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THE POINT SPECTRUM OF THE LINEARIZED BOLTZMANN OPERATOR WITH AN EXTERNAL-FORCE POTENTIAL IN AN UNBOUNDED DOMAIN

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ABSTRACT. We will investigate the point spectrum on the imaginary axis and the corresponding eigenspaces of the linearized Boltzmann operator with an external-force potential in an unbounded domain $\subset \mathbb{R}^3$. The boundary condition is the perfectly reflective boundary condition. We suppose that the boundary is a piecewise C^2-class surface, but we do not assume the convexity of the complement of the domain. The point spectrum and the corresponding eigenspaces vary considerably not only with geometrical properties of the external-force potential but also with those of the boundary surface. Therefore we need to classify external-force potentials and domains appropriately.

§1 INTRODUCTION

The nonlinear Boltzmann equation with an external potential $\phi = \phi(x)$,

$$f_t + \Lambda f = Q(f,f), \tag{1.1}$$

describes the time evolution of rarefied gas which is acted upon by the force $\mathbb{F} = -\nabla\phi$. $f = f(t,x,\xi)$ is an unknown function denoting the density of gas particles at time $t \geqq 0$, at a point $x \in \Omega$, and with a velocity $\xi \in \mathbb{R}^3$. Ω is a domain $\subseteq \mathbb{R}^3$ in which the rarefied gas is confined. Λ and $Q(\cdot,\cdot)$ are the following operators:

$$\Lambda \equiv \xi\cdot\nabla_x - \nabla_x\phi\cdot\nabla_\xi,$$

$$Q(g,h) \equiv (1/2)\int_{\xi'\in\mathbb{R}^3, s\in S^2} B(\theta,|\xi-\xi'|)\times$$

$$\times\{g(\eta)h(\eta')+g(\eta')h(\eta)-g(\xi)h(\xi')-g(\xi')h(\xi)\}d\xi' ds,$$

where $g(\eta) = g(t,x,\eta)$, etc., $\eta = \xi-((\xi-\xi')\cdot s)s$, $\eta' = \xi'+((\xi-\xi')\cdot s)s$, and $\cos\theta = (\xi-\xi')\cdot s/|\xi-\xi'|$. $B(\theta,V)$ is a nonnegative given function of $(\theta,V) \in [0,\pi]\times[0,+\infty)$. We will impose the following:[1]

Assumption 1.1. $B(\theta,V)/|\sin\theta\cos\theta| \leqq c_{1.1}(V+V^{\varepsilon-1})$, for any (θ,V), where $c_{1.1}$ and $\varepsilon<1$ are positive constants independent of (θ,V).

Under this assumption we can linearize (1.1) around the absolute Maxwellian state $M \equiv \exp(-\phi(x)-|\xi|^2/2)$. Substituting $f = M + M^{1/2}u$ in (1.1), and dropping the nonlinear term, we obtain the linearized Boltzmann equation,

$$u_t = Bu, \tag{1.2}$$

where $B \equiv A + L_1$, $A \equiv -\Lambda+(\exp(-\phi))(-\nu)$, and $L_1 \equiv (\exp(-\phi))K$. $\nu = \nu(\xi)$ is a multiplication operator, and K is an integration operator with a symmetric kernel; ν and K act on ξ only. These operators satisfy the following:[1,2]

Lemma 1.2. (i) There exists a positive constant $c_{1.2}$ such that for any ξ $0 < \nu(\xi) \leqq c_{1.2}(1+|\xi|)$.

(ii) K is a self-adjoint compact operator on $L^2(\mathbb{R}_\xi^3)$.

(iii) $(-\nu+K)$ is a self-adjoint nonpositive operator on $L^2(\mathbb{R}_\xi^3)$.

(iv) $(-\nu+K)f = 0$ iff f is a linear combination of $\xi_j\omega^{1/2}$, $j = 1,2,3$, $\omega^{1/2}$, and $|\xi|^2\omega^{1/2}$, where ξ_j is the j-th component of ξ, $j = 1,2,3$, i.e., $\xi = (\xi_1, \xi_2, \xi_3)$; $\omega \equiv \exp(-|\xi|^2/2)$.

It is important to investigate the asymptotic behavior of solutions of (1.2). In order to study this subject, we need to inspect the point spectrum of B and the corresponding eigenspaces. In [6] we have already investigated this subject when $\Omega = \mathbb{R}^3$, and by making use of the result in [6], we have obtained decay estimates for solutions to the Cauchy problem for (1.2) (see [3-5]). In the present paper, we will study that subject when Ω is an unbounded domain $\subset \mathbb{R}^3$. The main result is Theorem 4.1. Our boundary condition is the perfectly reflective boundary condition.

In [6] we are confronted with the difficulties arising from the fact that the point spectrum of B and the corresponding eigenspaces exhibit a complicated structure which varies with geometrical properties of the external-force potential. For this reason, it is necessary to classify the external-force potentials (see [6, pp. 185, 189]). However, in studying the subject of the present paper, we find the difficulty caused by the fact that the point spectrum of B and the corresponding eigenspaces vary considerably not only with geometrical properties of the external-force potential but also with those of the boundary surface. Hence, we need to classify both the external-force potentials and the boundary surfaces.

In this paper, we suppose that the boundary $\partial\Omega$ is a piecewise C^2-class surface, but we do not assume the convexity of the complement of the domain.

§ 2 ASSUMPTIONS

We impose the following on the domain Ω:

Assumption 2.1. (i) Ω is an unbounded domain $\subset \mathbb{R}^3$.

(ii) There exist a family of bounded domains $\{O_j\}_{j\in\mathbb{N}}$ and a family of functions $\{\psi_j(\underline{x})\}_{j\in\mathbb{N}}$ which satisfy the following (1-4):

(1) $\{O_j\}_{j\in\mathbb{N}}$ is a covering of $\partial\Omega$, and $J(\rho) \equiv \{j \in \mathbb{N};\ O_j(\rho) \equiv O_j \cap \{x;\ |x| \leqq \rho\}$ is not empty$\}$ is a finite set for any $\rho > 1$.

(2) For each $j \in \mathbb{N}$ there exists an orthogonal coordinate system in terms of which $\partial\Omega \cap O_j$ and $\Omega \cap O_j$ are represented as follows:

$$\partial\Omega \cap O_j = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ x_3 = \psi_j(\underline{x}),\ \underline{x} = (x_1, x_2) \in p_j(O_j)\},$$

$$\Omega \cap O_j \subseteq \{x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ x_3 < \psi_j(\underline{x}),\ \underline{x} = (x_1, x_2) \in p_j(O_j)\},$$

where $p_j(\cdot)$ is the orthogonal projection operator from $\mathbb{R}^3$ to the x_1x_2-plane.

(3) For each $j \in \mathbb{N}$, $\psi_j(\underline{x})$ is a piecewise C^2-class function of $\underline{x} \in p_j(O_j)$.

(4) $\{\psi_j(\underline{x})\}_{j\in\mathbb{N}}$ satisfies that for any $\rho > 1$ and $k, \ell = 1, 2$,

$$\sup_{\underline{x}\in M(j,\rho),\, j\in J(\rho)} |\partial^2\psi_j(\underline{x})/\partial x_k \partial x_\ell| \leqq c_{2.1}(\rho), \tag{2.1}$$

where $M(j, \rho)$ denotes the set of all points of $p_j(O_j(\rho))$ at which $\psi_j(\underline{x})$ is 2-times partially differentiable; $c_{2.1} = c_{2.1}(\rho)$ is a monotone increasing, positive-valued function of $\rho > 1$.

We make the following assumption on $\phi = \phi(x)$:

Assumption 2.2. (i) $\phi = \phi(x)$ is a real-valued function of $x \in \Omega$.

(ii) $\phi = \phi(x)$ belongs to $C^1(\Omega_x)$, and for any $\rho > 1$,

$$\sup_{|x| \leqq \rho, x \in \Omega} |\nabla \phi(x)| \leqq c_{2.2}(\rho), \tag{2.2}$$

where $c_{2.2} = c_{2.2}(\rho)$ is a monotone increasing, positive-valued function of $\rho > 1$.

(iii) $L^2(\Omega_x)$ contains $\exp(-\phi(x)/2)$, $\phi(x)\exp(-\phi(x)/2)$, and $|x|\exp(-\phi(x)/2)$.

(iv) There exists a constant $c_{2.3}$ such that for any $x \in \Omega$ $\phi(x) \geqq c_{2.3}$.

We define $\mathcal{D}(A) \equiv \{v = v(x, \xi) \in L^2 \equiv L^2(\Omega_x \times \mathbb{R}_\xi{}^3);\ Av \in L^2$, and $v = v(x, \xi)$ satisfies the perfectly reflective boundary condition,

$$(\gamma_1 v(\cdot, \cdot))(x, \xi) = (\gamma_2 v(\cdot, \cdot))(x, \xi - 2(n(x) \cdot \xi)n(x)), \tag{2.3}$$

for any $(x, \xi) \in S_1\}$. γ_j, $j = 1, 2$, denote the trace operators along the characteristic curves defined by the following:

$$dx/dt = \xi, \quad d\xi/dt = -\nabla \phi(x); \tag{2.4}$$

γ_j, $j = 1, 2$, make functions defined in $\Omega_x \times \mathbb{R}_\xi{}^3$ correspond to those defined in S_j, $j = 1, 2$, respectively. We can define $\mathcal{D}(B) \equiv \mathcal{D}(A)$.

For a differentiable real-valued function $f = f(x)$, $x \in \mathbb{R}^3$, we define

$$\ell(f) \equiv \{(\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4;\ \Sigma_{j=1}^3 \theta_j \partial f(x)/\partial x_j = \theta_4$$
$$\text{for any } x = (x_1, x_2, x_3) \in \mathbb{R}^3\}.$$

§3 CLASSIFICATIONS OF $\mathbb{P}$ AND $\mathbb{D}$

Denote by $\mathbb{P}_1$ the set of all potentials of the form

$$\phi(x) = m|x|^2 + \Sigma_{j=1}^{3} n_j x_j + n_4, \qquad (3.1)$$

where $m > 0$ and $n_j \in \mathbb{R}$, $j = 1,\cdots,4$, are constants. By $\mathbb{P}_2'$ we denote the set of all potentials $\phi = \phi(x) \in \mathbb{P}$ such that $c(\phi) \equiv \{m>0;\ \ell(\phi(x)-m|x|^2) \neq \{(0,0,0,0)\}\}$ is not empty; we easily see that $\mathbb{P}_1 \subset \mathbb{P}_2'$. Set $\mathbb{P}_2 \equiv \mathbb{P}_2'\backslash\mathbb{P}_1$, $\mathbb{P}_3 \equiv \mathbb{P}\backslash\mathbb{P}_2'$. We will decompose $\mathbb{P}$ as follows:

$$\mathbb{P} = \cup_{j=1}^{3}\mathbb{P}_j. \qquad (3.2)$$

Let $\phi \in \mathbb{P}_1$. Let us classify $\mathbb{D}$. By E_{1j}, $j = 1,\ldots,4$, we denote the sets of all domains $\Omega \in \mathbb{D}$ whose boundaries $\partial\Omega$ satisfy the following (1-4), respectively:

(1) $\partial\Omega$ is cylindrical, but is not a bent plane whose edge contains the vertex of $\phi = \phi(x)$.

(2) $\partial\Omega$ is a conical surface whose vertex is equal to that of $\phi = \phi(x)$, but $\partial\Omega$ is not a bent plane.

(3) $\partial\Omega$ is a bent plane whose edge includes the vertex of $\phi = \phi(x)$.

(4) $\partial\Omega$ is neither a cylindrical surface nor a conical surface whose vertex is equal to that of $\phi = \phi(x)$.

We see that E_{1j}, $j = 1,\ldots,4$, are disjoint, and that

$$\mathbb{D} = \cup_{j=1}^{4}E_{1j}. \qquad (3.3)$$

If $\phi = \phi(x) \in \mathbb{P}_2$ and if $\partial\Omega$ is cylindrical, we define

$$c(\phi,\partial\Omega) \equiv \{m > 0;\ \text{there exists } (\theta_1,\theta_2,\theta_3,\theta_4) \neq (0,0,0,0) \text{ satisfying } (\theta_1,\theta_2,\theta_3,\theta_4) \in \ell(\phi(x)-m|x|^2)$$

and $(\theta_1, \theta_2, \theta_3)//\partial\Omega\}$.

Let $\phi \in \mathbb{P}_2$. We will classify $\mathbb{D}$. By E_{2j}, $j=1,2,3$, we designate the sets of all domains $\Omega \in \mathbb{D}$ whose boundaries $\partial\Omega$ satisfy the following (5-7), respectively:

(5) $\partial\Omega$ is not cylindrical.

(6) $\partial\Omega$ is cylindrical, but $c(\phi, \partial\Omega)$ is empty.

(7) $\partial\Omega$ is cylindrical, and $c(\phi, \partial\Omega)$ is not empty.

We easily see that E_{2j}, $j = 1,2,3$, are disjoint, and that

$$\mathbb{D} = \cup_{j=1}^{3} E_{2j}. \tag{3.4}$$

§4 THE MAIN THEOREM

Theorem 4.1. (I) If $\phi = \phi(x) \in \mathbb{P}$ and $\Omega \in \mathbb{D}$, then $\sigma_P \ni 0$, and $e(0)$ is the set of all functions of the form

$$v = (\Sigma_{j=1}^{3} a_j \xi_j + a_4|\xi|^2 + a_5)M^{1/2}, \tag{4.1}$$

where $M \equiv \exp(-\phi(x)-|\xi|^2/2)$. $a_j = a_j(x)$, $j = 1,\ldots,5$, are complex-valued functions of $x \in \Omega$ satisfying the following (I-II):

(I) If $\mu = 0$, then $a_j = a_j(x)$, $j = 1,\ldots,5$, are such that

$$\begin{cases} a_j = \alpha_j + \Sigma_{k=1}^{3} \alpha_{jk} x_k, \quad j = 1,2,3, & (4.2) \\ a_4 \text{ is a complex constant}, & (4.3) \\ a_5 = 2a_4\phi(x) + \beta_0, & (4.4) \end{cases}$$

where β_0 is a constant $\in \mathbb{C}$. α_{jk}, $j,k = 1,2,3$, are complex constants satisfying the following (i-ii):

$$\text{(i)} \quad \alpha_{jk} + \alpha_{kj} = 0, \quad j,k = 1,2,3. \tag{4.5}$$

(ii) If we set

$$(\alpha, \beta) = ((\mathrm{Re}\,\alpha_1, \mathrm{Re}\,\alpha_2, \mathrm{Re}\,\alpha_3), (\mathrm{Re}\,\alpha_{23}, \mathrm{Re}\,\alpha_{31}, \mathrm{Re}\,\alpha_{12})), \\ ((\mathrm{Im}\,\alpha_1, \mathrm{Im}\,\alpha_2, \mathrm{Im}\,\alpha_3), (\mathrm{Im}\,\alpha_{23}, \mathrm{Im}\,\alpha_{31}, \mathrm{Im}\,\alpha_{12})), \quad (4.6)$$

then (α, β) satisfies the following:

(4.7): If $a(\phi) \cap a(\Omega)$ is empty, then $(\alpha, \beta) = (0, 0)$.

(4.8): If $a(\phi) \cap a(\Omega)$ is not empty, then there exists an $\ell \in a(\phi) \cap a(\Omega)$ such that $\beta // \ell$ and $\alpha = -\gamma \times \beta$ for any $\gamma \in \ell$.

(II) Let $\phi = \phi(x) \in \mathbb{P}_2$.

(i) If $\Omega \in E_{21} \cup E_{22}$, then $\sigma_p \cap \mathbb{C}_+ = \{0\}$.

(ii) Suppose that $\Omega \in E_{23}$. Then,

$$\sigma_p \cap \mathbb{C}_+ \setminus \{0\} = \{(-1)^k (2m)^{1/2} i;\ m \in c(\phi, \partial\Omega),\ k = 0, 1\}. \quad (4.9)$$

For any $m \in c(\phi, \partial\Omega)$ and for $k = 0, 1$, the eigenspace $e((-1)^k (2m)^{1/2} i)$ is the set of all functions of the form (4.1) whose $a_j = a_j(x)$, $j = 1, \ldots, 5$, satisfy the following:

$$\begin{cases} a_j = \beta_j, \quad j = 1, 2, 3, & (4.10) \\ a_4 = 0, & (4.11) \\ a_5 = (-1)^{k+1} (2m)^{1/2} \Sigma_{j=1}^{3} \beta_j x_j i + \beta_4, & (4.12) \end{cases}$$

where β_j, $j = 1, \ldots, 4$, are complex constants satisfying

$$(\mathrm{Re}\,\beta_1, \mathrm{Re}\,\beta_2, \mathrm{Re}\,\beta_3), (\mathrm{Im}\,\beta_1, \mathrm{Im}\,\beta_2, \mathrm{Im}\,\beta_3) // \partial\Omega, \quad (4.13)$$

$$(\mathrm{Re}\,\beta_1, \mathrm{Re}\,\beta_2, \mathrm{Re}\,\beta_3, \mathrm{Re}(-1)^k (2m)^{1/2} \beta_4 i), \\ (\mathrm{Im}\,\beta_1, \mathrm{Im}\,\beta_2, \mathrm{Im}\,\beta_3, \mathrm{Im}(-1)^k (2m)^{1/2} \beta_4 i) \in \ell(\phi(x) - m|x|^2). \quad (4.14)$$

(III) Suppose that $\phi = \phi(x) \in \mathbb{P}_1$, i.e., that $\phi = \phi(x)$ has the form (3.1).

(i) Assume that $\Omega \in E_{11}$. Then,

$$\sigma_p \cap \mathbb{C}_+ \backslash \{0\} = \{(-1)^k (2m)^{1/2} i;\ \ k = 0,1\}, \tag{4.15}$$

where m is that of (3.1). For $k = 0,1$, the eigenspace $e((-1)^k (2m)^{1/2} i)$ is the set of all functions of the form (4.1) whose $a_j = a_j(x)$, $j = 1,\dots,5$, satisfy (4.10-12), where β_j, $j = 1,\dots,4$, are complex constants satisfying (4.13) and

$$\Sigma_{j=1}^{3} \beta_j n_j = (-1)^k (2m)^{1/2} \beta_4 i, \tag{4.16}$$

where n_j, $j = 1,2,3$, are those of (3.1).

(ii) Let $\Omega \in E_{12}$. Then,

$$\sigma_p \cap \mathbb{C}_+ \backslash \{0\} = \{(-1)^k (8m)^{1/2} i;\ \ k = 0,1\}. \tag{4.17}$$

For $k = 0,1$, the eigenspace $e((-1)^k (8m)^{1/2} i)$ is the set of all functions of the form (4.1) whose $a_j = a_j(x)$, $j = 1,\dots,5$, satisfy the following:

$$\begin{cases} a_j = (-1)^{k+1} (8m)^{1/2} a_4 x_j i + \beta_j, \quad j = 1,2,3, & (4.18) \\ a_4 \ \text{ is a nonzero complex constant,} & (4.19) \\ a_5 = -2ma_4 |x|^2 + (-1)^{k+1} (2m)^{1/2} \Sigma_{j=1}^{3} \beta_j x_j i \\ \qquad\qquad + \Sigma_{j=1}^{3} \beta_j^2 / 4a_4, & (4.20) \end{cases}$$

where β_j, $j = 1,2,3$, are complex constants such that

$$\beta_j / a_4 = (-1)^{k+1} (2/m)^{1/2} n_j i, \quad j = 1,2,3. \tag{4.21}$$

(iii) Let $\Omega \in E_{13}$. Then,

$$\sigma_p \cap \mathbb{C}_+ \setminus \{0\} = \{(-1)^k (jm)^{1/2} i; \quad j = 2, 8, \quad k = 0, 1\}.$$

$e((-1)^k (jm)^{1/2} i)$, $j = 2, 8$, $k = 0, 1$, are the same as those described in (i-ii).

(iv) If $\Omega \in E_{14}$, then $\sigma_p \cap \mathbb{C}_+ = \{0\}$.

(IV) If $\phi = \phi(x) \in \mathbb{P}_3$ and $\Omega \in \mathbb{D}$, then $\sigma_p \cap \mathbb{C}_+ = \{0\}$.

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