数理解析研究所講究録
第 858 巻 1994 年 32-46

Interpolation in Analytic Crossed Products

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1. Introduction

Nonselfadjoint operator algebras have been studied since the paper of Kadison and Singer in 1960. In [1], Arveson introduced the notion of subdiagonal algebras as the generalization of weak*-Dirichlet algebras and studied the analyticity of operator algebras. After that, we have many papers about nonselfadjoint algebras in this direction: nest algebras, CSL algebras, reflexive algebras, analytic operator algebras, analytic crossed products and so on. Since the notion of subdiagonal algebras is the analogue of weak*-Derichlet algebras, subdiagonal algebras have many fruitful properties from the theory of function algebras. Thus, we have several attempts in this direction: Beurling-Lax-Halmos theorem for invariant subspaces, maximality, factorization theorem and so on.

Our aim in this note is to study the interpolation theory of subdiagonal algebras, in particular, analytic crossed products. In [15], Sarason studied the theory of generalized interpolation in $\mathbb{H}^\infty$ over the unit circle and proved two classical interpolation theorems due to Caratheodory and Pick. In fact, the interpolation is an operator dilation. In this note, we introduce the notion of interpolation in finite maximal subdiagonal algebras, in particular, analytic crossed products.

Our setting is the following. Let $\mathfrak{A}$ be a von Neumann algebra with a faithful normal tracial state τ. Let $\mathbb{H}^{\infty}$ be a finite maximal subdiagonal algebra of $\mathfrak{A}$ with respect to Φ and τ, where Φ is the faithful normal conditional expectation of $\mathfrak{A}$ onto the diagonal of $\mathbb{H}^{\infty}$. We consider the noncommutative L^2-space $L^2(\mathfrak{A},\tau)$ associated with τ and define the noncommutative Hardy space $\mathbb{H}^2$ by the closure of $\mathbb{H}^{\infty}$ in $L^2(\mathfrak{A},\tau)$. For every $a \in \mathfrak{A}$, we define operators L_a and R_a by $L_a x = ax$ and $R_a x = xa$ for $x \in L^2(\mathfrak{A},\tau)$. For every subset $S \subset \mathfrak{A}$, we put $L(S) = \{L_a\}_{a\in S}$ and $R(S) = \{R_a\}_{a\in S}$, respectively. If $\mathbb{H}^{\infty}$ is an analytic crossed product determined by a finite von Neumann algebra N and an automorphism α on N, then we study the invariant subspace structure under $\mathbb{H}^{\infty}$ (cf. [6], [7], [13] and so on). As in [6], we proved that if $\mathbb{H}^{\infty}$ is an anlytic crossed product and the diagonal N of $\mathbb{H}^{\infty}$ is a factor, then every two-sided invariant subspace of $\mathbb{H}^2$ is of the form $L_w\mathbb{H}^2$ for some unitary operator w in $\mathbb{H}^{\infty}$. Thus, we now consider a two-sided invariant subspace of $\mathbb{H}^2$ which is of the form $L_w\mathbb{H}^2$ for some unitary operator w in $\mathbb{H}^{\infty}$. Put $K = \mathbb{H}^2 \ominus L_w\mathbb{H}^2$. The orthogonal projection in $L^2(\mathfrak{A},\tau)$ with range K will be denoted by P_K. For any operator A in $L(\mathbb{H}^{\infty})$ or $R(\mathbb{H}^{\infty})$, we define the operators S_A on K by $S_A f = P_K(Af)$, $f \in K$. When an operator T on K can be written S_A for some A in $L(\mathbb{H}^{\infty})$ or $R(\mathbb{H}^{\infty})$, we shall say that this operator A interpolates T. We now set $\mathfrak{L}_+(K) = \{S_A: A \in L(\mathbb{H}^{\infty})\}$ and $\mathfrak{R}_+(K) = \{S_A: A \in R(\mathbb{H}^{\infty})\}$, respectively. Then we show that $\mathfrak{L}_+(K)$ and $\mathfrak{R}_+(K)$ are weakly closed subalgebras of $B(K)$ and $\mathfrak{L}_+(K)$ commutes with $\mathfrak{R}_+(K)$. Furthermore, if $\mathbb{H}^{\infty}$ is an analytic crossed product determined by a finite von Neumann algebra and a $*$-automorphism, then we prove that, if T is a bounded linear operator on K that commutes with $\mathfrak{R}_+(K)$, then there exists an operator A in $L(\mathbb{H}^\infty)$ such that $T = S_A$ and $\|T\| = \|A\|$. That is, $\mathfrak{L}_+(K)' = \mathfrak{R}_+(K)$ and $\mathfrak{R}_+(K)' = \mathfrak{L}_+(K)$ (Theorem 3.2). As the corollary, we prove that the generalization of Caratheodory-Fejer Theorem (Theorem 4.1).

2. Interpolation in finite maximal subdiagonal algebras

Let $\mathfrak{M}$ be a von Neumann algebra with a faithful normal tracial state τ. First, we recall the definition of finite subdiagonal algebras of $\mathfrak{M}$.

Definition 2.1. Let $\mathbb{H}^\infty$ be a subalgebra of M containing the unit and let Φ be a faithful normal expectation from $\mathfrak{M}$ onto $\mathfrak{D}$ ($= \mathbb{H}^\infty \cap \mathbb{H}^{\infty *}$). Then $\mathbb{H}^\infty$ is called a finite subdiagonal algebra of $\mathfrak{M}$ with respect to Φ and τ in case the following conditions are satisfied: (1) $\mathbb{H}^\infty + \mathbb{H}^{\infty *}$ is σ-weakly dense in $\mathfrak{M}$; (2) $\Phi(xy) = \Phi(x)\Phi(y)$, for all $x, y \in \mathbb{H}^\infty$; (3) $\tau \circ \Phi = \tau$.

Let $\mathbb{H}^\infty$ be a σ-weakly closed finite subdiagonal subalgebra of $\mathfrak{M}$ with respect to Φ and τ. By [3, Theorem 7], then $\mathbb{H}^\infty$ is maximal among those subalgebras of $\mathfrak{M}$ satisfying (1) and (2) of Definition 2.1. We shall denote the noncommutative L^p-spaces associated with $\mathfrak{M}$ and τ by $L^p(\mathfrak{M},\tau)$, $1 \le p < \infty$. For $1 \le p < \infty$, the closure of $\mathbb{H}^\infty$ in $L^p(\mathfrak{M},\tau)$ is denoted by $\mathbb{H}^p$ and the closure of $\mathbb{H}_0^\infty = \{x \in \mathbb{H}^\infty : \Phi(x) = 0\}$ is denoted by $\mathbb{H}_0^p$. We refer for the properties of $\mathbb{H}^p$ and $\mathbb{H}_0^p$ to [9, 10].

We denote the operators in the left regular representation of $\mathfrak{M}$ on $L^2(\mathfrak{M},\tau)$ by L_x, $x \in \mathfrak{M}$, and those in the right regular representaion by R_x, $x \in \mathfrak{A}$. Put $\mathfrak{L} = \{L_x : x \in \mathfrak{A}\}$, $\mathfrak{R} = \{R_x : x \in \mathfrak{A}\}$, $\mathfrak{L}_+ = \{L_x : x \in \mathbb{H}^\infty\}$ and $\mathfrak{R}_+ = \{R_x : x \in \mathbb{H}^\infty\}$, respectively.

Definition 2.2. Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathfrak{A}, \tau)$. We shall say that $\mathfrak{M}$ is: left-invariant, if $\mathfrak{L}_+\mathfrak{M} \subset \mathfrak{M}$; left-reducing, if $\mathfrak{L}\mathfrak{M} \subset \mathfrak{M}$; left-pure, if $\mathfrak{M}$ contains no left-reducing subspaces; and left-full, if the smallest left-reducing subspace containing $\mathfrak{M}$ is all of $L^2(\mathfrak{A}, \tau)$. The right-hand versions of these concepts are defined similarly, and a subspace which is both left-invariant and right-invariant will be called two-sided invariant.

In this note, we consider a two-sided invariant subspace $\mathfrak{M}$ of $\mathbb{H}^2$ which is of the form $L_w\mathbb{H}^2$ $(= w\mathbb{H}^2)$ for some unitary operator w in $\mathbb{H}^\infty$. At first, we study the interpolation for $L_w\mathbb{H}^2$. Let K be the subspace $\mathbb{H}^2 \ominus L_w\mathbb{H}^2$ of $\mathbb{H}^2$. The orthogonal projection in $L^2(\mathfrak{A}, \tau)$ with range K will denoted by P_K. Then, for every $a \in \mathbb{H}^\infty$, the operators S_{L_a} and S_{R_a} is defined by

$$S_{L_a} f = P_K(L_a f) \quad \text{and} \quad S_{R_a} f = P_K(R_a f), \qquad f \in K.$$

When an operator T on K can be written as S_A for some operator A in $\mathfrak{L}_+$ or $\mathfrak{R}_+$, we shall say that this operator A interpolates T.

Proposition 2.3. If $A \in \mathfrak{L}_+$ and $B \in \mathfrak{R}_+$, then $S_A S_B = S_B S_A$.

Proof. Let Σ be the σ-weakly closed subalgebra of $B(L^2(\mathfrak{A}, \tau))$ generated by $\mathfrak{L}_+$ and $\mathfrak{R}_+$. Since $\mathfrak{M}$ is two-sided invariant, $\mathfrak{M}$ is Σ-invariant. By [14, Lemma 0], K is semi-invariant under Σ in the sense that the projection P_K satisfies $P_K A_1 P_K A_2 P_K = P_K A_1 A_2 P_K$ for all $A_1, A_2 \in \Sigma$. Therefore, we have for every $A \in \mathfrak{L}_+$ and $B \in \mathfrak{R}_+$,

$$S_A S_B f = P_K A P_K B P_K f = P_K A B P_K f = P_K B A P_K f = P_K B P_K A P_K f = S_B S_A f,$$

where $f \in K$. This completes the proof.

We now put $\mathfrak{L}_+(K) = \{S_{L_a} : a \in \mathbb{H}^\infty\}$ and $\mathfrak{R}_+(K) = \{S_{R_a} : a \in \mathbb{H}^\infty\}$, respectively. Since K is semi-invariant as in the proof of Proposition 2.3, $\mathfrak{L}_+(K)$ and $\mathfrak{R}_+(K)$ are subalgebras of $B(K)$. Further, we define the map φ from $\mathbb{H}^\infty$ onto $\mathfrak{L}_+(K)$ by $\varphi(a) = S_{L_a}$, $a \in \mathbb{H}^\infty$. Then we have

Proposition 2.4. The map φ from $\mathbb{H}^\infty$ onto $\mathfrak{L}_+(K)$ is a homomorphism with $\ker\varphi = w\mathbb{H}^\infty$.

Proof. Since K is semi-invariant under Σ as in the proof of Proposition 2.3, it is clear that φ is homomorphic. Take any $a \in \mathbb{H}^\infty$ such that $\varphi(a) = 0$. Then $P_K L_a f = 0$ for every $f \in K$ and so $L_a f = af \in L_w \mathbb{H}^2 \subset \mathbb{H}^2$. On the other hand, since $L_w\mathbb{H}^2$ is two-sided invariant, $L_a L_w \mathbb{H}^2 \subset L_w\mathbb{H}^2$. Thus, $L_a\mathbb{H}^2 \subset L_w\mathbb{H}^2$. This implies that $a \in L_w\mathbb{H}^2 \cap \mathfrak{A} = L_w\mathbb{H}^\infty$ by [10, Theorem]. Hence we have $\ker\varphi \subset w\mathbb{H}^\infty$. Conversely, taking any $a \in w\mathbb{H}^\infty$, we have $L_a\mathbb{H}^2 \subset L_w\mathbb{H}^2$ and so $L_a K \subset L_w\mathbb{H}^2$. This implies that $\varphi(a) = 0$. This completes the proof.

By Proposition 2.4, the map φ induces the isomorphism $\tilde{\varphi}$ from $\mathbb{H}^\infty/_{w\mathbb{H}^\infty}$ onto $\mathfrak{L}_+(K)$. By [9, Proposition 3], we have

$$\mathbb{H}^1_0 = (\mathbb{H}^\infty)^\perp = \{f \in L^1(\mathfrak{A},\tau) : \tau(fa) = 0,\ a \in \mathbb{H}^\infty\}.$$

Thus $\mathbb{H}^\infty$ is isomorphic to the dual of $L^1(\mathfrak{A},\tau)/_{\mathbb{H}^1_0}$. Further, it is clear that $(w\mathbb{H}^\infty)^\perp = \mathbb{H}^1_0 w^*$. By [11, Lemma 5.5], we have

Lemma 2.5. Let $x \in \mathbb{H}^1$. Then, for every $n \in \mathbb{N}$, there exist two elements y_n and $z_n \in \mathbb{H}^2$ such that $x = y_n z_n$, $\|y_n\|_2^2 \le n^{-2} + \|x\|_1$ and $\|z_n\|_2^2 \le n^{-2} + \|x\|_1$. Further, if $x \in \mathbb{H}^1_0$, then we can choose the element z_n such that $z_n \in \mathbb{H}^2_0$.

Lemma 2.6. Let $f \in \mathbb{H}^1_0$. Then, for every $n \in \mathbb{N}$, there exist two elements g_n, $h_n \in K$ such that $\|g_n\|_2^2 \le \|f\|_1 + n^{-2}$, $\|h_n\|_2^2 \le \|f\|_1 + n^{-2}$ and $\tau(afw^*) = (S_{L_a} g_n, h_n)$, $a \in \mathbb{H}^\infty$. Conversely, if g, $h \in K$, then there exists an element $f \in \mathbb{H}_0^1$ such that $\tau(afw^*) = (S_{L_a} g, h)$, $a \in \mathbb{H}^\infty$.

Proof. Let $f \in \mathbb{H}_0^1$. By Lemma 2.5, for every $n \in \mathbb{N}$, there exist two elements $y_n \in \mathbb{H}^2$ and $z_n \in \mathbb{H}_0^2$ such that $f = y_n z_n$, $\|y_n\|_2^2 \le \|f\|_1 + n^{-2}$ and $\|z_n\|_2^2 \le \|f\|_1 + n^{-2}$. Hence we have for every $a \in \mathbb{H}^\infty$,

$$\tau(afw^*) = \tau(L_a y_n z_n w^*) = (L_a y_n, wz_n{}^*).$$

Since $z_n{}^* \in (\mathbb{H}_0^2)^* = (\mathbb{H}^2)^\perp$, we have $wz_n{}^* \in w(\mathbb{H}^2)^\perp = (w\mathbb{H}^2)^\perp = K \oplus (\mathbb{H}^2)^\perp$. Hence we have $(I-P_K)wz_n{}^* \in (\mathbb{H}^2)^\perp$. Putting $h_n = P_K wz_n{}^*$, then we have, for every $a \in \mathbb{H}^\infty$,

$$(L_a y_n, wz_n{}^*) = (L_a y_n, P_K wz_n{}^*) = (L_a y_n, h_n).$$

Further, since $y_n - P_K y_n \in w\mathbb{H}^2$, we have $L_a(I-P_K)y_n \in w\mathbb{H}^2$. We now put $g_n = P_K y_n$. Then $(L_a y_n, h_n) = (L_a(I-P_K)y_n + L_a P_K y_n, h_n) = (L_a g_n, h_n) = (S_{L_a} g_n, h_n)$. Therefore, we have $\|g_n\|_2^2 \le \|y_n\|_2^2 \le \|f\|_1 + n^{-2}$, $\|h_n\|_2^2 \le \|z_n\|_2^2 \le \|f\|_1 + n^{-2}$ and $\tau(afw^*) = (S_{L_a} g_n, h_n)$, $a \in \mathbb{H}^\infty$.

Conversely, let g, $h \in K$. Then $w^*h \in (\mathbb{H}^2)^\perp$ and so $h^*w \in \mathbb{H}_0^2$. Putting $f = gh^*w$, then we have, for every $a \in \mathbb{H}^\infty$,

$$\tau(afw^*) = \tau(L_a gh^* ww^*) = \tau(L_a gh^*) = (L_a g, h) = (S_{L_a} g, h).$$

This completes the proof.

Proposition 2.7. The natural isomorphism $\tilde{\varphi}$ of $\mathbb{H}^\infty/_{w\mathbb{H}^\infty}$ onto $\mathfrak{L}_+(K)$ is norm preserving.

Proof. Let any $a \in \mathbb{H}^\infty$. For every $b \in \mathbb{H}^\infty$, we have $\|S_{L_a}\| = \|P_K L_a P_K\| = \|P_K(L_a + L_w L_b)P_K\| \le \|a+wb\|$ and so $\|\varphi(a)\| \le \|a+w\mathbb{H}^\infty\|$. Take any $a \in \mathbb{H}^\infty$ such that $\|a+w\mathbb{H}^\infty\| = 1$. Since $\mathbb{H}^\infty/_{w\mathbb{H}^\infty}$ is isomorphic to the dual of $\mathbb{H}_0^1 w^*/_{\mathbb{H}_0^1}$, for every $\varepsilon > 0$, there exists an element $f \in \mathbb{H}_0^1$ such that $\|f\|_1 = 1$ and $|\tau(afw^*)| > 1-\varepsilon$. By Lemma 2.6, for any $n \in \mathbb{N}$, there exist two elements $g_n, h_n \in K$ such that $\|g_n\|_2^2 \le 1 + n^{-2}$, $\|h_n\|_2^2 \le 1 + n^{-2}$ and $\tau(cfw^*) = (S_{L_c} g_n, h_n)$, $c \in \mathbb{H}^\infty$. Thus

$$1 - \varepsilon < |\tau(afw^*)| = |(S_{L_a} g_n, h_n)| \le \|S_{L_a}\|\|g_n\|\|h_n\|$$
$$\le \|S_{L_a}\|(1 + n^{-2}).$$

Therefore we have $1-\varepsilon < \|S_{L_a}\|$ and so $\|\varphi(a)\| = \|S_{L_a}\| = 1$. This completes the proof.

Proposition 2.8. For any $a \in \mathbb{H}^\infty$, there exists an element $b \in \mathbb{H}^\infty$ such that $\|a + wb\| = \|a + w\mathbb{H}^\infty\|$.

Proof. For every $n \in \mathbb{N}$, there exists $b_n \in \mathbb{H}^\infty$ such that $\|a + wb_n\| \le \|a + w\mathbb{H}^\infty\| + 1/n$. Since $\{b_n\}$ is bounded in $\mathbb{H}^\infty$ and the unit ball of $\mathbb{H}^\infty$ is compact with respect to the weak*-topology, there exist an element $b_0 \in \mathbb{H}^\infty$ and a subsequence $\{b_{n(k)}\}$ such that weak*-limit $b_{n(k)} = b_0$. Then it is clear that $\|a + wb_0\| = \|a + w\mathbb{H}^\infty\|$. This completes the proof.

By Propositions 2.7 and 2.8, we have

Corollary 2.9. Let $T \in B(K)$ such that $T = S_{L_a}$ for some $a \in \mathbb{H}^\infty$. Then there exists an element $b \in \mathbb{H}^\infty$ such that $T = S_{L_b}$ and $\|T\| = \|b\|$.

Proposition 2.10. The natural isomorphism $\tilde{\varphi}$ of $\mathbb{H}^\infty/_{w\mathbb{H}^\infty}$ onto $\mathfrak{L}_+(K)$ is a homeomorphism relative to the weak*-topology on $\mathbb{H}^\infty/_{w\mathbb{H}^\infty}$ and the weak operator topology on $\mathfrak{L}_+(K)$.

Proof. Suppose that $\{a_j\}_{j\in\Lambda}$ is a net in $\mathbb{H}^\infty$ and a_0 an operator in $\mathbb{H}^\infty$ such that $\varphi(a_j) \to \varphi(a_0)$ in the weak operator topology. By Lemma 2.6, for any $f \in \mathbb{H}_0^1$, there exist two elements g, $h \in K$ such that $\tau(afw^*) = (\varphi(a)g,h)$, $a \in \mathbb{H}^\infty$. Hence we have

$$(fw^*+\mathbb{H}_0^1,\ a_j+w\mathbb{H}^\infty) = (fw^*,\ a_j) = \tau(a_jfw^*)$$
$$= (\varphi(a_j)g,h) \to (\varphi(a_0)g,h) = \tau(a_0fw^*) = (fw^*,a_0)$$
$$= (fw^*+\mathbb{H}_0^1,\ a_0+w\mathbb{H}^\infty).$$

This implies that $a_j+w\mathbb{H}^\infty \to a_0+w\mathbb{H}^\infty$ in the weak$*$-topology.

Conversely, suppose that $\{a_j\}_{j\in\Lambda}$ is a net in $\mathbb{H}^\infty$ and a_0 an operator in $\mathbb{H}^\infty$ such that $a_j+w\mathbb{H}^\infty \to a_0+w\mathbb{H}^\infty$ in the weak$*$-topology. For any g, $h \in K$, put $f = gh^*w$. By Lemma 2.6, we have

$$\tau(afw^*) = (\varphi(a)g,\ h), \quad a \in \mathbb{H}^\infty.$$

Therefore, it is clear that $(\varphi(a_j)g,\ h) \to (\varphi(a_0)g,\ h)$. Thus $\varphi(a_j) \to \varphi(a_0)$ in the weak operator topology. This completes the proof.

Our aim in this section is the following

Theorem 2.11. $\mathfrak{L}_+(K)$ is a weakly closed subalgebra of $B(K)$.

Proof. Suppose that $\{a_j\}$ is a net in $\mathbb{H}^\infty$ such that $\{\varphi(a_j)\}$ converges weakly to an operator T in $B(K)$. If $f \in \mathbb{H}_0^1$, by Lemma 2.6, then there exist operators g, $h \in K$ such that $\|g\|_2^2 \le \|f\|_1+1$, $\|h\|_2^2 \le \|f\|_1+1$ and $\tau(afw^*) = (\varphi(a)g,\ h)$, $a \in \mathbb{H}^\infty$. Thus we have

$$(*) \qquad (Tg,\ h) = \lim\ (\varphi(a_j)g,\ h)) = \lim\ \tau(a_jfw^*).$$

It follows that $\lim \tau(a_jfw^*)$ exists for all $f \in \mathbb{H}_0^1$ and is no larger in absolute value than $\|T\|(\|f\|_1+1)$. Moreover, the limit $(*)$ depends only on the coset of fw^* in $\mathbb{H}_0^1w^*/_{\mathbb{H}_0^1}$. Thus $(*)$ defines the bounded linear functional on $\mathbb{H}_0^1w^*/_{\mathbb{H}_0^1}$ and this functional induced by an operator a_0 in $\mathbb{H}^\infty$. Thus, we have $a_j+w\mathbb{H}^\infty \to a_0+w\mathbb{H}^\infty$ in the weak$*$-topology. By Proposition 2.10, $\varphi(a_j) \to \varphi(a_0)$ in the weak operator topology. Therefore, we have $\varphi(a_0) = T$ and so $\mathfrak{L}_+(K)$ is weakly closed. This completes the proof.

3. Interpolation in analytic crossed products

In this section, we study the interpolation theorem in the case of analytic crossed products. At first, we define the algebras. Let M be a σ-finite finite von Neumann algebra. That is, there exists a faithful normal tracial state ϕ on M. Let $L^2(M,\phi)$ be the noncommutative L^2-space associated with M and ϕ. For every $x \in M$, let ℓ_x (resp. r_x) be the left multiplication on $L^2(M,\phi)$: that is, $\ell_x y = xy$ (resp. $r_x y = yx$). Put $\ell(M) = \{\ell_x\colon x \in M\}$ and $r(M) = \{r_x\colon x \in M\}$, respectively. Also, we fix once and for all a $*$-automorphism α of M which preserves ϕ. Then there is a unitary operator u on $L^2(M,\phi)$ induced by α. To construct a crossed product, we consider the Hilbert space $\mathbb{L}^2$ defined by $\{f\colon \mathbb{Z} \longrightarrow L^2(M,\phi)\colon \sum_{n\in\mathbb{Z}} \|f(n)\|_2^2 < \infty\}$, where $\|\cdot\|_2$ is the norm of $L^2(M,\phi)$. For $x \in M$, we define operators L_x, R_x, L_δ and R_δ on $\mathbb{L}^2$ by the formulae $(L_x f)(n) = \ell_x f(n)$, $(R_x f)(n) = r_{\alpha^n(x)} f(n)$, $(L_\delta f)(n) = uf(n-1)$ and $(R_\delta f)(n) = f(n-1)$. Put $L(M) = \{L_x\colon x \in M\}$ and $R(M) = \{R_x\colon x \in M\}$. We set $\mathfrak{L} = \{L(M), L_\delta\}''$ and $\mathfrak{R} = \{R(M), R_\delta\}''$ and define the left (resp. right) analytic crossed product $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) to be the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by $L(M)$ (resp. $R(M)$) and L_δ (resp. R_δ). Let E_n be the projection on $\mathbb{L}^2$ defined by the formulae $(E_n f)(k) = f(n)$, if $k = n$, and 0, if $k \neq n$. We also define the integral $\varepsilon_n(T) = \int_0^1 e^{-2\pi int}\beta_t(T)dt$, $T \in \mathfrak{L}$ or $\mathfrak{R}$, where $\{\beta_t\}_{t\in\mathbb{R}}$ is the dual action of $\{\alpha^n\}_{n\in\mathbb{Z}}$. Furthermore, we define $\mathbb{H}^2 = \{f\in\mathbb{L}^2: f(n) = 0, n < 0\}$. We refer to the reader to [6, 7] for discussions of these algebras including some of their elementary properties. Putting $\tau = \phi\cdot\varepsilon_0$, then τ is a faithful normal tracial state on $\mathfrak{L}$ and $\mathfrak{R}$. Thus we have

Proposition 3.1 (cf. [6]). $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) is a finite maximal subdiagonal algebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) with respect to ε_0 and τ.

As in §2, we take a unitary operator W in $\mathfrak{L}_+$ such that $W\mathbb{H}^2$ is two-sided invariant. Put $K = \mathbb{H}^2\ominus W\mathbb{H}^2$. The orthogonal projection in $\mathbb{L}^2$ with range K will denote by P_K. Then, for every $A \in \mathfrak{L}_+$ or $\mathfrak{R}_+$, the operator S_A is defined by

$$S_A f = P_K(Af), \quad f \in K.$$

Then our goal in this section is the following

Theorem 3.2. If T is a bounded linear operator on K that commutes with $\mathfrak{R}_+(K)$ (resp. $\mathfrak{L}_+(K)$), then there exists an operator A in $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) such that $T = S_A$ and $\|T\| = \|A\|$. Therefore, $\mathfrak{L}_+(K)' = \mathfrak{R}_+(K)$ and $\mathfrak{R}_+(K)' = \mathfrak{L}_+(K)$.

To prove this theorem, we need some notations and some preliminaries. For a positive integer r, we consider the Hilbert spaces $\mathbb{L}^2\otimes\mathbb{C}^r$ and $\mathbb{H}^2\otimes\mathbb{C}^r$, where $\mathbb{C}^r$ is an r-dimensional Hilbert space. If $A \in B(\mathbb{L}^2)$, then $A\otimes I$ is the ampliated operator of A on $\mathbb{L}^2\otimes\mathbb{C}^r$. Then the commutant of $\mathfrak{L}\otimes I$ is isomorphic to $\mathfrak{R}\otimes M_r$, where M_r is the algebra of $r\times r$-matrices.

We now study the form of invariant subspaces under $\mathfrak{L}_+\otimes I$ in $\mathbb{L}^2\otimes\mathbb{C}^r$. That is, we say that a closed subspace $\mathfrak{M}$ of $\mathbb{L}^2\otimes\mathbb{C}^r$ is $\mathfrak{L}_+\otimes I$-invariant if $(\mathfrak{L}_+\otimes I)\mathfrak{M} \subset \mathfrak{M}$. Let $\mathfrak{M}$ be an $\mathfrak{L}_+\otimes I$-invariant subspace of $\mathbb{L}^2\otimes\mathbb{C}^r$. Since $L_\delta\otimes I$ is the shift oprerator on $\mathbb{L}^2\otimes\mathbb{C}^r$, we consider the wondering subspace $\mathfrak{F} = \mathfrak{M} \ominus (L_\delta \otimes I)\mathfrak{M}$. Since $\mathfrak{M}$ is $L(M)\otimes I$-invariant, the orthogonal projection $P_{\mathfrak{F}}$ onto $\mathfrak{F}$ is contained in $(L(M)\otimes I)' = L(M)'\otimes M_r$. Then we have

Lemma 3.3. $(E_0\otimes I)(L(M)'\otimes M_r)(E_0\otimes I) = (R(M)\otimes M_r)(E_0\otimes I)$.

Proof. Since $E_0 L(M)'E_0 = R(M)E_0$ by [13, Lemma 2.3], we have this lemma.

Let $\mathfrak{M}$ be an $\mathfrak{L}_+\otimes I$-invariant subspace of $\mathbb{L}^2\otimes\mathbb{C}^r$. Then we shall say that $\mathfrak{M}$ is pure if $\bigcap_{n\geq 0} (L_\delta\otimes I)^n\mathfrak{M} = \{0\}$. As in the proof of [13, Theorem 3.1], we have the following theorem.

Proposition 3.4. Let $\mathfrak{M}$ be a pure $\mathfrak{L}_+\otimes I$-invariant subspace of $\mathbb{L}^2\otimes\mathbb{C}^r$. Then there exists a sequence $\{V_n\}_{n=0}^{\infty}$ of partial isometries in $\mathfrak{R}\otimes M_r$ with mutually orthogonal ranges such that $\mathfrak{M} = \sum_{n=0}^{\infty} \oplus V_n(\mathbb{H}^2\otimes\mathbb{C}^r)$.

We consider the closed subspace $K\otimes\mathbb{C}^r$ of $\mathbb{H}^2\otimes\mathbb{C}^r$. Put $K_r = K\otimes\mathbb{C}^r$. The orthogonal projection in $\mathbb{L}^2\otimes\mathbb{C}^r$ with range K_r will denote by P_{K_r}. Then, it is clear that $K_r = (\mathbb{H}^2\otimes\mathbb{C}^r)\ominus(W\otimes I)(\mathbb{H}^2\otimes\mathbb{C}^r)$. Further, for every $A \in B(\mathbb{L}^2)\otimes M_r$, we define the operator S_A on K_r by

$$S_A f = P_{K_r}(Af), \quad \text{for every } f \in K_r.$$

Then we have the following lemma and the proof is routine and will be omitted.

Lemma 3.5. If T is an operator on K that commutes with $\mathfrak{R}_+(K)$, then $T\otimes I$ commutes with S_A for all $A \in \mathfrak{R}_+\otimes M_r$.

We put $\tilde{\mathfrak{L}}_+(K) = \{S_A\colon A \in \mathfrak{L}_+\otimes I\}$. Then we have

Proposition 3.6. Let $\mathfrak{M}$ be a closed subspace of K_r that is invariant under $\tilde{\mathfrak{L}}_+(K)$. Then there exists a sequence $\{V_n\}_{n=0}^{\infty}$ of partial isometries in $\mathfrak{R}_+ \otimes M_r$ with mutually orthogonal range such that $\mathfrak{M} = \sum_{n=0}^{\infty} \oplus S_{V_n} K_r$.

Proof. Put $\mathfrak{M}_0 = \mathfrak{M} \ominus (W\otimes I)(\mathbb{H}^2\otimes\mathbb{C}^r)$. Then it is clear that $\mathfrak{M}_0$ is a pure invariant subspace under $\mathfrak{L}_+\otimes 1$. Therefore, by Proposition 3.4, the exists a sequence $\{V_n\}_{n=0}^{\infty}$ of partial isometries in $\mathfrak{R}_+\otimes M_r$ with mutually orthogonal range such that $\mathfrak{M}_0 = \sum_{n=0}^{\infty} \oplus V_n(\mathbb{H}^2\otimes\mathbb{C}^r)$. Then we can prove that $\left(\sum_{n=0}^{\infty} \oplus V_n(\mathbb{H}^2\otimes\mathbb{C}^r)\right) \cap K_r = \sum_{n=0}^{\infty} \oplus S_{V_n} K_r$. We take any element $f \in \left(\sum_{n=0}^{\infty} \oplus V_n(\mathbb{H}^2\otimes\mathbb{C}^r)\right) \cap K_r$. Then we can write $f = \sum_{n=0}^{\infty} V_n g_n$ for some sequence $\{g_n\}_{n=0}^{\infty}$ in $\mathbb{H}^2\otimes\mathbb{C}^r$. Since $V_n \in \mathfrak{R}_+\otimes M_r$, we have $V_n(W\otimes I)(\mathbb{H}^2\otimes\mathbb{C}^r) \subset (W\otimes I)(\mathbb{H}^2\otimes\mathbb{C}^r)$. Hence $V_n(1-P_{K_r})(\mathbb{H}^2\otimes\mathbb{C}^r) \subset (W\otimes I)(\mathbb{H}^2\otimes\mathbb{C}^r)$ and so $P_{K_r} V_n (1-P_{K_r}) g_n = 0$. Thus we have

$$f = \sum_{n=0}^{\infty} V_n g_n = P_{K_r}\left(\sum_{n=0}^{\infty} V_n g_n\right) = \sum_{n=0}^{\infty} P_{K_r} V_n P_{K_r} g_n$$

$$= \sum_{n=0}^{\infty} S_{V_n}\left(P_{K_r} g_n\right) \in \sum_{n=0}^{\infty} \oplus S_{V_n} K_r.$$

Conversely, we take any $f \in \sum_{n=0}^{\infty} \oplus S_{V_n} K_r$. Then there exists a sequence $\{g_n\}_{n=0}^{\infty}$ in K_r such that $f = \sum_{n=0}^{\infty} S_{V_n} g_n$. Since $(1-P_{K_r}) V_n g_n \in (W\otimes I)(\mathbb{H}^2\otimes\mathbb{C}^r)$, we have

$$(1-P_{K_r})\left(\sum_{n=0}^{\infty} V_n g_n\right) \in (W\otimes I)(\mathbb{H}^2\otimes\mathbb{C}^r) \subset \sum_{n=0}^{\infty} \oplus V_n(\mathbb{H}^2\otimes\mathbb{C}^r).$$

Therefore, there exists a sequence $\{h_n\}_{n=0}^{\infty}$ in $\mathbb{H}^2\otimes\mathbb{C}^r$ such that $\left(1-P_{K_r}\right) \sum_{n=0}^{\infty} V_n g_n = \sum_{n=0}^{\infty} V_n h_n$. Hence we have

$$f = \sum_{n=0}^{\infty} S_{V_n} g_n = \sum_{n=0}^{\infty} P_{K_r} V_n g_n$$

$$= \sum_{n=0}^{\infty} V_n(g_n - h_n) \in \left(\sum_{n=0}^{\infty} \oplus V_n(\mathbb{H}^2 \otimes \mathbb{C}^r)\right) \cap K_r.$$

Thus we have $\mathfrak{M} = \sum_{n=0}^{\infty} \oplus V_n K_r$. This completes the proof.

Proof of Theorem 3.2. Let T be a bounded linear operator on K that commutes with $\mathfrak{R}_+(K)$. We take any elements $g_1, g_2, \cdots, g_r, h_1, h_2, \cdots, h_r$ in K. Put $G = g_1 \oplus g_2 \oplus \cdots \oplus g_r$. Let $\mathfrak{M}$ be the closed subspace of K_r generated by $\{(A \otimes I)(K)G : A \in \mathfrak{L}_+\}$. Then $\mathfrak{M}$ is invariant subspace under $\tilde{\mathfrak{L}}_+(K)$. By Proposition 3.6, there exists a sequence $\{V_n\}_{n=0}^{\infty}$ in $\mathfrak{R}_+ \otimes M_r$ such that $\mathfrak{N} = \sum_{n=0}^{\infty} \oplus S_{V_n} K_r$. Since $T \otimes I$ commutes with S_{V_n}, we have $(T \otimes I)\mathfrak{M} \subset \mathfrak{M}$. Since $(T \otimes I)G \in \mathfrak{M}$, there exists a sequence $\{A_n\}_{n=0}^{\infty}$ in $\mathfrak{L}_+$ such that $(T \otimes I)G = \lim_{n \to \infty} S_{A_n \otimes I}G$. Hence there exists an element $A_n \in \mathfrak{L}_+$ such that $|(Tg_k, h_k) - (S_{A_n} g_k, h_k)| < 1$ for every k $(1 \le k \le r)$. This implies that T is in the weak closure of $\mathfrak{L}_+(K)$. By Theorem 2.11, we have $T \in \mathfrak{L}_+(K)$. This completes the proof.

4. Caratheodry-Fejer Theorem

In this section, we generalize the Caratheodry-Fejer Theorem to analytic crossed products as an application of Theorem 3.2. We consider two-sided invariant subspace $L_\delta{}^n\mathbb{H}^2$ $(= R_\delta{}^n\mathbb{H}^2)$ of $\mathbb{H}^2$. Then put $K = \mathbb{H}^2 \ominus L_\delta{}^n\mathbb{H}^2$. As a generalization of Caratheodry-Fejer Theorem, we have

Theorem 4.1. For every $x_0, x_1, \cdots, x_n \in M$, we define the operator T on K by

$$T = S_{\left(\sum_{k=0}^{n} L_{x_k} L_\delta{}^k\right)}.$$

Then $\|T\| \leq 1$ if and only if there exists an element $A \in \mathfrak{L}_+$ such that $T = S_A$ and $\|A\| \leq 1$.

Proof. Since $T = S_{\left(\sum_{k=0}^{n} L_{x_k} L_\delta{}^k\right)} = \sum_{k=0}^{n} S_{L_{x_k}} S_{L_\delta}{}^k$, T commutes with $\mathfrak{R}_+(K)$. Therefore, by Theorem 3.2, we have this theorem.

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