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A GENERALIZED CUNTZ ALGEBRA $\mathcal{O}_N^M$

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Dedicated to Professor Masamichi Takesaki on his sixty-th birthday

Let M be a von Neumann algebra with a faithful normal tracial state τ and N be a von Neumann subalgebra of M. We construct a tensor algebra $T_N(M)$ relative to N ;

$$\begin{cases} T_N^p(M) = L^2(M) \otimes L^2(M) \otimes \cdots \otimes L^2(M) \\ T_N^0(M) = L^2(N) \end{cases}$$

where $L^2(M)$ and $L^2(N)$ are Hilbert spaces with respect to the trace τ and $\otimes$ means the N-relative tensor product $\otimes_N$ and

$$T_N(M) = \sum_{p=0}^{\infty\oplus} T_N^p(M).$$

Then $T_N(M)$ is also N-bimodule.

For $x \in M$, a creation operator $o(x)$ is defined by

$$\begin{cases} o(x)x_1 \otimes \cdots \otimes x_p = x \otimes x_1 \otimes \cdots \otimes x_p, & x_1 \otimes \cdots \otimes x_p \in T_N^p(M) \\ o(x)x_0 = xx_0, & x_0 \in T_N^0(M). \end{cases}$$

Then an annihilation operator $o(x)$ is the following;

$$\begin{cases} o(x)^* x_1 \otimes x_2 \otimes \cdots \otimes x_p = E(x^* x_1) x_2 \otimes \cdots \otimes x_p \\ o(x)^* x_0 = 0. \end{cases}$$

where E is the conditional expectation of M onto N with respect to τ.

A N-rank one operator $(x_1 \otimes \cdots \otimes x_p) \boxtimes (y_1 \otimes \cdots \otimes y_q)$ is defined by

$$\{(x_1 \otimes \cdots \otimes x_p) \boxtimes (y_1 \otimes \cdots \otimes y_q)\}(z_1 \otimes \cdots \otimes z_r)$$

$$= \delta_{q,r} x_1 \otimes \cdots \otimes x_p < z_1 \otimes \cdots \otimes z_r, y_1 \otimes \cdots \otimes y_q >_N$$

where

$$< z_1 \otimes \cdots \otimes z_q, y_1 \otimes \cdots \otimes y_q >_N = E(y_q^* \cdots (E(y_2^*(E(y_1^* z_1) z_2) \cdots z_q).$$

Let $\{u_i\}_{i=1}^n$ be a Pimsner-Popa bases for $M \supset N$. Then $o(u_i)$ are isometries for $0 \le i \le n-1$ and $o(u_n)$ may be a partially isometry such that

$$\sum_{i=1}^{n} o(u_i)o(u_i)^* = 1_{T_N(M)} - 1 \boxtimes 1$$

A N-compact operator algebra $K_N(M)$ is the C^*-algebra generated by all of N-rank one operators. A C^*-algebra $\mathcal{P}_N^M$ is generated by all creation operators and an identity operator. Then the N-compact operator algebra $K_N(M)$ turns out to be a closed ideal of $\mathcal{P}_N^M$. A quotient C^*-algebra $\mathcal{O}_N^M$ of $\mathcal{P}_N^M$ by $K_N(M)$ is called a generalized Cuntz algebra. The coset of $o(x)$ in $\mathcal{O}_N^M = \mathcal{P}_N^M/K_N(M)$ is also denoted by $o(x)$ without any confusion. Note that if $M = \mathbf{C}^n$ and $N = \mathbf{C}$, $\mathcal{O}_N^M$ is a Cuntz algebra $\mathcal{O}_n$([3]). A guage action α of the torus T into $Aut(\mathcal{O}_N^M)$ can be defined by

$$\alpha_t(o(x)) = o(e^{it}x), t \in T.$$

By the use fo Pimsner-Popa bases, the fixed point algebra $(\mathcal{O}_N^M)^T$ is isomorphic to a inductive limit algebra of a reduced von Neumann algebra of $M_n(\mathbf{C}) \otimes \cdots \otimes M_n(\mathbf{C}) \otimes N$.

Theorem 1. *If $M \supset N$ is a factor-subfactor pair with a finite index [M N], then there is a guage invariant state ϕ on $\mathcal{O}_N^M$ such that*

$$\phi(o(x_1)\cdots o(x_n)o(y_m)^*\cdots o(y_1)^*)$$
$$= \delta_{n.m}[MN]^{-n}\tau(E(x_1\cdots E(x_{n-1}E(x_n y_n^*)y_{n-1}^*)\cdots y_1^*)$$

and ϕ is a unique KMS-state with respect to the guage action and inverse temperature -log [M N].

Let G be a finite group. We consider the two following cases

$$M = L^\infty(G) \rtimes_\alpha G, \qquad N = L^\infty(G), \qquad \text{cannonical trace } \tau \text{ on } M$$

and

$$M = W^*(G) \rtimes_\delta G, \qquad N = W^*(G), \qquad \text{cannonical trace } \tau \text{ on } M$$

where α is translation on G and δ is a canonical co-action of G.

The Cuntz algebra $\mathcal{O}_{|G|}$ is generated by isometries $S_g, g \in G$. A canonical co-action δ_1 of G on $\mathcal{O}_{|G|}$ is defined by $\delta_1(S_g) = S_g \otimes \lambda(g)([1])$. A canonical action α^1 of G on $\mathcal{O}_{|G|}$ is defined by $\alpha_h^1(S_g) = S_{hg}$.

Proposition 2. *The generalized Cuntz algebras $\mathcal{O}_{L^\infty(G)}^{L^\infty(G)\rtimes_\alpha G}$ and $\mathcal{O}_{W^*(G)}^{W^*(G)\rtimes_\delta G}$ are isomorphic to $\mathcal{O}_{|G|} \rtimes_{\delta_1} G$ and $\mathcal{O}_{|G|} \rtimes_{\alpha^1} G$ respectively.*

Proposition 3. *([2]) The two crossed products $\mathcal{O}_{|G|} \rtimes_{\delta_1} G$ and $\mathcal{O}_{|G|} \rtimes_{\alpha^1} G$ are isomorphic to $\mathcal{O}_{|G|}$.*

For Coxeter graph A_l, we construct finite dimensional von Neumann algebras M and N as follows. For l=2m+1(resp. l=2m) let N be the m-direct sum of $\mathbf{C}$ and M be m+1-direct sum $\mathbf{C} \oplus M_2 \oplus \cdots \oplus M_2 \oplus \mathbf{C}$ (resp. M be m-direct sum $\mathbf{C} \oplus M_2 \oplus \cdots \oplus M_2$). The inclusion of $M \supset N$ is given by the bicolored graph of A_l and a trace τ on M is defined by Perron-Frobenius eigen vector. An element of N with only i-th component 1 and otherwise 0 is denoted by e_i.

Proposition 4. *Let M and N be finite dimensional algebras associated with Coxeter graph A_l. Thenwe have*

(1) for odd l,(resp even l),we obtain quasi Pimsner-Popa base $\{u_i\}_{i=1}^{m+1}$ such that

$$(a) \quad E(u_i^* u_j) = \begin{cases} \delta_{i,j}1, & \text{i=1,m+1 (resp. } 1 - e_m \text{ only for i=m+1)} \\ \delta_{i,j}(e_{i-1} + e_i), \in N, & otherwise \end{cases}$$

$$M = \sum_{i=1}^{m+1} u_i N, \quad and \quad \sum_{i=1}^{m+1} u_i u_i^* = \lambda 1$$

where λ is the Perron-Frobenius eigen value of $X^t X$ where X is adjacent matrix of A_l

$$(b) \quad \begin{cases} e_j u_i = u_i e_j, & \text{for i=1,} \cdots \text{,m+1,j=1,} \cdots, m \\ e_{i-1} u_i = u_i e_i, & \text{i=2,} \cdots, m. \end{cases}$$

(2) all non zero elements of $\{e_j o(u_i)\}_{i=1,j=1}^{m+1,m}$ are non zero partially isometries which generate $\mathcal{O}_N^M$.

(3) $(\mathcal{O}_N^M)^T$ is AF-algebra which Bratteli diagram is the repetition of labelled bicolored graph associated with $X^t X$

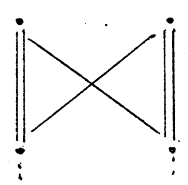
(4) $\mathcal{O}_N^M$ is isomorphic to Cuntz-Krieger algebra O_A ([4]) where A is the adjacent matrix of the line graph $\ell(A_l)$ of A_l.

C. Sutherland pointed out to us the above relation between the line graph of A_l and the matrix A

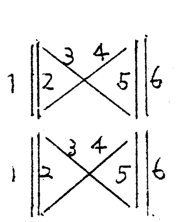
In the case of A_5, we define the base $\{u_i\}$ by

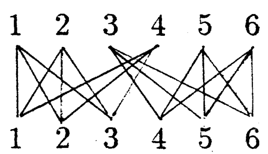
$$\begin{cases} u_1 = \sqrt{3} \oplus \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{3/2} \end{pmatrix} \oplus 0 \\ u_2 = 0 \oplus \begin{pmatrix} 0 & \sqrt{3/2} \\ \sqrt{3/2} & 0 \end{pmatrix} \oplus 0 \\ u_3 = 0 \oplus \begin{pmatrix} \sqrt{3/2} & 0 \\ 0 & 0 \end{pmatrix} \oplus \sqrt{3}. \end{cases}$$

Bratteli diagram of the fixed point algebra $(\mathcal{O}_N^M)^T$ with unique tracial state is



and $\mathcal{O}_N^M = O_A$ is as follows





$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

the line graph $\ell(A_5)$ the adjacent matrix of $\ell(A_5)$.

The K-group $K_0(\mathcal{O}_N^M)$ for A_5 is integer which show that it is different from Cuntz algebras .

Remark 5. When von Neumann algebras $M \supset N$ have quasi-Pimzner-Popa base ($E(u_i^* u_i)$ may be a projection in N instead of $E(u_i^* u_i) = 1$) Theorem 1 holds true even for non-factor N if the fixed point algebra has a unique tracial state.

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