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Simple setting for white noise calculus using Bargmann space

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§1. Notations

Let E_0 be a real separable Hilbert space with dim $E_0 = \infty$ and $(\cdot, \cdot)_0$ be its inner product. Let D be a densely defined selfadjoint operator of E_0 such that $D > 1$ and D^{-1} is of Hilbert-Schmidt type. Further we assume that the eigen system of D^{-1};

$$\{(\lambda_j, \zeta_j)\}_{j=0}^{\infty} \quad \text{with} \quad D^{-1}\zeta_j = \lambda_j\zeta_j \ (j = 0, 1, 2, \cdots)$$

satisfies

$$1 > \lambda_j \geq \lambda_{j+1} \quad (j = 0, 1, \cdots)$$

and that $\{\zeta_j;\ j = 0, 1, 2, \cdots\}$ is an orthonormal basis of E_0. The following constants t_0, s_0, and p_0 will appear frequently;

$$t_0 = -\log 2/2\log\lambda_0, \text{ i.e., } \quad 1/2 = \lambda_0^{2t_0},$$

$$s_0 = \inf\{s;\ \textstyle\sum_{j=0}^{\infty} \lambda_j^{2s} < \infty\},$$

$$p_0 = \max(t_0, s_0).$$

Since $\|D^{-1}\|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \lambda_j^2 < \infty$ is finite, s_0 is in $[0, 1]$.

For any real number $p > 0$ write E_p = the domain of D^p and define the inner product $(x, y)_p$ for $x, y \in E_p$ by

$$(x, y)_p = (D^p x, D^p y)_0.$$

Then $(E_p, (\cdot, \cdot)_p)$ is a Hilbert space. If $0 \leq q < p$, then $E_p \subset E_q$. Every E_p contains ζ_j's, and so $E \equiv \cap_{p>0} E_p$ is not empty. Set $\|\xi\|_p = \sqrt{(\xi, \xi)_p}$ for $\xi \in E$. The system of norms $\{\|\xi\|_p;\ p \geq 0\}$ is compatible. Since D^{-1} is of Hilbert-Schmidt type, the space E equipped with the projective limit topology of

$\{(E_p, \|\cdot\|_p);\ p > 0\}$ is a nuclear space. We can easily see that $D^p(E_p) = E_0$ for $p > 0$. For $p > 0$, let E_{-p} be the completion of E_0 with respect to the norm $\|\cdot\|_{-p} \equiv \|D^{-p}\cdot\|_0$. Clearly, if $0 \le q < p$, then $E_0 \subset E_{-q} \subset E_{-p}$. Let $E^* = \cup_{p>0} E_{-p}$ and let it be equipped with the inductive limit topology of $\{(E_{-p}, \|\cdot\|_{-p});\ p > 0\}$. We have $E \subset E_0 \subset E^*$. Once the increasing family $\{E_p;\ p \in \mathbb{R}\}$ of Hilbert spaces is set, the operator D^q $(q \in \mathbb{R})$ acts naturally and isometrically as:

$$D^q\colon E_p \longrightarrow E_{p-q} \text{ (surjective) } (p \in \mathbb{R}),$$

and so it acts continuously on E^* with respect to the inductive limit topology. We can naturally identify the dual space of E_p with E_{-p} $(p \in \mathbb{R})$.

Let H_p be the complexification of E_p, i.e., $H_p = E_p + \sqrt{-1}E_p$. Then D^q extends to an isometry from H_p onto H_{p-q} naturally by setting

$$D^q(x + \sqrt{-1}y) = D^q x + \sqrt{-1}D^q y \quad \text{for } x,\ y \in E_p\ (p,\ q \in \mathbb{R}).$$

Accordingly the real spaces E and E^* also have their complexifications H and H^*, respectively. The letters w and z are often used for elements of H^* or H_{-p} and letters x and y for ones of E^* or E_{-p}, where $p \ge 0$. Like in the real case, the operator D^q acts as an isometry from H_p onto H_{p-q}. Hence D^q acts on H^* continuously. Obviously we see that

$$\langle D^q w,\ \zeta\rangle = \langle w,\ D^q \zeta\rangle$$

holds for any $w \in H^*$ and any $\zeta \in H$. Where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form. That is, suppose that X is a locally convex topological vector space and X^* the dual of X. Then the value of x^* at x defined by each pair $(x, x^*) \in X \times X^*$is always denoted by $\langle x^*, x\rangle$. This is linear in both arguments x and x^*.

Denote by $\mathcal{P}(X^*)$ the space of all polynomials in $\{\langle x^*, x\rangle;\ x$

$\in X\}$ over $\mathbb{C}$; that is,

$$\mathcal{P}(X^*) = \{\text{any finite sum of } c\Pi_j\langle x^*, x_j\rangle;\ x_j \in X,\ c \in \mathbb{C}\}.$$

If X is a nuclear space or Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, then the n-fold symmetric tensor product of X is denoted by $X^{\hat{\otimes} n}$. If $x_1, x_2, \cdots, x_n \in X$, then $\hat{\otimes}_{j=1}^n x_j$ is the symmetrization of $x_1 \otimes x_2 \cdots \otimes x_n$. In particular the n-fold tensor product of x is denoted by $x^{\hat{\otimes} n}$.

The following notations on infinite-dimensional indices of nonnegative integers will be used.

$$\mathcal{N} = \{\text{all sequences of nonnegative integers}\}.$$

$$\mathcal{N}_0 = \{\mathbb{n} = (n_0, n_1, n_2, \cdots);\ \mathbb{n} \in \mathcal{N},\ n_j = 0 \text{ for almost all } j\}.$$

For $\mathbb{n}, \mathbb{k} \in \mathcal{N}_0$, write $\mathbb{n} \geq \mathbb{k}$ if and only if $n_j \geq k_j$ $(j \geq 0)$. For $\mathbb{n}, \mathbb{k} \in \mathcal{N}_0$ and a nonnegative integer p, define

$$p\mathbb{n} = (pn_0, pn_1, pn_2, \cdots), \quad |\mathbb{n}| = n_0 + n_1 + n_2 + \cdots,$$

$$\mathbb{n}\wedge\mathbb{k} = (n_0\wedge k_0, n_1\wedge k_1, n_2\wedge k_2, \cdots),$$

$$\mathbb{n}! = \Pi_j\, n_j! \quad \text{and} \quad \binom{\mathbb{n}}{\mathbb{k}} = \Pi_j \binom{n_j}{k_j}.$$

For $r \in \mathbb{R}$ and $\mathbb{n} \in \mathcal{N}_0$ with $|\mathbb{n}| = n$, the symbols $\lambda^{r\mathbb{n}}$, $\zeta^{\hat{\otimes}\mathbb{n}}$, $h_{\mathbb{n}}$ and $\mathbb{z}^{\mathbb{n}}$ are defined as follows:

$$\lambda^{r\mathbb{n}} = \Pi_j\, \lambda_j^{rn_j},$$

$$\zeta^{\hat{\otimes}\mathbb{n}} = \hat{\otimes}_{n_j \neq 0}\, \zeta_j^{\hat{\otimes} n_j} = \text{the symmetrization of } \otimes_{n_j \neq 0}\, \zeta_j^{\hat{\otimes} n_j},$$

$$\mathbb{z}^{\mathbb{n}} = \mathbb{z}^{\mathbb{n}}(z) = (2^n\mathbb{n}!)^{-1/2}\, \langle z^{\hat{\otimes} n}, \zeta^{\hat{\otimes}\mathbb{n}}\rangle \text{ for } z \in H^*, \qquad (1.\ 1)$$

$$h_{\mathbb{n}} = h_{\mathbb{n}}(x) = (2^n\mathbb{n}!)^{-1/2}\, \Pi_j H_{n_j}\Big(\langle x, \zeta_j\rangle/\sqrt{2}\Big) \text{ for } x \in E^*, \qquad (1.\ 2)$$

where $\{(\lambda_j, \zeta_j)\}_{j=0}^{\infty}$ is the eigen system of D^{-1} and $H_n(u)$ is the Hermite polynomial of n degrees defined by

$$H_n(u) = (-1)^n \exp[u^2]\ (d/du)^n \exp[-u^2].$$

$\mathcal{B}$ is the smallest σ-algebra containing all cylindrical sets of E^*. Here, cylindrical sets of E^* are subsets of E^* of the form:

$$\{x \in E^*;\ (\langle x, \xi_1\rangle, \cdots, \langle x, \xi_n\rangle) \in B_n\}$$

where n is any integer ≥ 1, B_n is any n-dimensional Borel set, and $\xi_1, \cdots, \xi_n$ are any elements of E.

§2. The space of white noise functionals (L^2), the Bargmann space $(\mathfrak{F}_0)$ over a nuclear space, and Gauss transform G

The functional $C(\xi) = \exp[-\frac{1}{2}\|\xi\|_0^2]$ of ξ is positive definite and continuous on the nuclear space E. By Bochner-Minlos' theorem there exists a unique Gaussian probability measure μ in the measurable space $(E^*, \mathscr{B})$ such that

$$\int_{E^*} \exp[\sqrt{-1}\langle x, \xi\rangle]\ d\mu(x) = C(\xi),$$

(Minlos [M]). Since D^{-s} for $s > s_0$ is of Hilbert-Schmidt type, $\mu(E_{-s}) = 1$ holds. Hence, when a functional is defined on E_{-s} for $s > s_0$, then we may consider that it is given μ-a.e. on E^*.

The space $L^2(E^*, \mathscr{B}, \mu)$ is called the space of white noise functionals and denoted by (L^2) (Hida [H1], [H2]). Then $\mathscr{P}(E^*)$, the space of all polynomials in $\{\langle x, \xi\rangle;\ \xi \in E\}$ over $\mathbb{C}$, is dense in (L^2). $\{h_{\mathfrak{m}};\ \mathfrak{m} \in \mathscr{N}_0\}$ is a complete orthonormal system of (L^2). From now on let CONS stand for complete orthonormal system.

Let us consider the product measure $\nu = \mu \times \mu$ in the space $H^* = E^* + \sqrt{-1}E^*$. Then the system $\{\mathfrak{z}^{\mathfrak{m}};\ \mathfrak{m} \in \mathscr{N}_0\}$ of (1. 1) is orthonormal in the space $L^2(H^*, \nu)$. A *Bargmann space* $(\mathfrak{F}_0)$ is the closure of $\mathscr{P}(H^*)$ in $L^2(H^*, \nu)$, where $\mathscr{P}(H^*)$ is the space of all polynomials in $\{\langle z, \xi\rangle;\ \xi \in H\}$ over $\mathbb{C}$. It is well-known that the space of all entire functions, $\mathfrak{F}(\mathbb{C}^n)$, which are defined on $\mathbb{C}^n$ and square integrable with respect to

$$dg(z) = (2\pi)^{-n} \exp[-(z\bar{z})/2]\ (\sqrt{-1}/2)^n\ dz d\bar{z}$$

is closed in $L^2(\mathbb{C}^n,\ dg(z))$, (see Bargmann [B1]). $(\mathfrak{F}_0)$ is an analogue of $\mathfrak{F}(\mathbb{C}^n)$ in passing from $\mathbb{C}^n$ to the infinite dimensional space H^*. But the element of $(\mathfrak{F}_0)$ is in general not analytic in H^*. Nevertheless $(\mathfrak{F}_0)$ is isometrically isomorhic to a normed space consisting of specific analytic functionals in H_0 (see Kondrat'ev [K2]). If we introduce a nuclear rigging $(\mathfrak{F}) \subset (\mathfrak{F}_p) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-p}) \subset (\mathfrak{F}')$, we can see this situation more clearly. The construction of the nuclear rigging and the problem of analytic functionals will be discussed in detail in §3, (see also Berezansky and Kondrat'ev [B-K]).

Now let us dicuss the map G from $\mathscr{P}(E^*)$ onto $\mathscr{P}(H^*)$ defined as follows: every $\varphi(x) \in \mathscr{P}(E^*)$ can be naturally and analytically extended to $\varphi(z) \in \mathscr{P}(H^*)$ replacing $\langle x, \xi\rangle$ by $\langle z, \xi\rangle$. We can define a map G on $\mathscr{P}(E^*)$ by

$$G\varphi(w) = \int_{E^*} \varphi(x + w/\sqrt{2})\ d\mu(x) \quad \text{for } \varphi \in \mathscr{P}(E^*). \qquad (2.\ 1)$$

Then obviously, $G\varphi$ belongs to $\mathscr{P}(H^*)$. Its inverse map G^{-1} is given by

$$G^{-1}f(x) = \int_{E^*} f(\sqrt{2}(x + \sqrt{-1}y))\ d\mu(y) \quad \text{for } f \in \mathscr{P}(H^*). \ (2.\ 2)$$

Actually, we can see that

$$Gh_{\mathrm{n}} = \mathrm{z}^{\mathrm{n}} \quad \text{and} \quad G^{-1}\mathrm{z}^{\mathrm{n}} = h_{\mathrm{n}}. \qquad (2.\ 3)$$

Since $\{h_{\mathrm{n}};\ \mathrm{n} \in \mathscr{N}_0\}$ and $\{\mathrm{z}_{\mathrm{n}};\ \mathrm{n} \in \mathscr{N}_0\}$ are CONS' in (L^2) and $(\mathfrak{F}_0)$ respectively, the map G extends to an isometry from (L^2) onto $(\mathfrak{F}_0)$:

$$\|G\varphi\|_{(\mathfrak{F}_0)} = \|\varphi\|_{(L^2)} \quad \text{for } \varphi \in (L^2). \qquad (2.\ 4)$$

The map G given by (2. 1) is often called Gauss transform ([B&K],[H2],[K2]), so we also call this isometric isomorphism $G:\mathscr{P}(E^*) \longrightarrow \mathscr{P}(H^*)$ or its extension from (L^2) onto $(\mathfrak{F}_0)$ *Gauss* *transform*. The integral expression (2. 1) of G (resp. (2. 2) of G^{-1}) is not valid on (L^2) (resp. on $(\mathfrak{F}_0)$). But we will show in a forthcoming paper that these expressions can extend to the ones between much wider spaces than $\mathcal{P}(E^*)$ and $\mathcal{P}(H^*)$.

§3. The Gel'fand triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$ rigged by the operator $\Lambda(D^p)$

Let D be the self-adjoint operator of H_0 introduced in §1. Since D^p act on H^* naturally and continuously, we can define operators $\Lambda(D^p)$ on $\mathcal{P}(H^*)$ by

$$\Lambda(D^p)f(z) = f(D^p z) \text{ for } f \in \mathcal{P}(H^*), \tag{3. 1}$$

where $z \in H^*$ and $p \in \mathbb{R}$. Let $f(z) = \Pi_{j=1}^n \langle z, \xi_j\rangle \in \mathcal{P}(H^*)$. Then, by the relation

$$\Lambda(D^p)f(z) = \Pi_{j=1}^n \langle D^p z, \xi_j\rangle = \Pi_{j=1}^n \langle z, D^p\xi_j\rangle$$

we see that $\{(\lambda^{-p\mathfrak{n}}, \mathbb{z}^{\mathfrak{n}}); \mathfrak{n} \in \mathcal{N}_0\}$ is an eigen system of $\Lambda(D^p)$:

$$\Lambda(D^p)\mathbb{z}^{\mathfrak{n}}(z) = \left(\Pi_j \lambda_j^{-pn_j}\right) \mathbb{z}^{\mathfrak{n}}(z) = \lambda^{-p\mathfrak{n}}\mathbb{z}^{\mathfrak{n}}(z). \tag{3. 2}$$

As is easily seen, $\mathcal{P}(H^*)$ is a pre-Hilbert space with the inner product

$$(\Lambda(D^p)f, \Lambda(D^p)g)_{(\mathfrak{F}_0)} = \int_{H^*} \left(\Lambda(D^p)f(z)\right)\overline{\Lambda(D^p)g(z)}\, d\nu(z). \tag{3. 3}$$

We will denote its completion by $(\mathfrak{F}_p)$ and the inner product by $(f, g)_{(\mathfrak{F}_p)}$. As well as in the case of D^q, we can see that the oprator $\Lambda(D^q)$ is an isometry from the Hilbert space $(\mathfrak{F}_p)$ onto the Hilbrt space $(\mathfrak{F}_{p-q})$. We can easily see the following:

PROPOSITION 3. 1. *For any* $p \in \mathbb{R}$, $\{\lambda^{p\mathfrak{n}}\mathbb{z}^{\mathfrak{n}}; \mathfrak{n} \in \mathcal{N}_0\}$ *is a* CONS *of* $(\mathfrak{F}_p)$. *And hence any* $f \in (\mathfrak{F}_p)$ *can be expressed in the form*

$$f = \Sigma_{\mathfrak{n}\in\mathcal{N}_0} c_{\mathfrak{n}}\mathbb{z}^{\mathfrak{n}} \tag{3. 4}$$

with coefficients $\{c_{\mathfrak{n}}; \mathfrak{n} \in \mathcal{N}_0\}$ *satisfying*

$$\|f\|^2_{(\mathfrak{F}_p)} = \Sigma_{\mathfrak{n}\in\mathcal{N}_0} \lambda^{-2p\mathfrak{n}} |c_{\mathfrak{n}}|^2 < \infty. \tag{3. 5}$$

Furthermore, we have that for $f \in (\mathfrak{F}_p)$ *of the form* (3. 4)

$$\Lambda(D^q)f = \Sigma_{\mathfrak{n}\in\mathcal{N}_0} \lambda^{-q\mathfrak{n}} c_{\mathfrak{n}} \mathbb{z}^{\mathfrak{n}} \in (\mathfrak{F}_{p-q}). \tag{3. 6}$$

By the proposition, we can identify $(\mathfrak{F}_{-p})$ with the dual space of $(\mathfrak{F}_p)$ and get, for $p > q > 0$,

$$(\mathfrak{F}_p) \subset (\mathfrak{F}_q) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-q}) \subset (\mathfrak{F}_{-p}).$$

Since D^{-1} is of Hilbert-Schmidt type, it follows that for any $p \in \mathbb{R}$ and for any $s > s_0$

$$\Sigma_{\mathfrak{n}\in\mathcal{N}_0} \|\lambda^{(p+s)\mathfrak{n}} \mathbb{z}^{\mathfrak{n}}\|^2_{(\mathfrak{F}_p)} = \Pi_j (1 - \lambda_j^{2s})^{-1} < \infty. \tag{3. 7}$$

This shows that the canonical injection from $(\mathfrak{F}_{p+s})$ into $(\mathfrak{F}_p)$ is also of Hilbert-Schmidt type. If we write

$$(\mathfrak{F}) = \cap_{p=0}^{\infty} (\mathfrak{F}_p) \quad \text{and} \quad (\mathfrak{F}') = \cup_{p=0}^{\infty} (\mathfrak{F}_{-p}), \tag{3. 8}$$

then the dual space of $(\mathfrak{F})$ is $(\mathfrak{F}')$. Thus we obtain a Gel'fand triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$. The following is known: the triplet of this type has a "holomorphic realization" given by analytic functionals of at most order 2 (ref. [B-K],[K2]). Within our setting let us reform this as:

PROPOSITION 3. 2. *For any* $p \in \mathbb{R}$, $f \in (\mathfrak{F}_p)$ *with the expression* (3. 4),

$$\Sigma_{\mathfrak{n}\in\mathcal{N}_0} c_{\mathfrak{n}} \mathbb{z}^{\mathfrak{n}}(z) \tag{3. 9}$$

converges absolutely and uniformly to a functional $\tilde{f}(z)$ *on any bounded set of* H_{-p}. *The limit functional* $\tilde{f}(z)$ *satisfies*

$$|\tilde{f}(z)| \leq \exp[\tfrac{1}{4}\|z\|^2_{-p}] \, \|f\|_{(\mathfrak{F}_p)} \quad \text{for any } z \in H_{-p}. \tag{3. 10}$$

Further $\tilde{f}(z)$ *is not only continuous but analytic in* H_{-p} *in the sense of* [H-P] (E. Hille & R. S. Phillips).

PROOF. By Schwarz' inequality and (1. 1), we have that for any $z \in H_{-p}$

$$\sum_{n \in \mathcal{N}_0} |c_{\mathbb{n}} \mathbb{z}^{\mathbb{n}}(z)|$$
$$= \sum_{n=0}^{\infty} \sum_{|\mathbb{n}|=n} |c_{\mathbb{n}} \mathbb{z}^{\mathbb{n}}(z)|$$
$$= \sum_{n=0}^{\infty} \sum_{|\mathbb{n}|=n} |c_{\mathbb{n}} \langle z^{\hat{\otimes} n}, (2^n\ \mathbb{n}!)^{-1/2} \zeta^{\hat{\otimes}\mathbb{n}} \rangle|$$
$$= \sum_{n=0}^{\infty} (2^n n!)^{-1/2} \sum_{|\mathbb{n}|=n} |c_{\mathbb{n}}|\ \lambda^{-p\mathbb{n}} \left(\frac{n!}{\mathbb{n}!}\right)^{1/2} |\langle z^{\hat{\otimes} n}, \lambda^{p\mathbb{n}} \zeta^{\hat{\otimes}\mathbb{n}} \rangle|$$
$$\leq \|f\|_{(\mathfrak{F}_p)} \exp[\tfrac{1}{4}\|z\|_{-p}^2].$$

Therefore the series converges to a continuous functional $\tilde{f}$ on H_{-p} absolutely and uniformly on any bounded set of H_{-p} and $\tilde{f}$ satisfies (3. 9). The finite sums of the right hand side of (3. 4) are functionals analytic and locally uniformly bounded in H_{-p} in the sense of [H-P]. Applying Theorem 3. 18. 1 of [H-P], we have the analyticity of $\tilde{f}$ in H_{-p}. □

PROPOSITION 3. 3. *If $p > s_0$ and $f \in (\mathfrak{F}_p)$, then the functional $\tilde{f}$ in* PROPOSITION 3. 2 *is a unique continuous version of f in H_{-p}; that is $\tilde{f}(z) = f(z)$ holds for ν-a.e. $z \in H^*$. Besides if $p > q + s_0$, then $\tilde{f}(D^q z)$ coincides with the continuous version of $\Lambda(D^q)f(z)$ in H_{-p+q}.*

PROOF. f, as the L^2-limit of (3. 4), is ν-a.e. defined and square-integrable in H^*. Since $\nu(H_{-p}) = 1$ for $p > s_0$, $\tilde{f}$ is equal to f ν-a.e. in H^*. Since every non void open set in H_{-p} has strictly positive ν-measure, the continuous version of f is uniquely given in H_{-p}. If $p > s_0 + q$ and $z \in H_{-p+q}$, then $D^q z \in H_{-p}$ and $p - q > s_0$. Therefore we see that

$$\tilde{f}(D^q z) = \sum_{n\in\mathcal{N}_0} c_{\mathfrak{n}} \mathbb{z}^{\mathfrak{n}}(D^q z)$$
$$= \sum_{n\in\mathcal{N}_0}^{\infty} \lambda^{-q\mathfrak{n}} c_{\mathfrak{n}} \mathbb{z}^{\mathfrak{n}}(z)$$

converges uniformly on any bounded set in H_{-p+q}. Thus we have the last assertion. □

For $p < s_0$ and $f \in (\mathfrak{F}_p)$, the functional $\tilde{f}$ analytic in H_{-p} does not mean a version in the sense of ν-a.e. because of $\nu(H_{-p}) = 0$. However, the version $\tilde{f}$ recovers f by means of Taylor coefficients (ref. [B-K], [K2]).

If $f \in (\mathfrak{F})$, then $\tilde{f}(z)$ can be defined on H_{-p} for any $p > 0$ and so $\tilde{f}(z)$ is defined in H^*. Moreover, if $p > q$, then the continuity of $\tilde{f}(z)$ in H_{-p} implies the one in H_{-q}. It follows from this that $\tilde{f}(z)$ is continuous in $z \in H^*$ with the inductive limit topology of $H^* = \varinjlim H_{-p}$. But we omit the proof. Besides we can say that $\tilde{f}(z)$ is not merely entire of at most order 2 on any H_{-p} $(p > 0)$ but also of minimal type, as we see in the following as a corollary of PROPOSITION 3. 2 (ref. [B-K], [K2]).

COROLLARY 3. 1. *If* $f \in (\mathfrak{F})$, *then for any* $p > 0$, *any* $k > 0$, *and for any* $z \in H_{-p}$ *we have*

$$|\tilde{f}(z)| \le \|f\|_{(\mathfrak{F}_{p+k})} \exp[\tfrac{1}{4}\lambda_0^{2k} \|z\|_{-p}^2]. \qquad (3.\ 10)$$

PROOF. Let $z \in H_{-p}$. Then this is clear from (3. 10) and

$$\|z\|_{-(p+k)}^2 \le \lambda_0^{2k} \|z\|_{-p}^2. \ \square$$

§4. The triplet $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S}')$ derived by Gauss transform from the triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$

In this section, we begin by reconsidering G as a map from

$\mathscr{S}(E^*)$ onto $\mathscr{S}(H^*)$. Next, we define operators $\{\Gamma(D^p) \equiv G^{-1}\Lambda(D^p)G;\ p \in \mathbb{R}\}$ which act on $\mathscr{S}(E^*)$. Using these operators we construct the nuclear rigging of white noise functionals:

$$(\mathscr{S}) \subset (\mathscr{S}_p) \subset (L^2) \subset (\mathscr{S}_{-p}) \subset (\mathscr{S}'). \tag{4. 1}$$

It will turn out that the rigging (4. 1) is obtained as the image of

$$(\mathfrak{F}) \subset (\mathfrak{F}_p) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-p}) \subset (\mathfrak{F}')$$

by the extended G^{-1}.

Let us define the operator $\Gamma(D^p)$ from $\mathscr{S}(E^*)$ onto itself. G is an isometry from $\mathscr{S}(E^*)$ onto $\mathscr{S}(H^*)$:

$$\left(\mathscr{S}(E^*),\ \|\cdot\|_{(L^2)}\right) \xrightarrow[\text{isometric}]{G} \left(\mathscr{S}(H^*),\ \|\cdot\|_{(\mathfrak{F}_0)}\right); \tag{4. 2}$$

$\Lambda(D^p)$ maps $\mathscr{S}(H^*)$ onto $\mathscr{S}(H^*)$. Therefore we can define $\Gamma(D^p)$ for each $p \in \mathbb{R}$ by setting

$$\Gamma(D^p)\varphi = G^{-1}\Lambda(D^p)G\varphi \quad \text{for } \varphi \in \mathscr{S}(E^*). \tag{4. 3}$$

Then, it is easy to see that $\mathscr{S}(E^*)$ is a pre-Hilbert space with the inner product

$$\left(\Gamma(D^p)\varphi,\ \Gamma(D^p)\psi\right)_{(L^2)} = \int_{E^*} \left(\Gamma(D^p)\varphi(x)\right)\overline{\Gamma(D^p)\psi(x)}\ d\mu(x). \tag{4. 4}$$

Let us denote its completion by $(\mathscr{S}_p)$ and the inner product by $(\varphi,\ \psi)_{(\mathscr{S}_p)}$. We evidently see that $(\mathscr{S}_0) = (L^2)$. Corresponding to the eigen system of $\Lambda(D^p)$, $\Gamma(D^p)$ has the eigen system:

$$\Gamma(D^p)h_{\mathbb{n}}(x) = \left(\Pi_j\ \lambda_j^{-pn_j}\right)h_{\mathbb{n}}(x) = \lambda^{-p\mathbb{n}}h_{\mathbb{n}}(x). \tag{4. 5}$$

This follows from (2. 3) and (3. 2):

$$Gh_{\mathbb{n}} = \mathbb{z}^{\mathbb{n}},\ G^{-1}\mathbb{z}^{\mathbb{n}} = h_{\mathbb{n}}, \text{ and}$$

$$\Lambda(D^p)\mathbb{z}^{\mathbb{n}}(z) = \left(\Pi_j\ \lambda_j^{-pn_j}\right)\ \mathbb{z}^{\mathbb{n}}(z) = \lambda^{-p\mathbb{n}}\mathbb{z}^{\mathbb{n}}(z).$$

The system $\{h_{\mathbb{n}};\ \mathbb{n} \in \mathscr{N}_0\}$ is a CONS of (L^2), so we can easily see the following.

PROPOSITION 4. 1. *For any* $p \in \mathbb{R}$, $\{\lambda^{p\mathfrak{n}} h_{\mathfrak{n}};\ \mathfrak{n} \in \mathcal{N}_0\}$ *is a* CONS *of* $(\mathcal{S}_p)$. *And hence any* $\varphi \in (\mathcal{S}_p)$ *can be expressed in the form*

$$\varphi = \sum_{\mathfrak{n}\in\mathcal{N}_0} c_{\mathfrak{n}} h_{\mathfrak{n}} \tag{4. 6}$$

with coefficients $\{c_{\mathfrak{n}};\ \mathfrak{n} \in \mathcal{N}_0\}$ *satisfying*

$$\|\varphi\|^2_{(\mathcal{S}_p)} = \sum_{\mathfrak{n}\in\mathcal{N}_0} \lambda^{-2p\mathfrak{n}} |c_{\mathfrak{n}}|^2 < \infty. \tag{4. 7}$$

Furthermore, for any p *and* $q \in \mathbb{R}$, $\Gamma(D^q)$ *can extend its domain to* $(\mathcal{S}_p)$ *as an isometry from* $(\mathcal{S}_p)$ *to* $(\mathcal{S}_{p-q})$ *satisfying that for* $\varphi \in (\mathcal{S}_p)$ *of the form* (4. 5)

$$\Gamma(D^q)\varphi = \sum_{\mathfrak{n}\in\mathcal{N}_0} \lambda^{-q\mathfrak{n}} c_{\mathfrak{n}} h_{\mathfrak{n}} \in (\mathcal{S}_{p-q}). \tag{4. 8}$$

By the proposition above we can identify the dual space of $(\mathcal{S}_p)$ with $(\mathcal{S}_{-p})$ for $p \in \mathbb{R}$. In fact, the bilinear form, $\langle\Psi, \varphi\rangle$, of $(\varphi, \Psi) \in (\mathcal{S}_p)\times(\mathcal{S}_{-p})$ is realized by

$$\langle\Psi, \varphi\rangle := \int_{E^*} \Big(\Gamma(D^{-p})\Psi(x)\Big)\Gamma(D^p)\varphi(x)\ d\mu(x). \tag{4. 9}$$

Let us write

$$(\mathcal{S}) = \cap_{p=0}^{\infty} (\mathcal{S}_p) \text{ and } (\mathcal{S}') = \cup_{p=0}^{\infty} (\mathcal{S}_{-p}). \tag{4. 10}$$

From (3. 7) it follows that, for any $p \in \mathbb{R}$ and any $s > s_0$,

$$\sum_{\mathfrak{n}\in\mathcal{N}_0} \|\lambda^{(p+s)\mathfrak{n}} h_{\mathfrak{n}}\|^2_{(\mathcal{S}_p)} = \Pi_j (1 - \lambda_j^{2s})^{-1} < \infty.$$

Thus we have a nuclear rigging

$$(\mathcal{S}) \subset (\mathcal{S}_p) \subset (L^2) \subset (\mathcal{S}_{-p}) \subset (\mathcal{S}'),\ p > 0. \tag{4. 11}$$

Clearly $(\mathcal{S}')$ is a dual space of $(\mathcal{S})$. We call $(\mathcal{S})$ the space of test white noise functionals and $(\mathcal{S}')$ the space of generalized white noise functionals, as usual.

Let $p \in \mathbb{R}$. It follows from (4. 2) that for any $f \in \mathcal{F}(H^*)$

$$\|G^{-1}f\|_{(\mathcal{S}_p)} = \|\Gamma(D^p)G^{-1}f\|_{(L^2)} = \|\Lambda(D^p)f\|_{(\mathfrak{F}_0)} = \|f\|_{(\mathfrak{F}_p)}.$$

Therefore G^{-1} can extend uniquely to an isometric operator G_p^{-1}

from $(\mathfrak{F}_p)$ onto $(\mathcal{S}_p)$. The extensions $\{G_p^{-1};\ p \in \mathbb{R}\}$ are consistent. That is, if $p < q$, then G_p^{-1} coincides with G_q^{-1} on $(\mathfrak{F}_q)$. So we have a unique continuous extension from $(\mathfrak{F}')$ onto $(\mathcal{S}')$, which we denote by the same symbol G^{-1}. It satisfies that for any f, $g \in (\mathfrak{F}_p)$ and any $p \in \mathbb{R}$

$$(G^{-1}f,\ G^{-1}g)_{(\mathcal{S}_p)} = (f,\ g)_{(\mathfrak{F}_p)}. \qquad (4.\ 12)$$

Moreover, we can easily see that for $F \in (\mathfrak{F}_{-p})$ and $f \in (\mathfrak{F}_p)$

$$\langle G^{-1}F,\ G^{-1}f\rangle = \langle F,\ f\rangle. \qquad (4.\ 13)$$

The above nuclear rigging is the same as the usual rigging of white noise calculus, as we see in the following. Let us put for $n = 0, 1, 2, \cdots$

$$\mathcal{P}_n(E^*) = \{\varphi;\ \varphi \in \mathcal{P}(E^*),\ \text{the degree of } \varphi \le n\},$$

$$\overline{\mathcal{P}}_n = (L^2)\text{-closure of } \mathcal{P}_n(E^*),$$

$$\mathcal{H}_n = \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{(n-1)}\ (n \ge 1),\ \text{and } \mathcal{H}_0 = \mathbb{C},$$

where $\mathcal{P}_0(E^*) = \overline{\mathcal{P}}_0 = \mathbb{C}$. Then $\langle x^{\hat{\otimes}n},\ f_n\rangle$ is well-defined as an element of $\overline{\mathcal{P}}_n$ for any $f_n \in H_0^{\hat{\otimes}n}$ and $\{h_{\mathfrak{m}};\ \mathfrak{m} \in \mathcal{N}_0 \text{ and } |\mathfrak{m}| = n\}$ is an orthonormal basis of $\mathcal{H}_n$ for each $n \ge 0$. It is well-known that (L^2) has Wiener-Itô decomposition

$$(L^2) = \textstyle\sum_{n=0}^{\infty} \oplus\ \mathcal{H}_n,$$

Let us put

$$F_n(H) = \{\text{any finite sum of } \otimes_{j=1}^{n}\xi_j;\ \xi_j \in H\ (1 \le j \le n)\},$$

$$Z_n = \{\text{any linear combination of } \zeta^{\hat{\otimes}\mathfrak{m}}/\sqrt{\mathfrak{m}!};\ |\mathfrak{m}| = n\},$$

$$\Phi(H) = \{(f_n)_{n=0}^{\infty};\ f_n = 0 \text{ for almost all } n \ge 0 \text{ and } f_n \in F_n(H)\},$$

and

$$\Phi = \{(f_n)_{n=0}^{\infty};\ f_n = 0 \text{ for almost all } n \ge 0 \text{ and } f_n \in Z_n\}.$$

Clearly $\Phi \subset \Phi(H)$ and Φ is dense in Φ_0. Let Φ_p $(p \in \mathbb{R})$ be the

Fock space defined as a direct sum of Hilbert spaces $H_p^{\hat{\otimes} n}$ with weights $\sqrt{n!}$ $(n \geq 0)$. That is,

$$\Phi_p = \Sigma_{n=0}^{\infty} \oplus \sqrt{n!} H_p^{\hat{\otimes} n}.$$

Then it is easy to see that the Fock space Φ_p $(p \in \mathbb{R})$ coincides with the completion of $\Phi(H)$ by the inner product

$$(\overrightarrow{f}, \overrightarrow{g})_{\Phi_p} = \Sigma_{n=0}^{\infty} n!(f_n, g_n)_{H_p^{\hat{\otimes} n}}$$

$$= \Sigma_{n=0}^{\infty} n! \left((D^p)^{\otimes n} f_n, (D^p)^{\otimes n} g_n\right)_{H_0^{\hat{\otimes} n}} \tag{4. 14}$$

where $\overrightarrow{f} = (f_n)_{n=0}^{\infty}$ and $\overrightarrow{g} = (g_n)_{n=0}^{\infty} \in \Phi(H)$.

The Segal isomorphism I_s from Φ_0 to (L^2) is defined by

$$I_s((f_n)_{n=0}^{\infty}) = \Sigma_{n=0}^{\infty} :\langle x^{\hat{\otimes} n}, f_n\rangle:, \tag{4. 15}$$

where $:\langle x^{\hat{\otimes} n}, f_n\rangle:$ denotes the orthogonal projection of $\langle x^{\hat{\otimes} n}, f_n\rangle$ to the space $\mathcal{H}_n$ for each $f_n \in H_0^{\hat{\otimes} n}$. Let us assure ourselves that I_s is well-defined. First we note that the right hand side of (4. 15) is a finite sum if $\overrightarrow{f} = (f_n)_{n=0}^{\infty}$ is an element of Φ or $\Phi(H)$. If we apply the formula

$$(2u)^n = \Sigma_{2k \leq n} \binom{n}{2k} \frac{(2k)!}{k!} H_{n-2k}(u)$$

to $\langle x^{\hat{\otimes} n}, \zeta^{\hat{\otimes} \mathbb{n}}/\sqrt{\mathbb{n}!}\rangle$, then we have

$$\langle x^{\hat{\otimes} n}, \zeta^{\hat{\otimes} \mathbb{n}}/\sqrt{\mathbb{n}!}\rangle = \Sigma_{2\mathbb{k} \leq \mathbb{n}} \binom{\mathbb{n}}{2\mathbb{k}} \frac{(2\mathbb{k})!}{\mathbb{k}!} 2^{-k}((\mathbb{n}-2\mathbb{k})!/\mathbb{n}!)^{1/2} h_{\mathbb{n}-2\mathbb{k}}(x).$$

But $\{h_{\mathbb{n}};\ |\mathbb{n}| = n\}$ is an orthonormal basis of $\mathcal{H}_n$ for each $n \geq 0$ and these bases are mutually orthogonal for different $n \geq 0$; hence we have

$$:\langle x^{\hat{\otimes} n}, \zeta^{\hat{\otimes} \mathbb{n}}/\sqrt{\mathbb{n}!}\rangle: = h_{\mathbb{n}}(x) \quad (\mathbb{n} \in \mathcal{N}_0),$$

i.e.,
$$I_s\left((0, \cdots, 0, \zeta^{\hat{\otimes} \mathbb{n}}/\sqrt{\mathbb{n}!}, 0, \cdots)\right) = h_{\mathbb{n}}. \tag{4. 16}$$

In addition we have

$$\|(0, \cdots, 0, \zeta^{\hat{\otimes}\mathfrak{n}}/\sqrt{\mathfrak{n}!}, 0, \cdots)\|_{\Phi_0} = \|h_{\mathfrak{n}}\|_{(L^2)}. \qquad (4.\ 17)$$

This means that I_S can be well-defined as an isometry from Φ_0 to (L^2).

PROPOSITION 4. 2. *For $p \in \mathbb{R}$, the Hilbert space $(\mathcal{S}_p)$ coincides with the completion of $\{I_S(\vec{f}); \vec{f} \in \Phi(H)\}$ by the inner product*

$$(I_S(\vec{f}), I_S(\vec{g}))_p \equiv (\vec{f}, \vec{g})_{\Phi_p}. \qquad (4.\ 18)$$

PROOF. By PROPOSITION 4. 1, the set

$$\mathcal{P} \equiv \{\Sigma_{\mathfrak{n}\in J}\, c_{\mathfrak{n}} h_{\mathfrak{n}};\ \text{J (finite subset)} \subset \mathcal{N}_0,\ c_{\mathfrak{n}} \in \mathbb{C}\}$$

is dense in $(\mathcal{S}_p)$. Since $\{\lambda^{p\mathfrak{n}}(n!/\mathfrak{n}!)^{1/2}\zeta^{\hat{\otimes}\mathfrak{n}};\ |\mathfrak{n}| = n\}$ is a CONS of $H_p^{\hat{\otimes}n}$ for each $n \geq 0$, the set $\{I_S(\vec{f}); \vec{f} \in \Phi\}$ is dense in the completion of $\{I_S(\vec{f}); \vec{f} \in F(H)\}$ completed by (4. 17). Further by (4. 16) we have $I_S(\Phi) = \mathcal{P}$. Therefore the assertion is true. □

COROLLARY 4. 1. *If $p \geq 0$ and $\vec{f} \in \Phi_p$, then $I_S(\vec{f})$ is defined as an element of (L^2). Hence the space $(\mathcal{S}_p)$ coincides with the totality of $\{I_S(\vec{f}); \vec{f} \in \Phi_p)\}$.*

PROOF. This is clear from $\Phi_p \subset \Phi_0$ for $p \geq 0$. □

In the following we consider several properties of white noise functionals. We will see that our setting makes the computations easy and helps us obtain the sharper inequalities.

THEOREM 4. 1. *Let $p > p_0$. For any $\varphi \in (\mathcal{S}_p)$ with the expression* (4. 6), *the functional of $z \in H_{-p}$*

$$\Sigma_{\mathfrak{n}\in\mathcal{N}_0}\, c_{\mathfrak{n}} h_{\mathfrak{n}}(z) \qquad (4.\ 19)$$

converges absolutely and uniformly to a functional $\tilde{\varphi}(z)$ *on any bounded set of* H_{-p}. *The limit functional* $\tilde{\varphi}(z)$ *is continuous on* H_{-p}, *(is called the continuous continuation of* φ *in* H_{-p}*) and satisfies, for any* $z = x + \sqrt{-1}y \in H_{-p}$ $(x, y \in E_{-p})$,

$$|\tilde{\varphi}(z)|$$

$$\le \sqrt{\alpha_{2p}} \exp[\tfrac{1}{2} \Sigma_{j=0}^{\infty}\{\frac{\lambda_j^{2p}}{1+ \lambda_j^{2p}} |\langle x, \zeta_j\rangle|^2 + \frac{\lambda_j^{2p}}{1-\lambda_j^{2p}}|\langle y, \zeta_j\rangle|^2\}] \|\varphi\|_{(\mathscr{S}_p)}$$

$$\le \sqrt{\alpha_{2p}} \exp[\|z\|_{-p}^2] \|\varphi\|_{(\mathscr{S}_p)}. \tag{4. 20}$$

where $\alpha_p = \Pi_{j=0}^{\infty} (1 - \lambda_j^{2p})^{-1/2}$. *Therefore* $\tilde{\varphi}(z)$ *is analytic in* H_{-p} *in the sense of* [H-P] *(E. Hille & R. S. Phillips).*

PROOF. Let $p > p_0$ and $\varphi \in (\mathscr{S}_p)$ which has the expression

$$\varphi = \Sigma_{\mathbb{n}\in\mathscr{N}_0} c_{\mathbb{n}} h_{\mathbb{n}} \quad \text{with} \quad \Sigma_{\mathbb{n}\in\mathscr{N}_0} \lambda_j^{-2p\mathbb{n}}|c_{\mathbb{n}}|^2 < \infty.$$

By Schwartz' inequality and Mehler's formula: for $|s| < 1$

$$\Sigma_{n=0}^{\infty} s^n (2^n n!)^{-1} H_n(u)H_n(v)$$

$$= (1-s^2)^{-1/2}\exp[(1-s^2)^{-1}\{2suv-s^2(u^2+v^2)\}, \tag{4. 21}$$

we have

$$|\Sigma_{\mathbb{n}\in\mathscr{N}_0} c_{\mathbb{n}} h_{\mathbb{n}}(z)|$$

$$\le \left(\Sigma_{\mathbb{n}\in\mathscr{N}_0} \lambda^{-2p\mathbb{n}}|c_{\mathbb{n}}|^2\right)^{1/2} \left(\Sigma_{\mathbb{n}\in\mathscr{N}_0} \lambda^{2p\mathbb{n}}|h_{\mathbb{n}}(z)|^2\right)^{1/2}$$

$$= \|\varphi\|_{(\mathscr{S}_p)} \left(\Sigma_{\mathbb{n}\in\mathscr{N}_0} \lambda^{2p\mathbb{n}} h_{\mathbb{n}}(z)h_{\mathbb{n}}(\bar{z})\right)^{1/2}$$

$$= \|\varphi\|_{(\mathscr{S}_p)}\left(\Pi_j \Sigma_{n=0}^{\infty} \lambda_j^{2pn}(2^n n!)^{-1}H_n\left(\frac{\langle z, \zeta_j\rangle}{\sqrt{2}}\right)H_n\left(\frac{\langle \bar{z}, \zeta_j\rangle}{\sqrt{2}}\right)\right)^{1/2}$$

$$= \|\varphi\|_{(\mathscr{S}_p)}\sqrt{\alpha_{2p}} \exp[\tfrac{1}{2} \Sigma_{j=0}^{\infty}\{\frac{\lambda_j^{2p}}{1+\lambda_j^{2p}} |\langle x, \zeta_j\rangle|^2 + \frac{\lambda_j^{2p}}{1-\lambda_j^{2p}} |\langle y, \zeta_j\rangle|^2\}].$$

If $0 < u < 1/2$, then $u/(1-u) < 2u$. So we have, putting $u = \lambda_j^{2p}$,

$$|\Sigma_{\mathbb{n}\in\mathscr{N}_0} c_{\mathbb{n}} h_{\mathbb{n}}(z)| \le \|\varphi\|_{(\mathscr{S}_p)}\sqrt{\alpha_{2p}} \exp[\|z\|_{-p}^2], \tag{4. 22}$$

and (4. 19) converges absolutely and uniformly to a functional $\tilde{\varphi}(x)$ on any bounded set of H_{-p}. This inequality gives the locally uniformly boundedness of every finite sum of (4. 19) and it follows from this that $\tilde{\varphi}(z)$ is analytic in H_{-p} (in the sense of [H-P]). □

COROLLARY 4. 2. *Especially, if* $x \in E_{-p}$ *and* $\varphi \in (\mathcal{S}_p)$ *for* $p > s_0$, *then we have*

$$|\tilde{\varphi}(x)| \le \sqrt{\alpha_{2p}} \exp[\tfrac{1}{2} \textstyle\sum_{j=0}^{\infty} \dfrac{\lambda_j^{2p}}{1+\lambda_j^{2p}} |\langle x, \zeta_j\rangle|^2] \, \|\varphi\|_{(\mathcal{S}_p)}$$

$$\le \sqrt{\alpha_{2p}} \exp[\tfrac{1}{2}\|x\|_{-p}^2] \, \|\varphi\|_{(\mathcal{S}_p)} \qquad (4.\ 23)$$

and $\varphi(x) = \tilde{\varphi}(x)$ μ-a.e. $x \in E^*$. *(*$\tilde{\varphi}(x)$ *is called a continuous version of* φ.*)*

PROOF. Let $y = 0$ in (4. 20). Then for $p > s_0$ we can see that our assertion is true. □

§5. Other properties of two triplets

THEOREM 5. 1 *Let* $0 \le q < p - p_0$. *Then the functional* $\exp[\frac{1}{2}\|x\|_{-p}^2]$ *defined in* E_{-p} *belongs to* $(\mathcal{S}_q)$. *Actually, the* $(\mathcal{S}_q)$*-norm is evaluated as*

$$\|\exp[\tfrac{1}{2}\|\cdot\|_{-p}^2]\|_{(\mathcal{S}_q)} = \Pi_j\Big((1 - \lambda_j^{2p})^2 - \lambda_j^{4(p-q)}\Big)^{-1/4}.$$

PROOF. By a direct computation we can see that if $p > p_0$, then the functional $\exp[\frac{1}{2}\|x\|_{-p}^2]$ belongs to $(L^2) = (\mathcal{S}_0)$. So it is expanded into a Fourier series. Let us compute the Fourier coefficients

$$c_{\mathfrak{n}} = \int_{E^*} \exp[\tfrac{1}{2}\|x\|_{-p}^2] \, h_{\mathfrak{n}}(x) \, d\mu(x) \qquad (5.\ 1)$$

with respect to the CONS $\{h_{\mathbb{n}}(x);\ \mathbb{n} \in \mathcal{N}_0\}$ of (L^2). To get the values $c_{\mathbb{n}}$, if we note the equality

$$\exp[\tfrac{1}{2}\|x\|_{-p}^2] = \Pi_{j=0}^{\infty} \exp[\tfrac{1}{2}\lambda_j^{2p}\langle x,\ \zeta_j\rangle^2]$$

and independentness of $\langle x,\ \zeta_j\rangle$'s, we have only to calculate the integrals

$$\int_{E^*} \exp[\tfrac{1}{2}\lambda_j^{2p}\langle x,\ \zeta_j\rangle^2]\ H_n(\langle x,\ \zeta_j\rangle/\sqrt{2})\ d\mu(x).$$

But if n is odd, then the integral is equal to zero and if n is even, say $n = 2k$, then it is equal to

$$(1-\lambda_j^{2p})^{-1/2}\ \frac{n!}{k!}\ \left(\lambda_j^{2p}/(1 - \lambda_j^{2p})\right)^k.$$

So we have for $\mathbb{n} = 2\mathbb{k} = (2k_0,\ 2k_1,\ 2k_2,\ \cdots\cdot)$

$$c_{\mathbb{n}} = \alpha_p\ (2^n\mathbb{n}!)^{-1/2}\ \frac{\mathbb{n}!}{\mathbb{k}!}\ \Pi_j\left(\lambda_j^{2p}/(1 - \lambda_j^{2p})\right)^{k_j}$$

else $c_{\mathbb{n}} = 0$, where

$$\alpha_p = \Pi_{j=0}^{\infty}\ (1 - \lambda_j^{2p})^{-1/2}.$$

Therefore

$$\begin{aligned}\|\exp[\tfrac{1}{2}\|\cdot\|_{-p}^2]\|_{(\mathscr{S}_q)}^2 &= \Sigma_{\mathbb{n}\in\mathcal{N}_0}\ \lambda^{-2q\mathbb{n}}|c_{\mathbb{n}}|^2 \\ &= \alpha_p^2\ \Sigma_{\mathbb{k}\in\mathcal{N}_0} 2^{-2\mathbb{k}}\ \binom{2\mathbb{k}}{\mathbb{k}}\ \Pi_j\left(\lambda_j^{4(p-q)}/(1 - \lambda_j^{2p})^2\right)^{k_j}.\end{aligned} \tag{5. 2}$$

If we recall the definition of the constant p_0 and the formula

$$2^{-2k}\ \binom{2k}{k} = (-1)^k\ \binom{-1/2}{k}$$

(5. 2) is followed by

$$\alpha_p^2\ \Sigma_{\mathbb{k}\in\mathcal{N}_0}\ \Pi_j\binom{-1/2}{k_j}\left(-\lambda_j^{2(p-q)}/(1 - \lambda_j^{2p})\right)^{2k_j}.$$

But $0 \le q < p - p_0$ implies that $\lambda_j^{2(p-q)}/(1-\lambda_j^{2p}) < 1$ and so this infinite sum of the finite product is equal to

$$\alpha_p^2 \prod_j \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(-\lambda_j^{2(p-q)}/(1 - \lambda_j^{2p})\right)^{2k}$$

$$= \alpha_p^2 \prod_j \left(1 - \lambda_j^{4(p-q)}/(1 - \lambda_j^{2p})^2\right)^{-1/2}$$

$$= \prod_j \left((1 - \lambda_j^{2p})^2 - \lambda_j^{4(p-q)}\right)^{-1/2} < \infty. \quad \square$$

THEOREM 5. 2 *Let* $s_0 < s$ *and* $p_0 < p$. *Then we have*

$$(\mathfrak{F}_{s+p}) \cdot (\mathfrak{F}_{s+p}) \subset (\mathfrak{F}_s)$$

and for $f, g \in (\mathfrak{F}_{s+p})$

$$\|f \cdot g\|_{(\mathfrak{F}_s)} \le \gamma_p^2 \|f\|_{(\mathfrak{F}_{s+p})} \|g\|_{(\mathfrak{F}_{s+p})} \tag{5. 3}$$

where γ_p *is given in* LEMMA 5. 1. *Hence* $(\mathfrak{F})$ *is an algebra.*

PROOF. First we note that for $\mathbb{m}, \mathbb{n} \in \mathscr{N}_0$

$$\binom{\mathbb{m}+\mathbb{n}}{\mathbb{n}} \le 2^{|\mathbb{m}|+|\mathbb{n}|} \quad \text{and} \quad \mathbb{z}^{\mathbb{m}}(z) \cdot \mathbb{z}^{\mathbb{n}}(z) = \binom{\mathbb{m}+\mathbb{n}}{\mathbb{n}}^{1/2} \mathbb{z}^{\mathbb{m}+\mathbb{n}}(z).$$

Let $c_{\mathbb{n}} = (f, \mathbb{z}^{\mathbb{n}})_{(\mathfrak{F}_0)}$ and $d_{\mathbb{n}} = (g, \mathbb{z}^{\mathbb{n}})_{(\mathfrak{F}_0)}$. Then we have

$$\tilde{f}(z) = \sum_{\mathbb{n} \in \mathscr{N}_0} c_{\mathbb{n}} \mathbb{z}^{\mathbb{n}}(z) \quad \text{and} \quad \tilde{g}(z) = \sum_{\mathbb{n} \in \mathscr{N}_0} d_{\mathbb{n}} \mathbb{z}^{\mathbb{n}}(z).$$

By PROPOSITION 3. 1 these two series are absolutely convergent on H_{-s-p}. Therefore we have

$$\tilde{f}(z) \cdot \tilde{g}(z) = \sum_{\mathbb{m}, \mathbb{n} \in \mathscr{N}_0} c_{\mathbb{m}} d_{\mathbb{n}} \binom{\mathbb{m}+\mathbb{n}}{\mathbb{n}}^{1/2} \mathbb{z}^{\mathbb{m}+\mathbb{n}}(z)$$

and so, using the triangle inequality and Schwarz' one,

$$\|f \cdot g\|_{(\mathfrak{F}_s)}$$

$$\le \sum_{\mathbb{m}, \mathbb{n} \in \mathscr{N}_0} |c_{\mathbb{m}}| |d_{\mathbb{n}}| 2^{(|\mathbb{m}|+|\mathbb{n}|)/2} \lambda^{-s(\mathbb{m}+\mathbb{n})}$$

$$\le \left(\sum_{\mathbb{m} \in \mathscr{N}_0} |c_{\mathbb{m}}| \lambda^{-(s+p)\mathbb{m}} 2^{|\mathbb{m}|/2} \lambda^{p\mathbb{m}}\right) \left(\sum_{\mathbb{n} \in \mathscr{N}_0} |d_{\mathbb{n}}| \lambda^{-(s+p)\mathbb{n}} 2^{|\mathbb{n}|/2} \lambda^{p\mathbb{n}}\right)$$

$$\le \|f\|_{(\mathfrak{F}_{s+p})} \|g\|_{(\mathfrak{F}_{s+p})} \prod_j \sum_{n=0}^{\infty} \left(2\lambda_j^{2p}\right)^n$$

$$= \|f\|_{(\mathfrak{F}_{s+p})} \|g\|_{(\mathfrak{F}_{s+p})} \prod_j (1 - 2\lambda_j^{2p})^{-1}. \quad \square$$

Let us mention the fact that $(\mathscr{S})$ is an algebra. How to conclude this result was shown in [Ku-T2]. But our setting described above makes some computations a little bit simple. A rewritten form of this theorem within our framework is:

PROPOSITION 5. 1 *Let* $s_0 < s$ and $2p_0 < p$. *If the functionals* φ *and* ψ *are in* $(\mathscr{S}_{s+p})$, *then* $\varphi\cdot\psi$ *belongs to* $(\mathscr{S}_s)$ *and*

$$\|\varphi\cdot\psi\|_{(\mathscr{S}_s)} \le \beta_s\ \kappa_p\ \|\varphi\|_{(\mathscr{S}_{s+p})}\ \|\psi\|_{(\mathscr{S}_{s+p})}$$

where $\beta_s = \Pi_j(1-\lambda_j^{4s}/4)^{-1/2}$ *and* $\kappa_p = \Pi_j(1-4\lambda_j^{2p})^{-1}$.

PROOF. Let $\varphi,\ \psi \in (\mathscr{S}_{s+p})$. Suppose that φ and ψ have the expansions as elements of $(\mathscr{S}_s)$:

$$\varphi = \Sigma_{\mathbb{n}\in\mathscr{N}_0} c_{\mathbb{n}} h_{\mathbb{n}} \quad \text{with} \quad \Sigma_{\mathbb{n}\in\mathscr{N}_0}\lambda^{-2s\mathbb{n}}|c_{\mathbb{n}}|^2 < \infty$$

and

$$\psi = \Sigma_{\mathbb{n}\in\mathscr{N}_0} d_{\mathbb{n}} h_{\mathbb{n}} \quad \text{with} \quad \Sigma_{\mathbb{n}\in\mathscr{N}_0}\lambda^{-2s\mathbb{n}}|d_{\mathbb{n}}|^2 < \infty.$$

The absolute convergence for $x \in E_{-s}$ of the serieses

$$\tilde{\varphi}(x) = \Sigma_{\mathbb{n}\in\mathscr{N}_0} c_{\mathbb{n}} h_{\mathbb{n}}(x) \quad \text{and} \quad \tilde{\psi}(x) = \Sigma_{\mathbb{n}\in\mathscr{N}_0} d_{\mathbb{n}} h_{\mathbb{n}}(x)$$

imlies the absolute convergence of

$$\tilde{\varphi}(x)\cdot\tilde{\psi}(x) = \Sigma_{\mathbb{m},\mathbb{n}\in\mathscr{N}_0} c_{\mathbb{m}} d_{\mathbb{n}} h_{\mathbb{m}}(x) h_{\mathbb{n}}(x).$$

Therefore we have

$$\|\varphi\psi\|_{(\mathscr{S}_s)} \le \Sigma_{\mathbb{m},\mathbb{n}\in\mathscr{N}_0} |c_{\mathbb{m}} d_{\mathbb{n}}|\, \|h_{\mathbb{m}} h_{\mathbb{n}}\|_{(\mathscr{S}_s)}$$

$$\le \Sigma_{\mathbb{m},\mathbb{n}\in\mathscr{N}_0} \lambda^{-(s+p)(\mathbb{m}+\mathbb{n})} |c_{\mathbb{m}} d_{\mathbb{n}}|\, \|\lambda^{s\mathbb{m}} h_{\mathbb{m}} \lambda^{s\mathbb{n}} h_{\mathbb{n}}\|_{(\mathscr{S}_s)} \lambda^{p(\mathbb{m}+\mathbb{n})}.$$

But if we apply the formula

$$H_m(u) H_n(u) = \Sigma_{k=0}^{m\wedge n}\ 2^k k!\binom{m}{k}\binom{n}{k} H_{m+n-2k}(u),$$

the fact that $\{(\lambda^{-s\mathbb{n}},\ h_{\mathbb{n}});\ \mathbb{n} \in \mathscr{N}_0\}$ is an eigen system of $\Lambda(D^s)$, and

$$\text{the inequality } \binom{m}{k} \le 2^{m}$$

to the norm $\|\lambda^{s\mathbb{m}}h_{\mathbb{m}}\cdot\lambda^{s\mathbb{n}}h_{\mathbb{n}}\|_{(\varphi_s)}$, we have

$$\|\lambda^{s\mathbb{m}}h_{\mathbb{m}}\lambda^{s\mathbb{n}}h_{\mathbb{n}}\|^2_{(\varphi_s)} = \Sigma_{\mathbb{k}\le\mathbb{m}\wedge\mathbb{n}} \binom{\mathbb{m}}{\mathbb{k}}\binom{\mathbb{n}}{\mathbb{k}}\binom{\mathbb{m}+\mathbb{n}-2\mathbb{k}}{\mathbb{n}-\mathbb{k}}\lambda^{4s\mathbb{k}} \le \beta_s^2\ 4^{\mathbb{m}+\mathbb{n}}.$$

After all we obtain

$$\begin{aligned} &\|\varphi\phi\|_{(\varphi_s)} \\ &\le \beta_s\ \Sigma_{\mathbb{m},\mathbb{n}\in\mathcal{N}_0}\lambda^{-(s+p)(\mathbb{m}+\mathbb{n})}|c_{\mathbb{m}}d_{\mathbb{n}}|(2\lambda^p)^{(\mathbb{m}+\mathbb{n})} \\ &= \beta_s\ \|\varphi\|_{(\varphi_{s+p})}\|\phi\|_{(\varphi_{s+p})}\ \Sigma_{\mathbb{n}\in\mathcal{N}_0}(2\lambda^p)^{2\mathbb{n}} \\ &= \beta_s\ \kappa_p\ \|\varphi\|_{(\varphi_{s+p})}\|\phi\|_{(\varphi_{s+p})}. \quad \square \end{aligned}$$

From this proposition we can easily conclude that $(\mathscr{S})$ is an algebra (cf. [L],[Y]).

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