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Construction of a Kac algebra action on the AFD factor of type II_1

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The purpose of this note is to announce the result obtained in [9]. Namely we describe a construction of an "outer" action of a finite-dimensional Kac algebra on the AFD factor of type II_1.

§ 1. Kac algebras and their actions

Throughout this note, fix a finite-dimensional Hopf C^*-algebra $\mathbf{K}=(\mathcal{M}, \Gamma, \kappa, \varepsilon)$, i.e.,

(i) $\mathcal{M}$ is a finite-dimensional C^*-algebra;

(ii) Γ is a coproduct of $\mathcal{M}$, i.e., an injective homomorphism from $\mathcal{M}$ into $\mathcal{M}\otimes\mathcal{M}$ satisfying the coassociativity: $(\Gamma\otimes\iota)\circ\Gamma=(\iota\otimes\Gamma)\circ\Gamma$;

(iii) ε is a counit of $\mathcal{M}$, i.e., a homomorphism from $\mathcal{M}$ into $\mathbf{C}$ satisfying $(\varepsilon\otimes\iota)\circ\Gamma=(\iota\otimes\varepsilon)\circ\Gamma=\iota$;

(iv) κ is an antipode of $\mathcal{M}$, i.e., a linear mapping from $\mathcal{M}$ into itself satisfying $m_{\mathcal{M}}\circ(\kappa\otimes\iota)\circ\Gamma(a)=m_{\mathcal{M}}\circ(\iota\otimes\kappa)\circ\Gamma(a)=\varepsilon(a)\cdot 1$, where $m_{\mathcal{M}}$ is the multiplication of $\mathcal{M}$;

(v) all the morphisms above are $*$-preserving.

Note that (1) $\kappa^2=\iota$, because of finite-dimensionality of $\mathcal{M}$; (2) if φ is a functional on $\mathcal{M}$ defined by

$$\varphi=\oplus_{i=1}^{k} n_i \mathrm{Tr}_{n_i}$$

along with a decomposition of $\mathcal{M}$:

$$\mathcal{M}\cong M_{n_1}(\mathbf{C})\oplus\cdots\oplus M_{n_k}(\mathbf{C}),$$

where $M_n(\mathbf{C})$ is the full matrix algebra of size n and Tr_n denotes the ordinary trace on $M_n(\mathbf{C})$, then φ is a left-invariant (hence, right-invariant) trace on $\mathcal{M}$: $(\varphi \otimes \iota) \circ \Gamma(a) = (\iota \otimes \varphi) \circ \Gamma(a) = \varphi(a) \cdot 1$. The system $(\mathcal{M}, \Gamma, \kappa, \varphi)$ is a Kac algebra in the sense of Enock-Schwartz, and φ is called the Haar weight. We shall mainly work with $\mathbf{K}=(\mathcal{M}, \Gamma, \kappa, \varphi)$ instead if $(\mathcal{M}, \Gamma, \kappa, \varepsilon)$, since we often consider $\mathcal{M}$ to be represented on the Hilbert space $L^2(\varphi)$ with respect to this specific φ. Once a Kac algebra $\mathbf{K}$ is given, we immediately obtain three new Kac algebras as follows:

(1) The commutant of $\mathbf{K}$, denoted by $\mathbf{K}' = (\mathcal{M}', \Gamma', \kappa', \varphi')$. Here $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $L^2(\varphi)$. The coproduct Γ' is defined by $\Gamma'(y) = (J \otimes J)\Gamma(JyJ)(J \otimes J)$ $(y \in \mathcal{M}')$ with J as the modular conjugation of φ. κ' and φ' are defined similarly.

(2) The reflection of $\mathbf{K}$, denoted by $\mathbf{K}^\sigma = (\mathcal{M}, \Gamma^\sigma, \kappa, \varphi)$. The coproduct Γ^σ is given by $\Gamma^\sigma = \sigma \circ \Gamma$, where σ is the flip: $\sigma(x \otimes y) = y \otimes x$.

(3) The dual of $\mathbf{K}$, denoted by $\mathbf{K}\hat{} = (\mathcal{M}\hat{}, \Gamma\hat{}, \kappa\hat{}, \varphi\hat{})$. This is constructed as follows. By considering the adjoint maps of Γ, κ, $m_{\mathcal{M}}$ and so on, the dual space $\mathcal{M}^*$ can be turned into a Kac algebra. Meanwhile, since φ is faithful, $\mathcal{M}^*$ can be identified with $\mathcal{M}$ by the correspondence $a \in \mathcal{M} \mapsto \varphi_a \in \mathcal{M}^*$, where $\varphi_a(b) = \varphi(ab)$. We write $\mathbf{K}\hat{} = (\mathcal{M}\hat{}, \Gamma\hat{}, \kappa\hat{}, \varphi\hat{})$ for $\mathcal{M}$ with this new Kac algebra structure through this identification, and use notation $f * g$, $f^\sharp$ for the multiplication and the involution of $\mathbf{K}\hat{}$. $\mathcal{M}\hat{}$ too is considered to be represented on $L^2(\varphi)$ via the representation λ: $\lambda(f)g = f * g$.

Combination of these Kac algebras (1) – (3) produces more new Kac algebras such as $\mathbf{K}\hat{}'$, $\mathbf{K}\hat{}^\sigma$ and so on.

Definition. (Nakagami-Takesaki, Enock) An action of $\mathbf{K}=(\mathcal{M}, \Gamma, \kappa, \varphi)$ on a von

Neumann algebra $\mathcal{A}$ is an injective unital $*$-homomorphism β from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{M}$ such that

$$(\beta \otimes \iota) \circ \beta = (\iota \otimes \Gamma) \circ \beta. \tag{$*$}$$

Here are some simple examples of Kac algebra actions.

(1) G is a (finite) group. Let $\alpha : G \longrightarrow \mathrm{Aut}(\mathcal{A})$ be an action of G in the ordinary sense. Then the map $\beta : s \in G \mapsto \alpha_s(a) \in \mathcal{A}$ $(a \in \mathcal{A})$ can be viewed as a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{A} \otimes \ell^\infty(G)$. Moreover, it enjoys property $(*)$ above. Thus β is an action of the commutative Kac algebra $\ell^\infty(G)$ on $\mathcal{A}$. In fact, it is an easy exercise to check that we have a bijective correspondence:

$$\{\alpha :\ \alpha : G \longrightarrow \mathrm{Aut}(\mathcal{A})\,\} \overset{\text{bijection}}{\longrightarrow} \{\beta :\ \beta \text{ is an action of the Kac algebra } \ell^\infty(G) \text{ on } \mathcal{A}\,\}.$$

(2) A map $a \in \mathcal{A} \mapsto a \otimes 1 \in \mathcal{A} \otimes \mathcal{M}$ is clearly an action of $\mathbf{K}$. This is called the trivial action.

(3) Due to coassociativity of a coproduct, Γ itself is an action of $\mathbf{K}$ on $\mathcal{M}$. This fact is crucial in the following discussion.

Definition. For an action β of $\mathbf{K}$ on $\mathcal{A}$, the crossed product $\mathcal{A} \times_\beta \mathbf{K}$ is by definition generated by $\beta(\mathcal{A})$ and $\mathbf{C}_\mathcal{H} \otimes \mathcal{M}^{\wedge\prime}$ (assuming that $\mathcal{A}$ is represented on $\mathcal{H}$). On the crossed product, there exists an action $\tilde{\beta}$ of $\mathbf{K}^{\wedge\prime}$, called the dual action of β. $\tilde{\beta}$ maps the generators $\beta(a)$ and $1 \otimes z$ of the crossed product as follows: $\tilde{\beta}(\beta(a)) = \beta(a) \otimes 1$, $\tilde{\beta}(1 \otimes z) = 1 \otimes \Gamma^{\wedge\prime}(z)$. Dual weight construction holds good also in the case of Kac algebra actions. Moreover, Takesaki duality is true.

§ 2. Construction of a pair of II_1 factors

Start with a Kac algebra $\mathbf{K}=(\mathcal{M}, \Gamma, \kappa, \varphi)$. Let $A_0 = \mathbf{C}$, $A_1 = \mathcal{M}$. Since Γ is an action of $\mathbf{K}$ on $\mathcal{M}$, we may take its crossed product. We set $A_2 = \mathcal{M} \times_\Gamma \mathbf{K}$. On A_2, there is the dual action $\tilde{\Gamma}$ of Γ. So define $A_3 = A_2 \times_{\tilde{\Gamma}} \mathbf{K}^{\hat{}\prime}$. By continuing this procedure, we obtain an increasing sequence $\{A_n\}$ of finite-dimensional C^*-algebras. Remark that we have in general $\mathbf{K}^{\hat{}\hat{}} = \mathbf{K}$, $\mathbf{K}^{\hat{}\sigma} = \mathbf{K}'^{\hat{}}$, $\mathbf{K}^{\sigma\,\prime} = \mathbf{K}^{\prime\,\sigma}$. From this, it follows that

$$A_{4n} = A_{4n-1} \times_{\Gamma^{(4n-2)}} \mathbf{K}^{\sigma\,\prime} \qquad (n \geq 1),$$

$$A_{4n+1} = A_{4n} \times_{\Gamma^{(4n-1)}} \mathbf{K}^{\hat{}\sigma} \qquad (n \geq 0),$$

$$A_{4n+2} = A_{4n+1} \times_{\Gamma^{(4n)}} \mathbf{K} \qquad (n \geq 0),$$

$$A_{4n+3} = A_{4n+2} \times_{\Gamma^{(4n+1)}} \mathbf{K}^{\hat{}\prime} \qquad (n \geq 0),$$

where $\Gamma^{(-1)} =$ the trivial action of $\mathbf{K}^{\hat{}\sigma}$ on $A_0 = \mathbf{C}$, $\Gamma^{(0)} = \Gamma$, and $\Gamma^{(n)} =$ the dual action of $\Gamma^{(n-1)}$. By Takesaki duality,

$$A_{2n} \cong \otimes^n M_{\dim \mathcal{M}}(\mathbf{C}) \qquad (n \geq 1).$$

Next we put $B_0 = \mathcal{M}^{\hat{}\sigma}$. Then define B_n inductively by

$$B_{4n} = B_{4n-1} \times_{\delta^{(4n-1)}} \mathbf{K}^{\sigma\,\prime} \qquad (n \geq 1),$$

$$B_{4n+1} = B_{4n} \times_{\delta^{(4n)}} \mathbf{K}^{\hat{}\sigma} \qquad (n \geq 0),$$

$$B_{4n+2} = B_{4n+1} \times_{\delta^{(4n+1)}} \mathbf{K} \qquad (n \geq 0),$$

$$B_{4n+3} = B_{4n+2} \times_{\delta^{(4n+2)}} \mathbf{K}^{\hat{}\prime} \qquad (n \geq 0),$$

where $\delta^{(0)} = \delta = \Gamma^{\hat{}\sigma}$, and $\delta^{(n)} =$ the dual action of $\delta^{(n-1)}$. Thus we get another increasing sequence $\{B_n\}$ of finite-dimensional C^*-algebras. Takesaki duality implies

$$B_{2n-1} \cong \otimes^n M_{\dim \mathcal{M}}(\mathbf{C}) \qquad (n \geq 1).$$

Observation 1. For each $n \geq 0$, A_n can be considered as a subalgebra of B_n. For example, if $n = 1, 2$, we have

$$A_1 = \mathcal{M}, \qquad B_1 = \delta(\mathcal{M}^{\wedge}) \vee \mathbf{C} \otimes \mathcal{M};$$

$$A_2 = \Gamma(\mathcal{M}) \vee \mathbf{C} \otimes \mathcal{M}^{\wedge\prime}, \qquad B_2 = \delta(\mathcal{M}^{\wedge}) \otimes \mathbf{C} \vee \mathbf{C} \otimes \Gamma(\mathcal{M}) \vee \mathbf{C} \otimes \mathbf{C} \otimes \mathcal{M}^{\wedge\prime}.$$

Hence $\pi_n(a) = 1 \otimes a \ \ (a \in A_n)$ in general embeds A_n into B_n so that the diagram

$$\begin{array}{ccc} B_n & \rightarrow & B_{n+1} \\ \uparrow & & \uparrow \\ A_n & \rightarrow & A_{n+1} \end{array}$$

commutes. Moreover, we have

Theorem 1. For each $n \geq 0$,

$$\begin{array}{ccc} B_n & \rightarrow & B_{n+1} \\ \uparrow & & \uparrow \\ A_n & \rightarrow & A_{n+1} \end{array}$$

forms a commuting square. Here, on each B_n, we consider the faithful trace obtained as the dual weight by crossed product construction.

Proof for $n = 0$. By Takesaki duality, $B_1 \overset{\pi}{\cong} \mathcal{L}(L^2(\varphi))$. By keeping track of how this isomorphism π was constructed, one has that

$$\pi(B_0) = \mathcal{M}^{\wedge}, \qquad \pi(A_1) = \mathcal{M}.$$

Thus π transforms the diagram in question into

$$\begin{array}{ccc} \mathcal{M}^{\wedge} & \rightarrow & \mathcal{L}(L^2(\varphi)) \\ \uparrow & & \uparrow \\ \mathbf{C} & \rightarrow & \mathcal{M}. \end{array}$$

Hence it suffices to show that this diagram is a commuting square. For this purpose, we need to recall the unitary canonically associated to every Kac algebra, called the fundamental unitary (or the Kac-Takesaki operator). It is defined in the following way. Since the Haar weight φ is left-invariant, the equation

$$W(f \otimes g) = \Gamma(g)(f \otimes 1) \qquad (f, g \in \mathcal{M})$$

defines an isometry on $L^2(\varphi) \otimes L^2(\varphi)$. It is actually a unitary that belongs to $\mathcal{M} \otimes \mathcal{M}\hat{}$. Moreover, W implements the coproduct Γ: $\Gamma(a) = W(a \otimes 1)W^*$, and the coassociativity is shown to be equivalent to the so-called the pentagon equation

$$W_{12}W_{23} = W_{23}W_{13}W_{12}.$$

We see below that W contains more information on the given Kac algebra $\mathbf{K}$. First, since $W \in \mathcal{M} \otimes \mathcal{M}\hat{}$, it has the form

$$W = \sum_{i=1}^{d} a_i \otimes \lambda(f_i),$$

where a_i, $f_i \in \mathcal{M}$ $(i = 1, 2, \ldots, n)$. We may assume that $\{f_1, f_2, \ldots, f_d\}$ is linearly independent in $\mathcal{M}$.

Proposition 1. With the above notation, we have $d = \dim \mathcal{M}$. Thus $\{f_1, f_2, \ldots, f_d\}$ is a basis for $\mathcal{M}$. In fact, for any $f \in \mathcal{M}$,

$$f = \sum_{i=1}^{d} \varphi(f a_i^*) f_i^\sharp = \sum_{i=1}^{d} \varphi(f^\vee a_i) f_i = \sum_{i=1}^{d} \varphi(f^\vee a_i^*) f_i^*.$$

Moreover, the set $\{a_1, a_2, \ldots, a_d\}$ also forms a basis for $\mathcal{M}$ and satisfies

$$a = \sum_{i=1}^{d} \varphi(a f_i^\vee) a_i = \sum_{i=1}^{d} \varphi(a f_i^\sharp) a_i^* = \sum_{i=1}^{d} \varphi(a^\vee f_i^\sharp) a_i^\sharp$$

for any $a \in \mathcal{M}$. Moreover,

$$\Gamma(a) = \sum_{i=1}^{d} a_i \otimes (f_i * a) \qquad (a \in \mathcal{M});$$

$$\hat{\Gamma}(\lambda(f)) = \sum_{i=1}^{d} \lambda(f_i^\sharp) \otimes \lambda(a_i^* f)$$

for any $f \in \mathcal{M}$. The algebra $\mathcal{L}(L^2(\varphi))$ coincides with $\mathrm{span}\{\lambda(f_i)a_j : 1 \le i, j \le d\}$. The unique conditional expectations $E_{\mathcal{M}}$ and $E_{\mathcal{M}\hat{}}$ from $\mathcal{L}(L^2(\varphi))$ onto $\mathcal{M}$ and $\mathcal{M}\hat{}$ with respect to the normalized trace on $\mathcal{L}(L^2(\varphi))$ is respectively given by

$$E_{\mathcal{M}}(\sum_{i=1}^{d}\lambda(f_i)b_i) = \sum_{i=1}^{d}\varepsilon(f_i)b_i \qquad (b_i \in \mathcal{M});$$

$$E_{\mathcal{M}^{\hat{}}}(\sum_{i=1}^{d}\lambda(k_i)a_i) = \sum_{i=1}^{d}\varphi(a_i)\lambda(k_i) \qquad (k_i \in \mathcal{M}).$$

In particular,

$$E_{\mathcal{M}}(\lambda(f)) = \varepsilon(f)\cdot 1,$$

$$E_{\mathcal{M}^{\hat{}}}(a) = \varphi(a)\cdot 1.$$

Thus the diagram

$$\begin{array}{ccc} \mathcal{M}^{\hat{}} & \rightarrow & \mathcal{L}(L^2(\varphi)) \\ \uparrow & & \uparrow \\ \mathbf{C} & \rightarrow & \mathcal{M}. \end{array}$$

is a commuting square.

Therefore, Proposition 1 proves the preceeding Theorem for the case $n = 0$.

Let A_∞ and B_∞ be the approximately finite-dimensional (AF) C^*- algebras obtained from the sequences $\{A_n\}$ and $\{B_n\}$, respectively. The algebra A_∞ is regarded as a C^*-subalgebra of B_∞ in an obvious way. B_∞ is the d^∞-UHF algebra and thus has the unique faithful factorial tracial state τ. We denote by $\mathcal{Q}$ the von Neumann algebra $\pi_\tau(B_\infty)''$ generated by the GNS representation π_τ of τ on B_∞, which is the AFD factor of type II_1. Set $\mathcal{P} = \pi_\tau(A_\infty)'' \subseteq \mathcal{Q}$. The algebra $\mathcal{P}$ is again the AFD factor of type II_1. Therefore, we have constructed a factor-subfactor pair of the AFD factors $\mathcal{P}$ and $\mathcal{Q}$.

§ 3. Construction of an action β on $\mathcal{P}$

To motivate an idea, we digress and consider a problem of constructing an action α of a group G on a von Neumann algebra $\mathcal{A}$ when G is given. One way to do this is

(i) to find a Hilbert space $\mathcal{H}$ on which G admits a unitary repesentation u so that $u(s)\mathcal{A}u(s)^* = \mathcal{A}$ for any $s \in G$;

(ii) then define $\alpha_s = \mathrm{Ad}u(s)$.

In terms of the correspondence

$$\{\alpha : \ \alpha : G \longrightarrow \mathrm{Aut}(\mathcal{A})\} \overset{\text{bijection}}{\longrightarrow} \{\beta : \ \beta \text{ is an action of the Kac algebra } \ell^\infty(G) \text{ on } \mathcal{A}\},$$

this procedure is the same as

(i) to find a Hilbert space $\mathcal{H}$ for which there exists a unitary $R \in \mathcal{L}(\mathcal{H}) \otimes \ell^\infty(G)$ satisfying $(\iota \otimes \Gamma_G)(R) = R_{12}R_{13}$ (Γ_G is the coproduct of $\ell^\infty(G)$) and $R(\mathcal{A} \otimes \mathbf{C})R^* \subseteq \mathcal{A} \otimes \ell^\infty(G)$;

(ii) then define $\beta(a) = R(a \otimes 1)R^*$.

For a general $\mathbf{K}$=($\mathcal{M}$, Γ, κ, φ), the idea is the same. Namely we

(i) find a unitary $R \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}$ satisfying $(\iota \otimes \Gamma)(R) = R_{12}R_{13}$ and $R(\mathcal{A} \otimes \mathbf{C})R^* \subseteq \mathcal{A} \otimes \mathcal{M}$;

(ii) then define $\beta(a) = R(a \otimes 1)R^*$.

So we will look for such a unitary R below to construct an action β on the factor $\mathcal{P}$. First, let us look at the embedding, say γ, of B_0 into $\mathcal{Q}$:

$$\gamma : B_0 = \mathcal{M}\hat{\ } \hookrightarrow B_\infty \subseteq \mathcal{Q}.$$

Secondly, with W as the fundamental unitary of $\mathbf{K}$, consider $S = \sigma W \sigma$ which lies in $\mathcal{M}\hat{\ } \otimes \mathcal{M}$. Put $R = (\gamma \otimes \iota_{\mathcal{M}})(S) \in \mathcal{Q} \otimes \mathcal{M}$.

Theorem 2. The unitary R satisfies $(\iota \otimes \Gamma^\sigma)(R) = R_{12}R_{13}$ and $R(\mathcal{P} \otimes \mathbf{C})R^* \subseteq \mathcal{P} \otimes \mathcal{M}$. Thus the equation

$$\beta(X) = R(X \otimes 1)R^* \qquad (X \in \mathcal{P})$$

defines an action of the reflection $\mathbf{K}^\sigma$ on $\mathcal{P}$. Moreover, the inclusion $\mathcal{P} \subseteq \mathcal{Q}$ is spatially isomorphic to $\mathcal{P} \subseteq \mathcal{P} \times_\beta \mathbf{K}^\sigma$.

To ensure that β is not a trivial action, we show that it is outer, i.e., the relative commutant $\beta(\mathcal{P})' \cap \mathcal{P} \times_\beta \mathbf{K}^\sigma$ is trivial. This is done by proving the following theorem.

Theorem 3. With the notation as before, we have

$$E_{B_n}(B_{n+1} \cap A'_{n+1}) \subseteq \mathbf{C},$$

where E_{B_n} is the unique conditional expectation from $\mathcal{Q}$ onto B_n with respect to the normalized trace on $\mathcal{Q}$.

The essential part of the proof of this theorem is to prove the assertion when $n = 0$. If $n = 0$, then, as we noted,

$$\begin{array}{ccc} \mathcal{M}^\wedge & \rightarrow & \mathcal{L}(L^2(\varphi)) \\ \uparrow & & \uparrow \\ \mathbf{C} & \rightarrow & \mathcal{M}. \end{array} \quad \cong \quad \begin{array}{ccc} B_0 & \rightarrow & B_1 \\ \uparrow & & \uparrow \\ \mathbf{C} & \rightarrow & A_1. \end{array}$$

From this, we see that the assertion of the theorem is equivalent to $E_{\mathcal{M}^\wedge}(\mathcal{M}') \subseteq \mathbf{C}$. Thus it suffices to prove that the diagram

$$\begin{array}{ccc} \mathcal{M}^\wedge & \rightarrow & \mathcal{L}(L^2(\varphi)) \\ \uparrow & & \uparrow \\ \mathbf{C} & \rightarrow & \mathcal{M}' \end{array}$$

is also a commuting square. But this can be verified exactly the same way as before.

§ 4. The Jones index of $\mathcal{P} \subseteq \mathcal{Q}$

To compute the Jones index $[\mathcal{Q} : \mathcal{P}]$, it is enough by Theorem 2 to calculate $[\mathcal{P} \times_\beta \mathbf{K}^\sigma : \mathcal{P}]$. For this purpose, we describe the Jones projection $e_{\mathcal{P}}$ of this inclusion. First, it can be shown that $\tilde{J}\beta(\mathcal{P})\tilde{J} = \mathcal{P}' \otimes \mathbf{C}$, where $\tilde{J}$ is the modular conjugation of the normalized trace on the crossed product. Hence the extension of $\mathcal{P} \subseteq \mathcal{P} \times_\beta \mathbf{K}^\sigma$ is $\mathcal{P} \otimes \mathcal{L}(L^2(\varphi))$. So $e_{\mathcal{P}}$ belongs to $\mathcal{P} \otimes \mathcal{L}(L^2(\varphi))$. It can be proven that it has the form

$$e_{\mathcal{P}} = 1 \otimes p,$$

where p is a minimal projection in $\mathcal{L}(L^2(\varphi))$. In fact, p is the projection corresponding to the one-dimensional representation of $\mathcal{M}$, i.e., the counit ε. Thus

$$\text{Trace}(e_{\mathcal{P}}) = (\dim \mathcal{M})^{-1}.$$

Therefore, $[\mathcal{P} \times_\beta \mathbf{K}^\sigma : \mathcal{P}] = \dim \mathcal{M}$.

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