数理解析研究所講究録
第 862 巻 1994 年 191-201

Notes on the periodic solutions of the 2-dimensional heat convection equations

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§1. Problems and assumptions.

We consider the heat convection equation of Boussinesq approximation in a time-dependent bounded domain $\Omega(t)$ of R^2.

$$(1)\quad \begin{cases} u_t + (u\cdot\nabla)u = -\nabla p/\rho + \nu\Delta u + \{1 - \alpha(\theta - T_0)\}g\ , \\ \operatorname{div} u = 0\ , \\ \theta_t + (u\cdot\nabla)\theta = \kappa\Delta\theta\ , \end{cases}$$

where u, θ and p are the velocity of the fluid, the temperature and the pressure, respectively. We assume the boundary $\partial\Omega(t)$ of the domain $\Omega(t)$ consists of two components, that is, $\partial\Omega(t) = \Gamma_0 \cup \Gamma(t)$ and $\partial\Omega(t)$ varies periodicaly in t with the period T. The boundary condition and the periodic one are as follows :

$$(2)\quad u|_{\partial\Omega(t)} = \beta(x,t)\ ,\quad \theta|_{\Gamma_0} = T_0 > 0\ ,\quad \theta|_{\Gamma(t)} = 0\ ,$$

$$(3)\quad u(t+T) = u(t)\ ,\quad \theta(t+T) = \theta(t)\ .$$

In the 3-dimensional case, we showed the existence, uniqueness and the stability of the periodic strong solution in [4] and [5] when the data are small. Recently, Inoue-Ôtani [2] obtained the periodic strong solution (space dimension n =2,3) under their small type condition. As for the periodic weak solution, we refer Morimoto [3]. Now, the purpose of our present work is to remove the small type condition on the boundary data of the fluid velocity. (The present result is an improvement of our previous one in [6].)

Here we state assumptions :

(A0) $\Omega(t)$ is included in a fixed open ball $B_1 = B(0,d)$ and the inner boundary Γ_0 and the outer one $\Gamma(t)$ do not intersect each other.

(A1) For each fixed $t \geqq 0$, $\Gamma(t)$ and Γ_0 are both simple closed curves. Moreover, They are smooth (of class c^∞) in x, t. Let K be a compact set surrounded by Γ_0, then K includes the origine 0 in its interior.

(A2) There exists $\Omega(r_0,r_1) = \{x \in R^2 : 0<r_0<|x|<r_1\}$ such that $\Omega(r_0,r_1) \subset \Omega(t)$ for all $t \geqq 0$. Moreover, there exists $\delta > 0$ such that

$$\mathrm{dist}\,(\Gamma_0\ ,\ \{|x| = r_0\}) \geqq \delta \qquad \text{and}$$

$$\mathrm{dist}\,(\Gamma(t),\ \{|x| = r_1\}) \geqq \delta \quad \text{for all } t \geqq 0.$$

(A3) $\Omega(t+T) = \Omega(t)$, $\Gamma(t+T) = \Gamma(t)$ and $\beta(x,t+T) = \beta(x,t)$ for all $t \geqq 0$.

(A4) $g(x)$ is a bounded continuous vector function in R^2-K.

(A5) There exists a function $b = b(x,t)$ of the form $b = \mathrm{rot}\ c(x,t)$ where $c = c(x,t) \in C^3$ on $(B_1-K) \times [0,\infty)$ and periodic in t with period T. Moreover, $b|_{\partial\Omega(t)} = \beta$ and

$$\int_{\Gamma_0} \beta\cdot n\ ds = \int_{\Gamma(t)} \beta\cdot n\ ds = 0\ .$$

We have two lemmas. (Put $B=B_1-K$.)

<u>Lemma 1.</u> For any $\varepsilon > 0$, there exists $b_\varepsilon = b_\varepsilon(x,t)$ such that $b_\varepsilon \in H^2(B)$, div $b_\varepsilon = 0$, $b_\varepsilon(\partial\Omega(t)) = \beta$ and

(4) $\quad |((u\cdot\nabla)b_\varepsilon,\ u)| \leqq \varepsilon\|\nabla u\|^2$ for any $u \in H^1_\sigma(\Omega(t))$.

(cf.Temam [7].)

Lemma 2. For any $\varepsilon > 0$, there exists $\overline{\theta}_\varepsilon = \overline{\theta}_\varepsilon(x,t)$ such that $\overline{\theta}_\varepsilon \in C(\overline{B}) \cap H^2(B)$ and

(5) $\|(u\cdot\nabla)\overline{\theta}_\varepsilon\| \leqq \varepsilon\|\nabla u\|$ for any $u \in H_0^1(\Omega(t))$.

To show Lemma 2, we use $\theta_0(x)$:

$$\theta_0(x) = \begin{cases} T_0 \quad , \ x \in B_{r_0} - K \\ T_0\cdot\log(r/r_1)/\log(r_0/r_1) \ , \quad x \in \Omega(r_0, r_1) \ . \\ 0 \quad , \ x \in B - B_{r_1} \quad . \end{cases}$$

where $B_{r_i} = \{x \in R^2 \ ; \ |x| \leqq r_i\}$ $(i = 0, 1)$.

On the other hand, according to Lemma 1.9 of Temam [7], for any $\varepsilon > 0$, there exists $\alpha_\varepsilon = \alpha_\varepsilon(x,t) \in C^2(\Omega(t))$ such that $\alpha_\varepsilon = 1$ in some neighbourhoods of Γ_0 and $\Gamma(t)$; $\alpha_\varepsilon = 0$ if $\rho(x) \geqq 2\delta(\varepsilon)$ and $|D_k\alpha_\varepsilon(x)| \leqq \varepsilon/\rho(x)$ if $\rho(x) \leqq 2\delta(\varepsilon)$ $(k = 1, 2)$, where $\rho(x,t) = \min\{\mathrm{dist}(x,\Gamma_0), \mathrm{dist}(x,\Gamma(t))\}$ and $\delta(\varepsilon) = \exp(-1/\varepsilon)$.

And we put $\overline{\theta}_\varepsilon = \alpha_\varepsilon\theta_0$. Then, thanks to the assumption (A2), we have, by retaking ε, if necessary, a function $\overline{\theta}_\varepsilon$ satisfying the condition of Lemma 2.

§2. Abstract heat convection equation.

We write $b = b_\varepsilon$ and $\overline{\theta} = \overline{\theta}_\varepsilon$. (Later we take an appropriate $\varepsilon > 0$.) Then we put $u = \hat{u} + b$, $\theta = \hat{\theta} + \overline{\theta}$; $(x,y) = d(x^*, y^*)$, $t = (d^2/\nu)t^*$, $\hat{u} = (\nu/d)u^*$, $\hat{\theta} = (\nu T_0/\kappa)\theta^*$ and $p = (\rho\nu^2/d^2)p^*$.

Here we abbreviate asterisks * and we have

$$(6) \begin{cases} u_t + (u\cdot\nabla)u = -\nabla p + \Delta u - (u\cdot\nabla)b - (b\cdot\nabla)u - R\theta \\ \qquad\qquad -b_t - (b\cdot\nabla)b + \Delta b + d^3g/\nu^2 - R(\overline{\theta} - \kappa/\nu), \\ \mathrm{div}\ u = 0 \end{cases}$$

$$\theta_t + (u\cdot\nabla)\theta = (\kappa/\nu)\Delta\theta + (\kappa/\nu)\Delta\bar{\theta} - (u\cdot\nabla)\bar{\theta} - (b\cdot\nabla)\theta - (b\cdot\nabla)\bar{\theta},$$

where $R = \alpha g T_0 d^3/\kappa\nu$.

(7) $u|_{\partial\Omega(t)} = 0$, $\theta|_{\partial\Omega(t)} = 0$,

(8) $u(t+T) = u(t)$, $\theta(t+T) = \theta(t)$.

We define proper lower semicontinuous convex functions for $U = (u,\theta)$:

$$\varphi_B(U) = \begin{cases} \frac{1}{2}\int_B(|\nabla u|^2 + \frac{\kappa}{\nu}|\nabla\theta|^2)\,dx, & U \in H^1_\sigma(B) \times H^1_0(B), \\ +\infty, & U \in (H_\sigma(B) \times L^2(B)) \setminus (H^1_\sigma(B) \times H^1_0(B)). \end{cases}$$

$$I_{K(t)}(U) = \begin{cases} 0, & U \in K(t), \\ +\infty, & U \in (H_\sigma(B) \times L^2(B)) \setminus K(t), \end{cases}$$

where $K(t) = \{U \in H_\sigma(B) \times L^2(B) \,;\, U = 0 \text{ a.e. } B \setminus \Omega(t)\}$,

and $\varphi^t_B(U) = \varphi_B(U) + I_{K(t)}(U)$ for $t \geqq 0$.

Then, we can consider the subdifferential operator $\partial\varphi^t$ of φ^t and we have :

(i) $D(\partial\varphi^t) = \{U \in H_\sigma(B) \times L^2(B) \,;\, U|_{\Omega(t)} \in (H^2(\Omega(t)) \cap H^1_\sigma(\Omega(t))) \times (H_2(\Omega(t)) \cap H^1_0(\Omega(t))),\ U|_{B-\Omega(t)} = 0\}$.

(ii) $\partial\varphi^t(U) = \{f \in H_\sigma(B) \times L^2(B); P(\Omega(t))f|_{\Omega(t)} = A(\Omega(t))U|_{\Omega(t)}\}$,

where $A(\Omega(t)) = (-P_\sigma(\Omega(t))\Delta, -(\kappa/\nu)\Delta)$, $P(\Omega(t)) = (P_\sigma(\Omega(t)), 1_{\Omega(t)})$, $P_\sigma(\Omega(t))$ is a projection $L^2(\Omega(t)) \longrightarrow H_\sigma(\Omega(t))$.

Now we introduce an abstract heat convection equation (AHC) :

(AHC) $\frac{dU}{dt} + \partial\varphi^t(U) + F(t)U(t) + M(t)U(t) \ni P(B)f(t)$, $t \geqq 0$,

where $F(t)U(t) = (P_\sigma(B)(u\cdot\nabla)u, (u\cdot\nabla)\theta)$,

$M(t)U(t) = (P_\sigma(B)((u\cdot\nabla)b + (b\cdot\nabla)u + R\theta),\ (u\cdot\nabla)\bar{\theta} + (b\cdot\nabla)\theta)$.

$$f(t) = (-b_t - (b\cdot\nabla)b + \Delta b + d^3 g/\nu^2 - R(\bar{\theta}-\kappa/\nu), (\kappa/\nu)\Delta\bar{\theta} - (b\cdot\nabla)\bar{\theta}).$$

Definition 1. Let U be a function $[0,S] \longrightarrow H_\sigma(B) \times L^2(B)$ where $S \in (0,\infty)$. Then U is called a strong solution of (AHC) on $[0,S]$ if it satisfies :

(i) $U \in C([0,S] ; H_\sigma(B) \times L^2(B))$ and dU/dt exists for a.e. $t \in (0,S]$.

(ii) $U(t) \in D(\partial\varphi^t)$ for a.e. $t \in [0,S]$ and there exists a function $G:[0,S] \longrightarrow H_\sigma(B)\times L^2(B)$ such that $G(t) \in \partial\varphi^t(U(t))$ for a.e. $t \in [0,S]$

and

$$\frac{dU}{dt} + G(t) + F(t)U(t) + M(t)U(t) = P(B)f(t)$$

holds for a.e. $t \in [0,S]$.

Definition 2. A strong solution of (AHC) is called a periodic strong solution (resp. a srong solution of the initial value problem) if it satisfies the condition (9)(resp.(10)) stated below :

(9) $U(t+T) = U(t)$ for $t \in [0,\infty)$ in $H_\sigma(B) \times L^2(B)$,

(10) $U(0) = (\tilde{a}, \tilde{h})$ in $H_\sigma(B) \times L^2(B)$,

where $(a, h) \in H_\sigma(\Omega(0)) \times L^2(\Omega(0))$ and $\tilde{a}$, $\tilde{h}$ are extensions of a, h to B with zero outside $\Omega(t)$, respectively.

§3. Results.

We make some assumptions.

(A6) $b \in L^\infty(0,\infty ; H^2(B))$, $b_t \in L^\infty(0,\infty ; L^2(B))$,
$\bar{\theta} \in L^\infty(0, \infty ; H^2(B))$.

Theorem. Suppose (A0) $\sim$ (A6) are satisfied. Then we have :

(i) For sufficiently small $R = \alpha g T_0 d^3/\kappa\nu$, there exists a periodic strong solution of (AHC) with period T.

(ii) Moreover, if $\|b\|_{L^\infty(0,\infty;H^2(B))}$, $\|b_t\|_{L^\infty(0,\infty;L^2(B))}$, $\|\overline{\theta}\|_{L^\infty(0,\infty;H^2(B))}$ are sufficiently small and ν is large enough, then the periodic strong solution is unique.

(iii) Under the same conditions on b, b_t, $\overline{\theta}$ and ν, the periodic strong solution $U_\pi(t)$ in (i) is stable, that is, $\|U(t)-U_\pi(t)\|_{L^2(\Omega(t))\times L^2(\Omega(t))} \longrightarrow 0$ as $t \longrightarrow \infty$, where u(t) is a strong solution of (AHC) with $U(0) = U_\pi(0) + U_0$ and $U_0 \in H_\sigma(\Omega(0)) \times L^2(\Omega(0))$ is an arbitrarily given data.

§4. Some lemmas.

Lemma 3. There exists a positive constant C_1 such that

(11) $\varphi^t(U) \geqq C_1\|U\|^2_{L^2(B)}$

for every $t \in [0,S]$ and $U \in H^1_\sigma(B) \times H^1_0(B)$.

The next lemma is a version of Lemma 2.1 of Foias, Manley and Temam [1].

Lemma 4. Let $U = (u,\theta)$ be a strong solution of (AHC). Then we have

(12) $\|\theta(t)\|_{L^2(B)} \leqq |B|^{1/2}\kappa/\nu + \|\theta(0)\|e^{-2\kappa t/\nu}$

for $t \in (0,\infty)$. Here $|B|$ is a volume of B.

To prove our theorem, the following lemma is useful.

Lemma 5.

(i) Let $U = (u,\theta)$ be a strong solution of (AHC). Then, for any $\delta \in (0,S)$, there exist positive constants $a_i(\delta)$ $(i = 1,2,3)$.

independent of S, depending on b and θ, such that

(13) $\varphi^t(U(t)) \leqq (a_2(\delta)/\delta + a_3(\delta))\exp a_1(\delta)$ for any $t \in [\delta, S]$.

(ii) Furthermore, if U is a periodic strong solution with period T, then the same estimate holds for all $t \in [0,T]$.

Here we give a sketch of proof of Lemma 5. Multiplying (AHC) by $G(t)$ and integrating on B, then we have

(14) $\frac{d}{dt}\varphi^t(U(t)) + \|G(t)\|^2$

$$\leqq C_4'\|U(t)\|^{1/2}\cdot\|U(t)\|_1\cdot\|G(t)\|^{3/2} + |(M(t)U(t),G(t))|$$

$$+ \|f(t)\|\cdot\|G(t)\| + C_2\|G(t)\|\cdot\varphi^t(U(t))^{1/2} + C_3\varphi^t(U(t)),$$

where $\|\cdot\|_1 = \|\cdot\|_{H^1(B)\times H^1(B)}$.

From this, we have

(15) $\frac{d}{dt}\varphi^t(U(t)) + \frac{1}{2}\|G(t)\|^2$

$$\leqq C_5\|U(t)\|^2\varphi^t(U(t))^2 + C_6M_1\varphi^t(U(t))$$

$$+ (2C_2^2 + C_3)\,\varphi^t(U(t)) + 2\|f\|_{\infty,2}^2,$$

where $M_1 = \|b\|_1\cdot\|b\|_2 + 2\|b\|\cdot\|b\|_2 + \|\bar\theta\|_1\cdot\|\bar\theta\|_2 + |R|^2$ and $\|f\|_{\infty,2} = \|f\|_{L^\infty(o,\infty;L^2(B))}$.

(Here we used (3.23) of Chap.III in Temam [8].)

On the other hand, multiplying (AHC) by $U(t)$ and integrating on B, then we get

(16) $\frac{d}{dt}\|U(t)\|^2 + 2C_1\|U(t)\|^2 \leqq (4|R|^2/C_1)\|\theta(t)\|^2 + 2\|f\|^2/C_1$

where we used Lemma 1 and Lemma 2 with suitable ε.

Thanks to Lemma 4, we get from (16)

(17) $\|U(t)\|^2 \leqq e^{-2C_1 t}\|U(0)\|^2 + \{(2|R|^2/C_1^2)(|B|\kappa^2/\nu^2 + \|\theta(0)\|^2)$

$+ \|f\|^2/C_1^2\}(1 - e^{-2C_1 t})$.

Hence, we get an apriori estimate :

(18) $\|U(t)\|^2 \leqq C_0 + C_0'\|U(0)\|^2$,

where $C_0 = (2|R|^2\cdot|B|\kappa^2/\nu^2 + \|f\|^2_{\infty,2}) / C_1^2$

and $C_0' = 1 + 2|R|^2/C_1^2$.

Using these inequalies (15), (18) and making use of the uniform Gronwall inequality, we get (13) of Lemma 5, where

$$
(19)\quad \begin{cases} a_1(\delta) = (2C_2^2 + C_3 + C_6 M_1)\delta, \\ \qquad\qquad + C_5(C_0 + C_0'\|U(0)\|^2)a_3(\delta)\delta , \\ a_2(\delta) = 2\delta\|f\|^2_{\infty,2} \\ a_3(\delta) = 2^{-1}(C_0 + C_0'\|U(0)\|^2) \\ \qquad\qquad + (\delta/C_1)\{2|R|^2(|B|^2\kappa^2/\nu^2 + \|\theta(0)\|^2) + \|f\|^2_{\infty,2}\} . \end{cases}
$$

Remark 1. When U is a periodic solution, we have

(20) $\|\theta(0)\|^2 \leqq |B|\kappa^2/\nu^2$,

(21) $\|U(0)\|^2 \leqq (1/C_1^2)(4|R|^2\cdot|B|\kappa^2/\nu^2 + \|f\|^2_{\infty,2})$.

Lemma 6. For any $U_0 = (a,h) \in H_\sigma(\Omega(0)) \times L^2(\Omega(0))$, there exists a unique strong solution U of (AHC) on [0,S] with $U(0) = U_0$. (See, [6].)

§5. Proof of Theorem.

First we prove (i) of Theorem. Here we assume $|R| \leqq C_1/4$.

Multiplying (AHC) by U(t), then we have

(22) $\frac{1}{2}\frac{d}{dt}\|U(t)\|^2 + 2\varphi^t(U(t))$

$\leqq |((u\cdot\nabla)b,u)| + |((b\cdot\nabla)u,u)| + |(R\theta,u)| + |((u\cdot\nabla)\overline{\theta},\theta)|$

$+ |((b\cdot\nabla)\theta,\theta)| + |(f(t),U(t))|$

$\leqq \varepsilon\|\nabla U\|^2 + |R|\cdot\|\theta\|\cdot\|u\| + \varepsilon'\|\nabla\theta\|\cdot\|\nabla u\| + \|f(t)\|\cdot\|U\|$

$\leqq \frac{1}{4}\varphi^t(U) \times 4 + \frac{1}{4\eta}\|f\|^2_{\infty,2}$,

where we used Lemma 1 with $\varepsilon = 1/8$ and Lemma 2 with

$\varepsilon' = (1/8)(\kappa/\nu)^{1/2}$; $\eta = C_1/4 \geqq |R|$.

From (22), we get

(23) $\frac{d}{dt}\|U(t)\|^2 + 2C_1\|U(t)\|^2 \leqq (2/C_1)\|f\|^2_{\infty,2}$

and we obtain

(24) $\|U(t)\|^2 \leqq e^{-2C_1t}\|U(0)\|^2 + (1/C_1^2)\|f\|^2_{\infty,2}(1-e^{-2C_1t})$.

Now we define a mapping τ as follows :

(25) $\tau : H = H_\sigma(\Omega(0)) \times L^2(\Omega(0)) \longrightarrow H$,

(26) $\tau U(0) = U(T)$ in H.

Here we used $\Omega(0) = \Omega(T)$ and Lemma 6. τ is continuous in H.

Moreover, τ is compact in H, because $\tau U(0) = U(T)$ is included

in a bounded set of $H^1_\sigma(\Omega(0)) \times H^1_0(\Omega(0))$ by Lemma 5.

On the other hand, if we take $r > 0$ such that $(1/C_1)\|f\|_{\infty,2} \leqq r$,

then for U(0) with $\|U(0)\| \leqq r$ we have by (23)

(27) $\|U(T)\|^2 \leqq e^{-2C_1T}r^2 + r^2(1-e^{-2C_1T}) = r^2$.

Therefore, $\tau B_r \subset B_r$, where

$B_r = \{\Phi \in H ; \|\Phi\|_H \leqq r\}$.

Hence, by Schander's fixed point theorem, there exists $V_0 \in H$

such that $\tau V_0 = V_0$.

Next we prove (ii). Let U_π be the periodic strong solution in (i) and U_1 be any periodic strong solution. Put $W = U_\pi - U_1$, then we have

$$(28)\quad \frac{1}{2}\frac{d}{dt}\|W(t)\|^2 + 2\varphi(W(t))$$

$$\leqq C_7\varphi^t(W(t))\cdot\varphi(U_\pi(t))^{1/2} + C_8N(t)\varphi^t(W(t))$$

for a.e. $t \in [0,T]$.

Here $N(t) = \|\nabla b(t)\| + \|\nabla\overline{\theta}(t)\| + |R|$. Noticing (ii) of Lemma 5, (19), (20), (21) and using the assumptions of (ii) of Theorem, then we see $2-C_7\varphi^t(U_\pi(t))^{1/2} - C_8N(t) > 0$ for $t \in [0,T]$. Thus, we can show the uniqueness of the periodic strong solution for small data. We omit the proof of (iii).

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