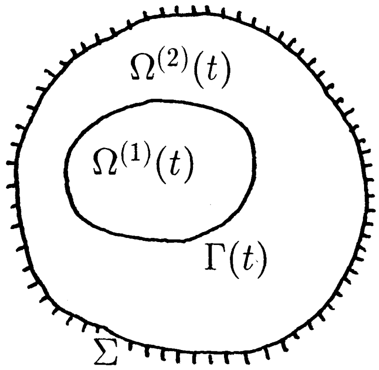
数理解析研究所講究録
第 862 巻 1994 年 111-117

Two-phase free boundary problem for viscous incompressible thermo-capillary convection

Naoto Tanaka
田中尚人
Department of Mathematics
Waseda University

1 Introduction



In this communication we are concerned with two-phase free boundary problem for incompressible viscous fluid which is formulated as follows: Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be two bounded domains in $\boldsymbol{R}^3$ which are filled with fluids (1) and (2), respectively, at the initial moment. We assume that $\partial\Omega^{(1)} = \Gamma$, $\partial\Omega^{(2)} = \Sigma \cup \Gamma$, $\Gamma \cap \Sigma = \emptyset$ ($\Gamma(0) \equiv \Gamma$ is the initial interface between fluids (1) and (2), Σ is fixed). Then, our problem consists in determining the domain $\Omega^{(j)}(t)$ occupied by the fluid (j) $(j = 1, 2)$ at the moment $t > 0$ together with the velocity vector field $v^{(j)}(x,t) = (v_1^{(j)}, v_2^{(j)}, v_3^{(j)})(x,t)$, the pressure $p^{(j)}(x,t)$ and with the absolute temperature $\theta^{(j)}(x,t)$ satisfying the system of Navier-Stokes equations:

$$(1.1)^{(1)} \qquad \begin{cases} \rho^{(1)}\left[\dfrac{D}{Dt}\right]^{(1)} v^{(1)} = \nabla\cdot \boldsymbol{P}^{(1)} + \rho^{(1)} f^{(1)}, \quad \nabla\cdot v^{(1)} = 0, \\ \left[\dfrac{D}{Dt}\right]^{(1)} \theta^{(1)} = \nabla\cdot(\kappa^{(1)}\nabla\theta^{(1)}) \quad x\in\Omega^{(1)}(t),\ t\in(0,T), \end{cases}$$

$$(1.1)^{(2)} \qquad \begin{cases} \rho^{(2)}\left[\dfrac{D}{Dt}\right]^{(2)} v^{(2)} = \nabla\cdot \boldsymbol{P}^{(2)} + \rho^{(2)} f^{(2)}, \quad \nabla\cdot v^{(2)} = 0, \\ \left[\dfrac{D}{Dt}\right]^{(2)} \theta^{(2)} = \nabla\cdot(\kappa^{(2)}\nabla\theta^{(2)}) \quad x\in\Omega^{(2)}(t),\ t\in(0,T), \end{cases}$$

$$(1.2) \qquad \begin{cases} (v^{(1)},\theta^{(1)})\big|_{t=0} = (v_0^{(1)},\theta_0^{(1)})(x) \quad x\in\Omega^{(1)}(0)\equiv\Omega^{(1)}, \\ (v^{(2)},\theta^{(2)})\big|_{t=0} = (v_0^{(2)},\theta_0^{(2)})(x) \quad x\in\Omega^{(2)}(0)\equiv\Omega^{(2)}, \end{cases}$$

$$(1.3) \qquad \begin{cases} v^{(1)} = v^{(2)}, \quad \boldsymbol{P}^{(1)}n - \boldsymbol{P}^{(2)}n = \sigma(\theta^{(s)})Hn + \nabla^{(s)}\sigma(\theta^{(s)}), \\ \theta^{(1)} = \theta^{(2)}, \quad \kappa^{(1)}\nabla\theta^{(1)}\cdot n - \kappa^{(2)}\nabla\theta^{(2)}\cdot n = 0, \\ x\in\Gamma(t),\ t\in(0,T), \end{cases}$$

$$(1.4) \qquad v^{(2)} = 0, \quad \theta^{(2)} = \theta_e \quad x\in\Sigma,\ t\in(0,T),$$

$$(1.5) \qquad \left[\frac{D}{Dt}\right]^{(1)} F(x,t) = 0 \quad x\in\Gamma(t),\ t\in(0,T)$$

(if $\Gamma(t)$ is given by $F(x,t)=0$), where $\left[\frac{D}{Dt}\right]^{(j)} = \frac{\partial}{\partial t} + (v^{(j)} \cdot \nabla)$ is the material derivative with respect to $v^{(j)}$, $\nabla = (\nabla_1, \nabla_2, \nabla_3)$, $\nabla_i = \frac{\partial}{\partial x_i}$ $(i = 1, 2, 3)$, $\boldsymbol{P}^{(j)} = \boldsymbol{P}^{(j)}(v^{(j)}, p^{(j)}) = -p^{(j)}\boldsymbol{I} + 2\mu^{(j)}\boldsymbol{D}(v^{(j)})$ is the stress tensor, $\boldsymbol{I}$ is the 3×3 unit matrix, $\boldsymbol{D}(v)$ is the velocity deformation tensor with (i, k) components $(\boldsymbol{D}(v))_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$ $(i, k = 1, 2, 3)$, $f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})(x, t)$ are given vector field of external mass forces. $\rho^{(j)}, \mu^{(j)}, \kappa^{(j)}$ are, respectively, the density of the fluid, the coefficient of viscosity and the coefficient of heat conductivity, which are all assumed to be positive constants. Here and in what follows we shall use the well-known notation of vector analysis and the summation convention. $n = n(x, t)$ is the unit normal vector pointing $\Omega^{(1)}(t)$ to $\Omega^{(2)}(t)$ at $x \in \Gamma(t)$, $H(x, t)$ is the twice mean curvature of $\Gamma(t)$, $\sigma = \sigma(\theta^{(s)})$, $(\theta^{(s)} = \frac{1}{2}(\theta^{(1)} + \theta^{(2)})|_{\Gamma(t)})$ is the coefficient of surface tension between fluids (1) and (2), $\nabla^{(s)}\sigma = \nabla\sigma - n(n \cdot \nabla\sigma)$ is the surface gradient on $\Gamma(t)$ and θ_e is a given temperture on fixed boundary Σ. The signature of H is chosen in such a way that $Hn = \Delta(t)x$, where $\Delta(t)$ is the Laplace-Beltrami operator on $\Gamma(t)$.

The aim of the present note is to announce various existence theorem to the problem $(1.1)^{(j)}$ – (1.5). Namely, in §2, we first discuss the temporarily local existence theorem and next it will be shown in §3 that the solution exists for all time near the equilibrium rest state provided that the data is sufficiently close to the rest state and finally the stationary motion of the problem will be studied in §4. For the proof of Theorems 1-3, see the original paper [1]-[3].

2 Local existence

In order to construct the temporarily local solution, it is convenient to choose $\xi = X(0; x, t) \in \Omega^{(j)}$ as new independent variables and reduces the problem to that of given initial domain $\Omega^{(j)}$, where $X(\tau; x, t)$ is the solution of the system of ordinary differential equations

$$(2.1)\qquad \begin{cases} \dfrac{d}{d\tau}X(\tau;x,t) = v^{(j)}(X(\tau;x,t),\tau), \\ X(t;x,t) = x \quad (0 \le \tau \le t). \end{cases}$$

If $v^{(j)}$ has suitable smoothness, then the fundamental existence theorem of ordinary differential equations yields that (2.1) has a unique solution curve $X(\tau;x,t)$, $x \in \Omega^{(j)}(t)$, $0 \le \tau \le t$. Whence this gives the relationship between so-called the Eulerian coordinate x and the Lagrangean coordinate ξ:

$$x = X(t;\xi,0) = \xi + \int_0^t \hat{v}^{(j)}(\xi,\tau)d\tau \equiv X_{\hat{v}^{(j)}}(\xi,t),$$

where $\hat{v}^{(j)}(\xi,t) \equiv v^{(j)}(X(t;\xi,0),t) = v^{(j)}(x,t)$. According to the kinematic condition on $\Gamma(t)$ and the boundary condition on Σ, this transformation is one-to-one mapping from $\Omega^{(j)}(t)$ [resp. $\Gamma(t)$, Σ] onto $\Omega^{(j)}$ [resp. Γ, Σ] for each t. Transforming the problem $(1.1)^{(j)}$ – (1.5) by this mapping and setting $(\hat{p},\hat{\theta})(\xi,t) = (p,\theta)(X_{\hat{v}}(\xi,t),t)$, we obtain

Theorem 1([1]) Let $\Gamma, \Sigma \in W_2^{7/2+l}$ with $\frac{1}{2} < l < 1$ and $\sigma \in W_2^{5+l}(\boldsymbol{R}_+)$, $\boldsymbol{R}_+ = \{x \in \boldsymbol{R},\ x > 0\}$, $(\sigma > 0)$. For arbitrary $(v_0^{(j)}, \theta_0^{(j)}) \in W_2^{2+l}(\Omega^{(j)})$, $f^{(j)} \in W_2^{5+l,5/2+l/2}(\boldsymbol{R}_T^3)$, $\theta_e \in W_2^{5/2+l,5/4+l/2}(\Sigma_T)$ satisfying $\theta_0^{(j)} > 0$, $\theta_e > 0$ and the natural compatibility conditions (we omit them here) the problem $(1.1)^{(j)}-(1.5)$ in Lagrangean coordinate system has the unique solution $(\hat{v}^{(j)}, \hat{p}^{(j)}, \hat{\theta}^{(j)})$ defined on $Q_{T_1}^{(j)} \equiv \Omega^{(j)} \times (0,T_1)$ for some $T_1 \in (0,T)$ such that $\hat{v}^{(j)}, \hat{\theta}^{(j)} \in W_2^{3+l,3/2+l/2}(Q_{T_1}^{(j)})$, $\nabla \hat{p}^{(j)} \in W_2^{1+l,1/2+l/2}(Q_{T_1}^{(j)})$, $\hat{p}^{(1)} - \hat{p}^{(2)}\big|_\Gamma \in W_2^{3/2+l,4/2+l/2}(\Gamma_{T_1})$ and

$$\begin{aligned} &\sum_{j=1}^{2} \Big(\|(\hat{v}^{(j)}, \hat{\theta}^{(j)})\|_{W_2^{3+l,3/2+l/2}(Q_{T_1}^{(j)})} + \|\nabla \hat{p}^{(j)}\|_{W_2^{1+l,1/2+l/2}(Q_{T_1}^{(j)})}\Big) + \\ &+ \|\hat{p}^{(1)} - \hat{p}^{(2)}\|_{W_2^{3/2+l,4/2+l/2}(\Gamma_{T_1})} \le \\ \le c &\Big[\sum_{j=1}^{2}\Big(\|(v_0^{(j)}, \theta_0^{(j)})\|_{W_2^{2+l}(\Omega^{(j)})} + \|f^{(j)}\|_{W_2^{5+l,5/2+l/2}(\boldsymbol{R}_T^3)}\Big) + \\ &+ \|\theta_e\|_{W_2^{5/2+l,5/4+l/2}(\Sigma_T)} + \|\sigma_0 H_0\|_{W_2^{3/2+l}(\Gamma)}\Big]. \end{aligned}$$

where $\sigma_0 = \sigma(\theta_0^{(s)})$, H_0 is the twice mean curvature of initial interface Γ.

Here, anisotropic Sobolev-Slobodetskiĭ space $W_2^{l,l/2}(Q_T)$ $(Q_T \equiv \Omega\times(0,T))$ is defined by

$$W_2^{l,l/2}(Q_T) = L_2((0,T);W_2^l(\Omega)) \cap L_2(\Omega;W_2^{l/2}(0,T))$$

3 Global existence

Let the domain $\Omega^{(1)}$ be deffeomorphic to a ball and its boundary Γ be given by the equation $|x| = r = R_0(\omega)$ $(\omega \in \boldsymbol{S}^2)$ in the spherical coordinate system (r,ω) with the origin at the center of gravity of $\Omega^{(1)} \cup \Omega^{(2)}$ and let $r = R(\omega,t)$ describes the interface $\Gamma(t)$. The equilibrium rest state of the problem $(1.1)^{(j)} - (1.5)$ is $(v^{(1)},p^{(1)},\theta^{(1)},v^{(2)},p^{(2)},\theta^{(2)},R) = (0,\bar{p},\bar{\theta},0,-\bar{p},\bar{\theta},\bar{R})$, where $\bar{\theta}$ are some positive constant, $\bar{R}$ is determined by $\frac{4}{3}\pi(\bar{R})^3 = |\Omega^{(1)}|$, $|\Omega^{(1)}|$ is the volume of $\Omega^{(1)}$, (note that $|\Omega^{(1)}(t)| = |\Omega^{(1)}|$ is true for all t) and $\bar{p} = \frac{\bar{\sigma}}{\bar{R}}$ $(\bar{\sigma} = \sigma(\bar{\theta}))$.
Define

$$\begin{aligned} E_0 \ \equiv\ & \sum_{j=1}^{2}\Big(\|(v_0^{(j)},\theta_0^{(j)}-\bar{\theta})\|_{W_2^{2+l}(\Omega^{(j)})} + \|f^{(j)}\|_{W_2^{5+l,5/2+l/2}(R^3_\infty)} + \\ + \ & \|f^{(j)}\|_{L_1(0,\infty;L_2(R^3))}\Big) + \|R_0 - \bar{R}\|_{W_2^{7/2+l}(S^2)}. \end{aligned}$$

Theorem 2([2]) Under the assumptions of Theorem 1, suppose also that $f^{(j)} \in W_2^{6+l,3+l/2}(\boldsymbol{R}^3_\infty)$, $f^{(j)} \in L_1(0,\infty;L_2(\boldsymbol{R}^3))$, and $\rho^{(1)} \neq \rho^{(2)}$. If E_0 be sufficiently small, then the solution $(v^{(1)},p^{(1)},\theta^{(1)},v^{(2)},p^{(2)},\theta^{(2)},R)$ of the problem $(1.1)^{(j)} - (1.5)$ exists for all $t > 0$ and satisfies

$$\begin{aligned} \sup_{t>0}\ \Big[& \sum_{j=1}^{2}\|(v^{(j)},\theta^{(j)}-\bar{\theta}\|_{W_2^{3+l}(\Omega^{(j)})(t)} + \|p^{(1)}-\bar{p}\|_{W_2^{2+l}(\Omega^{(1)}(t))} + \\ + \ & \|p^{(2)}+\bar{p}\|_{W_2^{2+l}(\Omega^{(2)}(t))} + \|R-\bar{R}\|_{W_2^{7/2+l}(S^2)}\Big] \leq cE_0. \end{aligned}$$

The similar theorem in the case of constant σ but non-homogeneous fluid is already established by the present author ([4]).

4 Stationary motion

Our final interest is the following stationary problem of $(1.1)^{(j)}$ – (1.5):

$$(4.1)^{(1)} \qquad \begin{cases} \rho^{(1)}(v^{(1)}\cdot\nabla)v^{(1)} = \nabla\cdot \boldsymbol{P}^{(1)} + \rho^{(1)} f^{(1)}, \quad \nabla\cdot v^{(1)} = 0, \\ (v^{(1)}\cdot\nabla)\theta^{(1)} = \nabla\cdot(\kappa^{(1)}\nabla\theta^{(1)}) \quad x\in\Omega^{(1)}, \end{cases}$$

$$(4.1)^{(2)} \qquad \begin{cases} \rho^{(2)}(v^{(2)}\cdot\nabla)v^{(2)} = \nabla\cdot \boldsymbol{P}^{(2)} + \rho^{(2)} f^{(2)}, \quad \nabla\cdot v^{(2)} = 0, \\ (v^{(2)}\cdot\nabla)\theta^{(2)} = \nabla\cdot(\kappa^{(2)}\nabla\theta^{(2)}) \quad x\in\Omega^{(2)}, \end{cases}$$

$$(4.2) \qquad \begin{cases} v^{(1)} = v^{(2)}, \quad \boldsymbol{P}^{(1)}n - \boldsymbol{P}^{(2)}n = \sigma(\theta^{(s)})Hn + \nabla^{(s)}\sigma(\theta^{(s)}), \\ \theta^{(1)} = \theta^{(2)}, \quad \kappa^{(1)}\nabla\theta^{(1)}\cdot n - \kappa^{(2)}\nabla\theta^{(2)}\cdot n = 0, \quad x\in\Gamma, \end{cases}$$

$$(4.3) \qquad v^{(2)} = 0, \quad \theta^{(2)} = \theta_e \quad x\in\Sigma,$$

$$(4.4) \qquad v^{(1)}\cdot n = 0 \quad x\in\Gamma.$$

Theorem 3([3]) Let l, Σ, σ be as in Theorem 1 and $f^{(j)} \in W_2^{5+l}(\boldsymbol{R}^3)$,

$\theta_e \in W_2^{5/2+l}(\Sigma)$, $\theta_e > 0$. If the quantity

$$E \equiv \sum_{j=1}^{2} \|f^{(j)}\|_{W_2^{5+l}(R^3)} + \|\theta_e - \bar{\theta}\|_{W_2^{5/2+l}(\Sigma)}$$

be sufficiently small, then the problem $(4.1)^{(j)}$ – (4.4) has the unique solution $(v^{(1)}, p^{(1)}, \theta^{(1)}, v^{(2)}, p^{(2)}, \theta^{(2)}, R)$ satisfying $v^{(j)}, \theta^{(j)} - \bar{\theta} \in W_2^{3+l}(\Omega^{(j)})$, $\nabla p^{(j)} \in W_2^{1+l}(\Omega^{(j)})$, $p^{(1)} - p^{(2)} - 2\bar{p}|_\Gamma \in W_2^{3/2+l}(\Gamma)$, $R \in W_2^{7/2+l}(\boldsymbol{S}^2)$ and

$$\sum_{j=1}^{2} \left(\|(v^{(j)}, \theta^{(j)} - \bar{\theta})\|_{W_2^{3+l}(\Omega^{(j)})} + \|\nabla p^{(j)}\|_{W_2^{1+l}(\Omega^{(j)})} \right) + \\ + \|p^{(1)} - p^{(2)} - 2\bar{p}\|_{W_2^{3/2+l}(\Gamma)} + \|R - \bar{R}\|_{W_2^{7/2+l}(S^2)} \leq cE,$$

where $\bar{p}$, $\bar{R}$ are same as Theorem 2 and the uniqueness of the interface R implies modulo all rotationary symmetric one.

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