)e, \qquad \phi(e) = \phi(f)$$

and the projections $\{e_1(= e), e_2(= f), e_3(= \gamma(e))\}$ satisfies Jones relation for $\phi(e) = \phi(f) = \lambda$, that is,

(2) $$e_ie_je_i = \lambda e_i,\ (|i-j| = 1), \qquad e_ie_j = e_je_i,\ (|i-j| \neq 1)$$

then γ is said to be *canonical.* We call the projection e a *basic* projection for γ.

Let γ be a canonical *-endomorphism of A and e a baic projection for γ. We define the *-endomorphism ρ on A by

(3) $$\rho(a) = e\gamma(a), \qquad (a \in A)$$

Then ρ satisfies that $\phi(\rho(x)) = \lambda\phi(x)$ for all $x \in A$, where $\lambda = \phi(e)$.

Lemma 5. Let ρ be a *-endomorphism defined by (**) for a canonical *-endomorphism γ of A. Then the isometry W defined by ρ satisfies that

$$W^*AW \subset A.$$

By Lemma 5, we can apply the result in §2 to the ρ defined by (**) and we consider the crossed product $C^*(A, W)$ in §2 which we denote by $< A, \gamma >$. We call $< A, \gamma >$ the *extension algebra* of A via a canonical *-endomorphism γ.

Proposition 6. Let A be a unital C^*–algebra with a faithful state ϕ and γ a canonical *-endomorphism of A. Let e and f be projections in Definition 4. Then there exists a canonical *-endomorphism $\hat{\gamma}$ of $< A, \gamma >$ which satisfies that

$$\hat{\gamma}(x) = \gamma(x),\ (\in \pi(A)) \qquad \hat{\gamma}(W) = vW,$$

where $v = \lambda^{-1}\gamma(e)fe$.

Many typical canonical *-endomorphisms γ are given as $\gamma = \sigma^2$ for some *-endomorphism σ. Such a *sigma* is also extended to a *-endomorphism $\hat{\sigma}$ of $< A, \gamma >$ which satisfies $\sigma(W) = \lambda^{-1/2} fe$.

§4. **Examples.** In this section we shall restrict ourselves to the case of concrete C^*-algebras and show relations between canonical *-endomorphisms.

Example 1. Let A_0 be the n by n matrix algebra $M_n(\mathbf{C})$ over the complex numbers $\mathbf{C}$. Put $A_i = A_0$ for all integer i. Let A be the infinite C*-tensor product $\bigotimes_i^\infty A_i$. Let γ be the 1-shift translation to the right on A. For a matrix units $e_{i,j}$ of $M_n(\mathbf{C})$. We identify $e_{i,j}$ and $e_{i,j} \otimes 1 \otimes \cdots$. Put $e = e_{1,1}$ and

$$u = \sum_{i=1}^{n} e_{i,i-1}, \qquad f = \sum_{i,j} u^{j-i} \otimes e_{i,j}/n.$$

Then γ, τ, e and f satisfies the conditions in Definition 4 for γ to be canonical.

Cuntz [Cu] defined the simple C^*-algebra O_n which generated by isometries $(S_j)_{1 \le j \le n}$ with $S_i S_j = \delta_{i,j} 1$ and $\sum_i S_i S_i^* = 1$. He obtained interesting results using basically his "canonical" inner *-endomorphism Φ on O_n defined by

$$\Phi(x) = \sum_j S_j x S_j^* \qquad \text{for all} \qquad a \in A.$$

Proposition 7. Let A and γ be the same as in Example 1. Then the extension algebra $< A, \gamma >$ is the Cuntz algebra O_n and the extension $\hat{\gamma}$ of γ to $< A, \gamma >$ is Cuntz's canonical inner *-endomorphism Φ.

Remark 8. By Proposition 5 and 6, Cuntz's endomorphism Φ is also canonical. Hence we have the extension algebra $< O_n, \Phi >$. By the definition, $< O_n, \Phi >$ is generated by $B = \pi(O_n)$ and an isometry $W(= W_\Phi$, which comes from $\Phi)$ with

$$WBW^* \subset B, \qquad W^*BW \subset B \tag{4}$$

Paschke proved that if B is strongly amenable and W is a non unitary isometry with the condition (4) then the C^*-algebra generated by B and W is always simple. By [Cu], O_n is simple and amenable but not strongly amenable. Following Proposition shows that his result does not hold without strong amenability. We also remark that O_n is generated by O_n and W which satisfy the condition (4), and O_n is simple. Hence O_n has two isometries with relation (4) one of which gives a simple C^*-algebra and the other gives a non simple algebra.

Proposition 9. Let Φ be Cuntz's *-endomorphism on O_n. Then $< O_n, \Phi >$ is isomorphic to the tensor product $O_n \otimes C^*(u)$. Here u is a unitary with $u^j \neq 1$ for all integer j.

Example 2. Let A, τ and γ be the same as in Example 1. Put

$$\eta(\gamma^m(e_{p,q})) = \sum_j e_{j,j} \gamma^{m+1}(e_{p+j,q+j})$$

for all $m \geq 0$. Then η is extended to the τ preserving *-endomorphism of A. Put

$$e = e_{1,1}, \qquad f = \sum_{i,j} \frac{e_{i,j}}{n}.$$

Then η, e and f satisfy the conditions for η to be canonical.

In this case,

$$\rho(x) = e\gamma(UxU^*), \qquad \text{for } U = \otimes_{i=1}^{\infty} u_i \text{ all } x \in A,$$

where u_i is the unitary u in the example 1. This implies the following :

In [Cu 2], Cuntz defined the *-endomorphism λ_R on $O(H)$ of a finite dimensional Hilbert space H defined by

$$\lambda_R(S) = RS, (S \in H)$$

for $R = FV$. Here F is the flip symmetry of $H \otimes H$ and V is a multiplicative unitary on $H \otimes H$. Example 2 is generalized to a general finite dimensional Hilbert space H by taking a suitable orthonormal basis. The following shows that Cuntz's λ is also canonical.

proposition 10. Let A, τ and η be the same as in Example 2. Then $< A, \eta >$ is O_n and the extension $\hat{\eta}$ of η to $< A, \eta >$ is the *-endomorphism λ_R due to Cuntz.

Example 3. Let $N \subset M$ be an inclusion of type II_1 factors with Jones index $[M : N] < \infty$. Put $\lambda = [M : N]^{-1}$. Iterating the basic construction for $N \subset M$, we have the tower of II_1 factors :

$$N \subset M_0 = M \subset M_1 \subset \cdots \subset M_j = < M_{j-1}, e_j > \subset \cdots.$$

Here, e_j is the Jones projection for $M_{j-2} \subset M_{j-1}$. Let τ_0 be the unieque trace of M. Then τ_0 is extended to the trace τ_j of M_j via $\tau(xe_j) = \lambda\tau(x)$ for all

$x \in M_{j-1}$. Let

$$A_j = M' \cap M_j \qquad \text{for all integer } j.$$

For an $x \in \bigcup_j A_j$, put $\tau(x) = \tau_j(x)$ when $x \in A_j$ for some j. Let A be the C^*-algebra obtained from the GNS construction of $\bigcup_j A_j$ by τ. Then τ induces a tracial state (which we denote by the same notation τ) on A. The antiautomorphism γ_j of A_{2j} defined by

$$\gamma_j(x) = J_j x^* J_j, \qquad x \in A_{2j},$$

where J_j is the canonical conjugation on $L^2(M_j, \tau_j)$. Then we have ([Ch-H], [O])

$$\gamma_{j+1} \cdot \gamma_j(x) = \gamma_j \cdot \gamma_{j-1}(x), \text{ for all } x \in A_{2j-2} \text{ and } j \geq 1.$$

Since γ_j is τ_j preserving, there is a τ preerving *-endomorphism Γ on A defined by $\Gamma(x) = \gamma_{j+1}\gamma_j(x)$ for all $x \in A_{2j}$.([O]) This Γ is called Ocneanu's 2-shift. The Jones projection $e = e_1$ and $f = e_2$ satisfy the conditions for Γ to be a canonical *-endomorphism for the pair $\{A, \tau\}$.

Example 6. Let G be a finite bipartite with a Ocneanu's biunitary connection ([O2]). Then we have the C^*-algebra A with a unique trace τ obtained from path algebras on G and a *-endomorphism σ on A induced by the connection. Put $\gamma = \sigma^2$. Then γ is canonical. the first Jones projection e_1 and and the second Jones projection e_2 in the path algebra satisfies the conditions in definition 4.

From these canonical *-endomorphisms γ on C^*-algebras A, we obtain simple C^*- algebras $< A, \gamma >$ because we have always a projection q depending the basic projection for these γ as an element x in the condition (*). M. Izumi told me these C^*-algebras are not always Cuntz algebras because they have different $K_0(< A, \gamma >$ from $K_0(O_n)$. Any way we have simple amenable but non strongle amenable C^*- algebras $< A,\gamma >$ and canonical *-endomorphisms on $< A,\gamma >$. Anyway, some results are published in a forsecoming paper. Also related algebras are obtained by Izumi and Katayama in this report.

The termonology "canonical" for *-endomorphisms on infinite factors are used in many papers by Longo (for instance, [L1,L2, L3]). We have similar results as in C^*-algabras for factors, which are descrived in [ch].

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