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Jackknife Methods and Higher Order Asymptotic Expansions

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Abstract

For a $\mathbf{R}^d$*-valued "smooth parameter"* $\theta = h(\eta)$*, where* h *is a d-dimensional smooth function of the mean of certain well-defined random variable* $\mathcal{X}$*, the "plug-in" estimator* $\widehat{\theta} = h(\overline{\mathcal{X}})$*, obtained by replacing the population mean* η *by its sample counterpart* $\overline{\mathcal{X}}$*, is typically biased. We consider the jackknife bias-adjusted estimator* $\widehat{\theta}_J$ *and the jackknife-t statistic, a version of* $\widehat{\theta}_J$ *further studentized by its jackknife estimate of standard deviation. Third order asymptotic theories are developed for both* $\widehat{\theta}_J$ *and jackknife-t version. Explicit formulae are given for Edgeworth approximations, with univariate jackknife-t being emphasized. We give explicit formula for third order Cornish-Fisher expansion, based on which we develope the confdence interval estimation for one-dimensional* θ *based on the jackknife-t statistic. Application to the inference of coefficient of variation is considered.*

Contents

1. Introduction

Interest for jackknife, a nonparametric method for estimating bias proposed by Quenouille(1949) and named after Tukey(1958) who showed the possibility for a wider application, has been seen an increase from the late 70's, particularly due to the advent of bootstrap invented by Efron(1979). As an alternative to asymptotic approximations, which are seldom simple enough for a ready application, the computer-intensive resampling methods has been widely accepted as powerful and versatile. More understanding of the rationale behind the simple and ready-for-use resampling methods is needed. Hall(1992) provided a comparatively successful explanation of the bootstrap story from the Edgeworth viewpoint.

Somewhat curiously, asympotic studies of jackknife, as far as the author knows, is quite a lack. Under so-called smooth function model, we, in this paper, give explicit asymptotic results in a multivariate setting for both the jackknife bias-adjusted estimator and the jackknife-t statistic., which are defined in Section 2. Section 3 and 4 present the main asymptotic results for these two estimators, with special concern for one-dimensional jackknife-t, for which we also give explicit third order Cornish-Fisher expansion.

Based on the results of Section 4, we study the inference for one-dimensional parameter by jackknife-t confidence estimation, which is the topic of Section 5. Particular applications in our mind are those relatively complicated inference for "ratio-type" estimators such as the ratio of two means, regression coefficient, correlation coefficient, etc.. The last section, Section 6, discusses the problem of estimating coefficient of variation from a Normal population, more detailed discussion of which can be found in Wang, Taguri and Ouchi(1994). For the complicated example of estimating correlation coefficient, see Wang, Ouchi and Taguri(1994), for a related discussion of asymptotic properties about jackknife, see also Hinkley and Wei(1984), and Nagao(1993).

2. Jackknifing

2.1. Smooth function model

Let $\{Z_p\}_{p\geq 1}$ be a sequence of i.i.d. r-dimensional random vectors, and $g_1, \cdots, g_d$ be real-valued Borel measurable functions on $\mathbf{R}^r$. Define

$$\mathcal{X}_p = (X_p^1, \cdots, X_p^i, \cdots, X_p^d) = (g_1(Z_p), \cdots, g_i(Z_p), \cdots, g_d(Z_p)),$$

and its mean by $\eta = \mathcal{E}[\mathcal{X}_p] = (\eta^1, \cdots, \eta^i, \cdots, \eta^d)$. For parameter of interest $\theta = h(\eta)$, consider statistics

$$\widehat{\theta} = h(\overline{\mathcal{X}}) = (h^1(\overline{\mathcal{X}}), \cdots, h^\alpha(\overline{\mathcal{X}}), \cdots, h^k(\overline{\mathcal{X}})) = (\widehat{\theta^1}, \cdots, \widehat{\theta^\alpha}, \cdots, \widehat{\theta^k}),$$

where h is a $\mathbf{R}^k$-valued smooth Borel measurable function on $\mathbf{R}^d$, and $\overline{\mathcal{X}} = \sum_{p=1}^N \mathcal{X}_p/N$, N being the sample size. Before going further, let's see several examples, each of which is of "ratio type", estimating such parameters is well-known to be difficult due to the possible high fluctuation caused by the denominator.

Example 1 (Coefficient of variation) Let $Z = W$, a univariate random variate. Define $\mathcal{X} = g(Z) = (W, W^2)$, i.e. $X^1 = W, X^2 = W^2$, so $\eta^1 = \mathcal{E}[X^1] = \mu_W, \eta^2 = \mathcal{E}[X^2] = \sigma_W^2 + \mu_W^2$. Coefficient of variation can thus be written as $\theta = \sigma_W/\mu_W = \sqrt{\eta^2 - (\eta^1)^2}/\eta^1 = h(\eta)$, with $r = 1, d = 2$ and $k = 1$.

Example 2 (Ratio of means) Take $Z = (V, W)$, $\mathcal{X} = g(Z) = (V, W)$, i.e. $X^1 = V, X^2 = W$, so $\eta^1 = \mu_V = \mathcal{E}[V], \eta^2 = \mu_W = \mathcal{E}[W]$. Thus the ratio of two means can be expressed as $\theta = \mu_W/\mu_V = \eta^2/\eta^1 = h(\eta)$, with $r = 2, d = 2$ and $k = 1$.

Example 3 (Ratio of variances) Similar to Example 2, let $Z = (V, W)$. Now define $\mathcal{X} = g(Z) = (V, V^2, W, W^2)$, i.e. $X^1 = V, X^2 = V^2, X^3 = W, X^4 = W^2$. So $\theta = \sigma_V^2/\sigma_W^2 = (\eta^2 - (\eta^1)^2)/(\eta^4 - (\eta^3)^2)$, with $r = 2, d = 4$ and $k = 1$.

Example 4 (Regression coefficient) Again let $Z = (V, W)$. Define $\mathcal{X} = g(Z) = (V, W, VW, V^2)$, so regression coefficient can be written as $\theta = Cov[V, W]/Var[V] = (\eta^3 - \eta^1\eta^2)/(\eta^4 - (\eta^1)^2)$, with $r = 2, d = 4$ and $k = 1$.

Example 5 (Correlation coefficient) As our last example, correlation coefficient, which may be one of the most challenging and at the same time practical example, is also fitted into this smooth function model. Here we define $Z = (V, W)$, a $\mathbf{R}^2$-valued random variable, $\mathcal{X} = g(Z) = (V, W, VW, V^2, W^2)$, then $\theta = \rho = (\eta^3 - \eta^1\eta^2)/\sqrt{(\eta^4 - (\eta^1)^2)(\eta^5 - (\eta^2)^2)}$, with $r = 2, d = 5$ and $k = 1$.

In fact, the class of such estimators is large enough to include all smooth functions of population moments.

2.2. Jackknife bias-adjusted estimator

The readily avaliable "plug-in" estimator $\widehat{\theta}$, by simply inserting $\overline{\mathcal{X}}$ in place of its population counterpart η, is usually biased, with order $O(N^{-1})$, N being the sample size. The ratio type estimators in our examples are particularly heavily biased. Knocking out the lower order bias by jackknife method is a simple job.

Omitting the p-th observation, we define the delete-one-sample mean by $\overline{\mathcal{X}}_{(-p)} = (N\overline{\mathcal{X}} - \mathcal{X}_p)/(N-1)$, and the estimator $\widehat{\theta}_{(-p)} = h(\overline{\mathcal{X}}_{(-p)})$ corresponding to it. The jackknife pseudo-values are thus defined as ${}_{pv}\widehat{\theta}_{(-p)} = N\widehat{\theta} - (N-1)\widehat{\theta}_{(-p)}$, based on which the jackknife bias-adjusted estimator is defined to be

$$\widehat{\theta}_J = \frac{1}{N}\sum_{p=1}^{N} {}_{pv}\widehat{\theta}_{(-p)} = N\widehat{\theta} - (N-1)\widehat{\theta}_{(\cdot)}, \tag{2.1}$$

where $\widehat{\theta}_{(\cdot)} = \sum_{p=1}^{N} \widehat{\theta}_{(-p)}/N$. $\widehat{\theta}_J$ can be shown, with no difficulty, to have bias of order $O(N^{-2})$, or smaller.

Higher order bias reduction can be achieved by iterating the jackknife procedure untill the desired order is realized. The iteration may be avoided by one-time jackknife at different levels, see Wang and Taguri(1994). However, the unfortunate nature of bias-reduction usually accompies with it an increase of variance or Mean Squared Error(MSE). Over-reduction of bias, when there exists no unbiased estimator, can result in divergence in MSE, as pointed out by Doss and Sethuraman(1989). For higher order bias reduction, see also Schucany, Gray and Owen(1971), and see Wang(1992), for a numerical procedure to reduce bias.

2.3. Jackknife-t statistic

While it is important to learn the nature of jackknifing by studying the properties of the estimator $\widehat{\theta}_J$((2.1)), it, however, suffers a serious draw back for not being asymptoticly pivotal. It is thus more usual to make statistical inference based on a pivotal version studentizing the original statistic. The way of studentization depends on the particular estimate of standard deviation one chooses. We consider here the jackknife estimate of standard deviation, for one thing, it provides a theoretical challenge, for another, investigation of the nature of jackknife is always one of our basic interests. More formally, the jackknife estimate for variance of

$$\widetilde{\theta}^{\alpha}_J = \sqrt{N}(\widehat{\theta}^{\alpha}_J - \theta^{\alpha}), \tag{2.2}$$

where $\alpha = 1, 2, \cdots, k$, is defined to be

$$\widehat{Var}^{\alpha}_J = (\widehat{Sd}^{\alpha}_J)^2 = \frac{1}{N-1}\sum_{p=1}^{N}({}_{pv}\widehat{\theta}^{\alpha}_{(-p)} - \widehat{\theta}^{\alpha}_J)^2, \tag{2.3}$$

where α also runs from 1 to k. There exist disputes over whether $\widehat{Var}^{\alpha}_J/N$((2.3)) should be considered as an estimator of variance of $\widehat{\theta}^{\alpha}$ or $\widehat{\theta}^{\alpha}_J$, see Hinkley(1978) and Efron(1982). We ignore this subtle issue, and go on to define the jackknife-t statistic by

$$T^{\alpha} = \widetilde{\theta}^{\alpha}_J/\widehat{Sd}^{\alpha}_J. \tag{2.4}$$

Our task in this paper is to construct a third order asymptotic theory for both $\widetilde{\theta}_J = (\widetilde{\theta}^{\alpha}_J)$ and $T = (T^{\alpha})$, based on which we make inference about the unknown θ.

3. Jackknife estimator

3.1. Stochastic expansion

Denote $\kappa^i = Cum[X^i]$, $\kappa^{i,j} = Cum[X^i, X^j]$, $\kappa^{i,j,k} = Cum[X^i, X^j, X^k]$, be the first, second and third cumulant of the related random variables. Denote $\eta^i = E[X^i]$, $\eta^{ij} = E[X^iX^j], \eta^{ijk} = E[X^iX^jX^k]$, be the mean of X^i, X^iX^j, $X^iX^jX^k$, respectively. By definition, $\kappa^i = \eta^i$, which will be used interchangeably wherever we feel convenient. The following two lemmas prove useful to obtain the stochastic expansion of $\widetilde{\theta}_J$.

Lemma 3.1. Define $X_p^{ij} = X_p^i X_p^j$, $\overline{X}^{ij} = \sum_{p=1}^N X_p^{ij}/N$, and $\widetilde{X}^{ij} = \sqrt{N}(\overline{X}^{ij} - \eta^{ij})$, then

$$\frac{1}{N}\sum_{p=1}^{N}(\overline{X}^i - X_p^i)(\overline{X}^j - X_p^j) = \kappa^{i,j} + \frac{1}{\sqrt{N}}(\widetilde{X}^{ij} - \underset{(i,j)}{[2]}\,\kappa^i\widetilde{X}^j) - \frac{1}{N}\widetilde{X}^i\widetilde{X}^j, \tag{3.1}$$

where $\widetilde{X}^i = \sqrt{N}(\overline{X}^i - \eta^i)$, and $\underset{(i,j)}{[2]}$ denotes symmetrization with respect to indices i, j, e.g. $\underset{(i,j)}{[2]}\,\eta^i\widetilde{X}^j = \eta^i\widetilde{X}^j + \eta^j\widetilde{X}^i$.

Define

$$\begin{aligned}
h_i^\alpha &= \frac{\partial}{\partial\overline{X}^i}h^\alpha(\overline{X})\mid_{\overline{X}=\eta},\\
h_{ij}^\alpha &= \frac{\partial^2}{\partial\overline{X}^i\partial\overline{X}^j}h^\alpha(\overline{X})\mid_{\overline{X}=\eta},\\
h_{ijk}^\alpha &= \frac{\partial^3}{\partial\overline{X}^i\partial\overline{X}^j\partial\overline{X}^k}h^\alpha(\overline{X})\mid_{\overline{X}=\eta}.
\end{aligned}$$

Applying Lemma 3.1., we have the following expansion for $\widetilde{\theta}_J$.

Proposition 3.1. The third order stochastic expansion of jackknife bias-adjusted estimator assumes the following form.

$$\begin{aligned}
\widetilde{\theta_J^\alpha} =& h_i^\alpha\widetilde{X^i} + \frac{1}{2\sqrt{N}}h_{ij}^\alpha(\widetilde{X^i}\widetilde{X^j} - \kappa^{i,j})\\
&+ \frac{1}{N}\left\{\frac{1}{6}h_{ijk}^\alpha\widetilde{X^i}\widetilde{X^j}\widetilde{X^k} + \frac{1}{2}h_{ij}^\alpha(2\eta^i\widetilde{X^j} - \widetilde{X^{ij}}) - \frac{1}{2}h_{ijk}^\alpha\kappa^{i,j}\widetilde{X^k}\right\}\\
&+ O_p(N^{-3/2}),
\end{aligned}\tag{3.2}$$

where Einstein summation convension is assumed, i.e., when the same index appears both as subscripts and superscripts, summation is automatically taken for that index, e.g.

$$h_{ij}^\alpha(\widetilde{X^i}\widetilde{X^j} - \kappa^{i,j}) = \sum_{i=1}^{d}\sum_{j=1}^{d}h_{ij}^\alpha(\widetilde{X^i}\widetilde{X^j} - \kappa^{i,j}).$$

As a comparision, the stabalized version of the usual bias-adjusted estimator

$$\widehat{\theta}_*^\alpha = \widehat{\theta}^\alpha - \frac{1}{2N}h_{ij}^\alpha(\overline{X})\kappa^{i,j}(\overline{X}) \tag{3.3}$$

admits expansion

$$
\begin{aligned}
\widetilde{\theta^\alpha_*} =& \sqrt{N}(\widehat{\theta}^\alpha_* - \theta^\alpha) \\
=& h^\alpha_i \widetilde{X}^i + \frac{1}{2\sqrt{N}} h^\alpha_{ij}(\widetilde{X}^i\widetilde{X}^j - \kappa^{ij}) \\
&+ \frac{1}{N}\left\{\frac{1}{6}h^\alpha_{ijk}\widetilde{X}^i\widetilde{X}^j\widetilde{X}^k - \frac{1}{2}h^\alpha_{ijk}\kappa^{i,j}\widetilde{X}^k - \frac{1}{2}h^\alpha_{ij}\partial_k\kappa^{i,j}\widetilde{X}^k\right\} \\
&+ O_p(N^{-3/2}),
\end{aligned}
\tag{3.4}
$$

where

$$
\partial_k\kappa^{i,j} = \frac{\partial}{\partial \overline{X}^k}\kappa^{i,j}(\overline{X}) \mid_{\overline{X}=\eta} .
$$

(3.2) and (3.4) tell us that the jackknife bias-adjusted estimator and the usual bias-adjusted estimator enjoy the same stochastic expansion up to order $O_p(N^{-1/2})$, from which we immediately have

Theorem 3.1. Edgeworth expansions of both the probability density functions and the cumulative distribution functions of $\widetilde{\theta}_J$ and $\widetilde{\theta}_*$ agree up to order $O(1/\sqrt{N})$.

3.2. Asymptotic cumulants

As a preparation for finding asymptotic moments and cumulants of $\widetilde{\theta}_J$, we first state the following two lemmas.

Lemma 3.2. Define $\kappa^{i,jk} = Cum\{X^i,\, X^{jk}\}$ to be the second cumulant of X^i and X^{jk}, then

$$
(a)\quad \kappa^{i,jk} = \kappa^{i,j,k} + \sum_{(j,k)}^{[2]} \kappa^{i,k}\kappa^j,
$$

$$
(b)\quad \mathcal{E}[\widetilde{X}^i\widetilde{X}^j\widetilde{X}^k\widetilde{X}^{lm}] = \sum_{(i,j,k)}^{[3]} \kappa^{i,j}\kappa^{k,lm} + O(N^{-1}).
$$

Lemma 3.3. If (3.2) is rewritten as

$$
\widetilde{\theta}^\alpha_J = A^\alpha_0 + \frac{1}{\sqrt{N}}A^\alpha_1 + \frac{1}{N}A^\alpha_2 + O_p(N^{-3/2}),
$$

then

$$
\mathcal{E}[A^\alpha_0(A^\beta_1 + \frac{1}{\sqrt{N}}A^\beta_2)] = 0,
$$

$$
Cum\{\widetilde{\theta}^\alpha_J,\, \widetilde{\theta}^\beta_J\} = Cov[A^\alpha_0,\, A^\beta_0] + \frac{1}{N}Cov[A^\alpha_1,\, A^\beta_1].
$$

Define

$$
\begin{aligned}
\kappa_A^{\alpha,\beta} &\stackrel{\text{def}}{=} h^\alpha_i h^\beta_j \kappa^{i,j,k}, \\
\kappa_A^{\alpha,\beta,\gamma} &\stackrel{\text{def}}{=} h^\alpha_i h^\beta_j h^\gamma_k \kappa^{i,j,k}, \\
\kappa_A^{\alpha,\beta,\gamma,\delta} &\stackrel{\text{def}}{=} h^\alpha_i h^\beta_j h^\gamma_k h^\delta_l \kappa^{i,j,k,l}.
\end{aligned}
$$

After some algebraic manipulations, we have the first four moments as

$$\left.\begin{aligned}
\mathcal{E}[\widetilde{\theta}^{\alpha}_{J}] &= O(N^{-3/2}),\\
\mathcal{E}[\widetilde{\theta}^{\alpha}_{J}\widetilde{\theta}^{\beta}_{J}] &= \kappa^{\alpha,\beta}_{A} + \frac{1}{2N}h^{\alpha}_{ij}h^{\beta}_{kl}\kappa^{i,k}\kappa^{j,l} + O(N^{-2}),\\
\mathcal{E}[\widetilde{\theta}^{\alpha}_{J}\widetilde{\theta}^{\beta}_{J}\widetilde{\theta}^{\gamma}_{J}] &= \frac{1}{\sqrt{N}}\left(\kappa^{\alpha,\beta,\gamma}_{A} + [3]h^{(\alpha}_{i}h^{\beta}_{j}h^{\gamma)}_{kl}\kappa^{i,k}\kappa^{j,l}\right) + O(N^{-3/2}),\\
\mathcal{E}[\widetilde{\theta}^{\alpha}_{J}\widetilde{\theta}^{\beta}_{J}\widetilde{\theta}^{\gamma}_{J}\widetilde{\theta}^{\delta}_{J}] &= \underset{(\alpha,\beta,\gamma,\delta)}{[3]}\kappa^{\alpha,\beta}_{A}\kappa^{\gamma,\delta}_{A} + \frac{1}{N}\Big\{\kappa^{\alpha,\beta,\gamma,\delta}_{A}\\
&\quad + \frac{\underset{(\alpha,\beta,\gamma,\delta)}{[4]}}{2}h^{\alpha}_{i}h^{\beta}_{j}h^{\gamma}_{k}h^{\delta}_{lm}(6\kappa^{i,l}\kappa^{j,k,m} + \kappa^{l,m}\kappa^{i,j,k})\\
&\quad + \underset{(\alpha,\beta,\gamma,\delta)}{[4]}h^{(\alpha}_{i}h^{\beta}_{j}h^{\gamma}_{k}h^{\delta)}_{lmn}\kappa^{i,l}\kappa^{j,m}\kappa^{k,n}\\
&\quad + \frac{\underset{(\alpha,\beta,\gamma,\delta)}{[6]}}{2}\kappa^{(\alpha,\beta}h^{\gamma}_{kl}h^{\delta)}_{mn}\kappa^{k,m}\kappa^{l,n}\Big\} + O(N^{-2}).
\end{aligned}\right\} \tag{3.5}$$

Calculating the second moment of $\widetilde{\theta}_{*}$ from (3.4) and comparing it with (3.5), we have

Proposition 3.2. In one-dimensional case, the deficiency of $\widehat{\theta}_{*}$ relative to $\widehat{\theta}_{J}$ is given by

$$2[h_i h_{jk}\kappa^{i,j,k} - h_i h_{jk}(\partial_l\kappa^{j,k})\kappa^{i,l}]/\kappa_{A,2},$$

where $\kappa_{A,2} = h_i h_j \kappa^{i,j}$.

Proposition 3.3. Let

$$\begin{aligned}
\kappa^{\alpha}_{N} &\overset{\text{def}}{=} Cum\{\widetilde{\theta}^{\alpha}_{J}\},\\
\kappa^{\alpha,\beta}_{N} &\overset{\text{def}}{=} Cum\{\widetilde{\theta}^{\alpha}_{J},\, \widetilde{\theta}^{\beta}_{J}\},\\
\kappa^{\alpha,\beta,\gamma}_{N} &\overset{\text{def}}{=} Cum\{\widetilde{\theta}^{\alpha}_{J},\, \widetilde{\theta}^{\beta}_{J},\, \widetilde{\theta}^{\gamma}_{J}\},\\
\kappa^{\alpha,\beta,\gamma,\delta}_{N} &\overset{\text{def}}{=} Cum\{\widetilde{\theta}^{\alpha}_{J},\, \widetilde{\theta}^{\beta}_{J},\, \widetilde{\theta}^{\gamma}_{J},\, \widetilde{\theta}^{\delta}_{J}\},
\end{aligned}$$

be the first, second, third and fourth cumulant of $\widetilde{\theta}_{J}$, then we have from (3.5)

$$\left.\begin{aligned}
\kappa^{\alpha}_{N} &= O(N^{-3/2}),\\
\kappa^{\alpha,\beta}_{N} &= \kappa^{\alpha,\beta}_{A} + \frac{1}{2N}h^{\alpha}_{ij}h^{\beta}_{kl}\kappa^{i,k}\kappa^{j,l} + O(N^{-2}),\\
\kappa^{\alpha,\beta,\gamma}_{N} &= \frac{1}{\sqrt{N}}\left(\kappa^{\alpha,\beta,\gamma}_{A} + \underset{(\alpha,\beta,\gamma)}{[3]}h^{\alpha}_{i}h^{\beta}_{j}h^{\gamma}_{kl}\kappa^{i,k}\kappa^{j,l}\right) + O(N^{-3/2}),\\
\kappa^{\alpha,\beta,\gamma,\delta}_{N} &= \frac{1}{N}\Big\{\kappa^{\alpha,\beta,\gamma,\delta}_{A} + \frac{\underset{(\alpha,\beta,\gamma,\delta)}{[4]}}{2}h^{\alpha}_{i}h^{\beta}_{j}h^{\gamma}_{k}h^{\delta}_{lm}(6\kappa^{i,l}\kappa^{j,k,m} + \kappa^{l,m}\kappa^{i,j,k})\\
&\quad + \underset{(\alpha,\beta,\gamma,\delta)}{[4]}h^{\alpha}_{i}h^{\beta}_{j}h^{\gamma}_{k}h^{\delta}_{lmn}\kappa^{i,l}\kappa^{j,m}\kappa^{k,n}\Big\} + O(N^{-2}).
\end{aligned}\right\} \tag{3.6}$$

3.3. Edgeworth expansions

Notation 3.1. Define differential operator, D^α, with respect to variance covariance matrix $\kappa_A^{\alpha,\beta}$ as

$$
\begin{aligned}
D^\alpha &= \kappa_A^{\alpha,\beta} \frac{\partial}{\partial y^\beta}, \\
(-D)^{\alpha_1 \cdots \alpha_r} &= (-1)^r D_1^\alpha \cdots D_r^\alpha.
\end{aligned}
$$

Notation 3.2. Let $\phi(y; \kappa_A^{\alpha,\beta})$ be the Normal density with mean 0 and variance-covariance matrix $\kappa_A^{\alpha,\beta}$:

$$\phi(y; \kappa_A^{\alpha,\beta}) = \| \kappa_A^{\alpha,\beta} \|^{-\frac{1}{2}} (2\pi)^{-\frac{k}{2}} \exp[-\frac{1}{2}\kappa_{\alpha,\beta} y^\alpha y^\beta],$$

where $\kappa_{\alpha,\beta}$ denotes the $(\alpha,\ \beta)$th element of the inverse of $\kappa_A^{\alpha\beta}$. Define Hermite polynomial of order r w.r.t. metric tensor $\kappa_A^{\alpha,\beta}$ by

$$(-D)^{\alpha_1 \cdots \alpha_r} \phi(y; \kappa_A^{\alpha,\beta}) = H^{\alpha_1 \cdots \alpha_r}(y) \phi(y; \kappa_A^{\alpha,\beta}).$$

Notation 3.3. Rewrite the cumulants of $\tilde{\theta}_J((3.6))$ as

$$\kappa_N^{\alpha_1,\cdots,\alpha_r} = N^{-\frac{r-2}{2}}(C_0^{\alpha_1,\cdots,\alpha_r} + \frac{1}{N} C_1^{\alpha_1,\cdots,\alpha_r} + \cdots) \quad (r \geq 1). \tag{3.7}$$

Define the covariant form of $C_j^{\alpha_1,\cdots,\alpha_r}$ by

$$C_{j;\alpha_1,\cdots,\alpha_r} = C_j^{\beta_1,\cdots,\beta_r} \kappa_{\alpha_1\beta_1} \cdots \kappa_{\alpha_r\beta_r}. \tag{3.8}$$

Theorem 3.2. Under certain regularity conditions, the prabability density function of $\tilde{\theta}_J$ can be expanded as

$$f(y) = \phi(y; (\kappa_A^{\alpha\beta}))\left(1 + \frac{1}{\sqrt{N}} q_1(y) + \frac{1}{N} q_2(y)\right) + O(N^{-3/2}), \tag{3.9}$$

where

$$
\left.
\begin{aligned}
q_1(y) &= \frac{1}{3!} C_{0;\alpha\beta\gamma} H^{\alpha\beta\gamma}(y), \\
q_2(y) &= \frac{1}{2} C_{1;\alpha\beta} H^{\alpha\beta}(y) + \frac{1}{4!} C_{0;\alpha\beta\gamma\delta} H^{\alpha\beta\gamma\delta}(y) \\
&\quad + \frac{\overset{[10]}{(\alpha,\beta,\gamma,\delta,\epsilon,\lambda)}}{6!} C_{0;\alpha\beta\gamma} C_{0;\delta\epsilon\lambda} H^{\alpha\beta\gamma\delta\epsilon\lambda}(y),
\end{aligned}
\right\} \tag{3.10}
$$

and $C_{0;\alpha\beta\gamma}$, $C_{1;\alpha\beta}$, $C_{0;\alpha\beta\gamma\delta}$ are defined by (3.8), which in turn are defined by (3.7).

Theorem 3.3. Again under certain regularity conditions, the cumulative distribution function of $\widetilde{\theta}_J$, $F(y)$, admits expansion

$$F(y) = \Phi(y; \kappa_A^{\alpha,\beta}) + \frac{1}{\sqrt{N}} Q_1(y) + \frac{1}{N} Q_2(y) + O(N^{-3/2}), \tag{3.11}$$

where

$$\left.\begin{aligned} Q_1(y) =& \frac{1}{3!} C_{0;\alpha\beta\gamma}(-D)^{\alpha\beta\gamma}\Phi(y;\kappa_A^{\alpha,\beta}), \\ Q_2(y) =& \Big[\frac{1}{2} C_{1;\alpha\beta}(-D)^{\alpha\beta} + \frac{1}{4!} C_{0;\alpha\beta\gamma\delta}(-D)^{\alpha\beta\gamma\delta} \\ &+ \frac{\overset{[10]}{(\alpha,\beta,\gamma,\delta,\epsilon,\lambda)}}{6!} C_{0;\alpha\beta\gamma} C_{0;\delta\epsilon\lambda}(-D)^{\alpha\beta\gamma\delta\epsilon\lambda}\Big]\Phi(y;\kappa_A^{\alpha,\beta}). \end{aligned}\right\} \tag{3.12}$$

Throughout our discussion, we have purposely avoided the regularily conditions for valid Edgeworth expansions, which are typically the requirements of the existence of finiteness of moments up to a sufficient order and smoothness condition of the underlying distribution, see, for instance, Bhattacharya and Rao(1976), Bhattacharya and Ghosh(1978), and Hall(1992).

4. Jackknife-t Statistic

4.1. Stochastic expansion

To aid our stochastic expansion, besides Lemma 3.1., we also need the following.

Lemma 4.1. Let $X_p^{ijk} = X_p^i X_p^j X_p^k$, $\overline{X^{ijk}} = \sum_{p=1}^N X_p^{ijk}/N$, $\widetilde{X}^{ijk} = \sqrt{N}(\overline{X^{ijk}} - \eta^{ijk})$, then

$$\begin{aligned} &\frac{1}{N}\sum_{p=1}^{N}(\overline{X}^i - X_p^i)(\overline{X}^j - X_p^j)(\overline{X}^k - X_p^k) \\ =& -\kappa^{i,j,k} + \frac{1}{\sqrt{N}} \overset{[3]}{\underset{(i,j,k)}{}} (\eta^{ij} - 2\eta^i\eta^j)\widetilde{X}^k \\ &+ \frac{1}{\sqrt{N}} \overset{[3]}{\underset{(i,j,k)}{}} \eta^i \widetilde{X}^{jk} - \frac{1}{\sqrt{N}}\widetilde{X}^{ijk} + O_p(\frac{1}{N}). \end{aligned} \tag{4.1}$$

Both Lemma 3.1. and 4.1. are simply refinements of the law of large numbers. We again omit the relatively simple proof.

Proposition 4.1. Third order stochastic expansion of studentized jackknife random variable assumes the following form.

$$
\begin{aligned}
T^\alpha =& \frac{1}{\sigma^\alpha} h_i^\alpha \widetilde{X}^i \\
&+ \frac{1}{\sqrt{N}} \Big\{ \frac{1}{2(\sigma^\alpha)^2} h_{ij}^\alpha (\widetilde{X}^i \widetilde{X}^j - \kappa^{i,j}) - \frac{1}{(\sigma^\alpha)^3} h_i^\alpha h_{jk}^\alpha h_l^\alpha \kappa^{k,l} \widetilde{X}^i \widetilde{X}^j \\
&- \frac{1}{2(\sigma^\alpha)^3} h_i^\alpha h_j^\alpha h_k^\alpha (\widetilde{X}^i \widetilde{X}^{jk} - 2\eta^j \widetilde{X}^i \widetilde{X}^k) \Big\} \\
&+ \frac{1}{N} \Big\{ - \frac{1}{2(\sigma^\alpha)^3} h_{ij}^\alpha h_{kl}^\alpha h_m^\alpha \kappa^{l,m} (\widetilde{X}^i \widetilde{X}^j \widetilde{X}^k - \kappa^{i,j} \widetilde{X}^k) \\
&- \frac{1}{4(\sigma^\alpha)^3} h_{ij}^\alpha h_k^\alpha h_l^\alpha (\widetilde{X}^i \widetilde{X}^j - \kappa^{i,j})(\widetilde{X}^{kl} - 2\eta^k \widetilde{X}^l) \\
&+ \frac{1}{6\sigma} h_{ijk}^\alpha \widetilde{X}^i \widetilde{X}^j \widetilde{X}^k + \frac{1}{2\sigma} h_{ij}^\alpha (2\eta^i \widetilde{X}^j - \widetilde{X}^{ij}) \\
&- \frac{1}{2\sigma} h_{ijk}^\alpha \kappa^{i,j} \widetilde{X}^k + \frac{3}{2(\sigma^\alpha)^5} h_i^\alpha h_{jl}^\alpha h_{km}^\alpha h_n^\alpha h_n^\alpha \kappa^{l,n} \kappa^{m,o} \widetilde{X}^i \widetilde{X}^j \widetilde{X}^k \\
&+ \frac{3}{2(\sigma^\alpha)^5} h_i^\alpha h_{jm}^\alpha h_k^\alpha h_l^\alpha h_n^\alpha \kappa^{m,n} (\widetilde{X}^i \widetilde{X}^j \widetilde{X}^{kl} - 2\eta^k \widetilde{X}^i \widetilde{X}^j \widetilde{X}^l) \\
&+ \frac{3}{8(\sigma^\alpha)^5} h_i^\alpha h_j^\alpha h_k^\alpha h_l^\alpha h_m^\alpha (\widetilde{X}^i \widetilde{X}^{jk} \widetilde{X}^{lm} - 4\widetilde{X}^i \widetilde{X}^{jk} \widetilde{X}^l \eta^m + 4\eta^i \eta^j \widetilde{X}^k \widetilde{X}^l \widetilde{X}^m) \\
&- \frac{1}{2(\sigma^\alpha)^3} h_i^\alpha h_{jkl}^\alpha h_m^\alpha \kappa^{l,m} \widetilde{X}^i \widetilde{X}^j \widetilde{X}^k - \frac{1}{2(\sigma^\alpha)^3} h_i^\alpha h_{jl}^\alpha h_{km}^\alpha \kappa^{l,m} \widetilde{X}^i \widetilde{X}^j \widetilde{X}^k \\
&+ \frac{1}{2(\sigma^\alpha)^3} h_i^\alpha h_j^\alpha h_k^\alpha (\widetilde{X}^i \widetilde{X}^j \widetilde{X}^k - \kappa^{i,j} \widetilde{X}^k) \\
&- \frac{1}{(\sigma^\alpha)^3} h_i^\alpha h_j^\alpha h_{kl}^\alpha (\widetilde{X}^i \widetilde{X}^{jk} \widetilde{X}^l - [2]\eta^{(j} \widetilde{X}^{k)} \widetilde{X}^i \widetilde{X}^l) \\
&+ \frac{1}{2(\sigma^\alpha)^3} h_i^\alpha h_j^\alpha h_{kl}^\alpha \kappa^{j,k,l} \widetilde{X}^i \Big\} + O_p(N^{-3/2}).
\end{aligned}
\tag{4.2}
$$

4.2. Asymptotic cumulants

The following lemmas help to evaluate the asymptotic cumulants required for the Edgeworth expansions.

Lemma 4.2. Define $\kappa^{i,j,kl} = Cum\{X^i, X^j, X^{kl}\}$, $\kappa^{kl,mn} = Cum\{X^{kl}, X^{mn}\}$, then

$$
\begin{aligned}
\kappa^{i,j,kl} &= \kappa^{i,j,k,l} + \underset{(k,l)}{[2]} \kappa^k \kappa^{l,i,j} + \underset{(i,j)}{[2]} \kappa^{k,i} \kappa^{j,l}, \\
\kappa^{kl,mn} &= \kappa^{k,l,m,n} + \underset{(k,l,m,n)}{[4]} \kappa^k \kappa^{l,m,n} + \underset{(k,l,m,n)}{[6]} \kappa^k \kappa^l \kappa^{m,n} \\
&\quad + \underset{(m,n)}{[2]} \kappa^{k,m} \kappa^{n,l} - \kappa^{k,l} \kappa^m \kappa^n - \kappa^k \kappa^l \kappa^{m,n}.
\end{aligned}
$$

Lemma 4.3. Moments of the following types are typical in dealing with jackknife-t statistics.

$$
\begin{aligned}
\mathcal{E}[\widetilde{X}^i\widetilde{X}^j\widetilde{X}^k\widetilde{X}^{lm}] &= [3]\kappa^{(i,j}\kappa^{k,),l,m} + [3]\kappa^{(i,j}\kappa^{k),(l}\kappa^{m)}[2] + O(N^{-1}),\\
\mathcal{E}[\widetilde{X}^i\widetilde{X}^j\widetilde{X}^{kl}\widetilde{X}^{mn}] &= \kappa^{i,j}\kappa^{k,l,m,n} + [4]\kappa^{i,j}\kappa^{(k}\kappa^{l,m,n)} + [6]\kappa^{i,j}\kappa^{(k}\kappa^{l}\kappa^{m,n)}\\
&\quad + [2]\kappa^{i,j}\kappa^{k,(m}\kappa^{n),l} - \kappa^{i,j}\kappa^{k,l}\kappa^{m}\kappa^{n} - \kappa^{i,j}\kappa^{k}\kappa^{l}\kappa^{m,n}\\
&\quad + (\kappa^{i,k,l} + [2]\kappa^{i,(k}\kappa^{l)})(\kappa^{j,m,n} + [2]\kappa^{j,(m}\kappa^{n)})\\
&\quad + (\kappa^{i,m,n} + [2]\kappa^{i,(m}\kappa^{n)})(\kappa^{j,k,l} + [2]\kappa^{j,(k}\kappa^{l)}) + O(N^{-1}),\\
\mathcal{E}[\widetilde{X}^i\widetilde{X}^j\widetilde{X}^k\widetilde{X}^l\widetilde{X}^{mn}] &= \frac{1}{\sqrt{N}}[6]\kappa^{(i,j}(\kappa^{k,l),m,n} + 2\kappa^{k,l),m}\kappa^{n} + [2]\kappa^{m,(k}\kappa^{l)),n})\\
&\quad + \frac{1}{\sqrt{N}}[4]\kappa^{(i,j,k}(\kappa^{l),m,n} + 2\kappa^{l),m}\kappa^{n}) + O(N^{-3/2}),\\
\mathcal{E}[\widetilde{X}^i\widetilde{X}^j\widetilde{X}^k\widetilde{X}^l\widetilde{X}^m\widetilde{X}^{no}] &= [15]\kappa^{(i,j}\kappa^{k,l}(\kappa^{m),n,o} + 2\kappa^{m),n}\kappa^{o}) + O(N^{-1}),\\
\mathcal{E}[\widetilde{X}^i\widetilde{X}^j\widetilde{X}^k\widetilde{X}^l\widetilde{X}^{mn}\widetilde{X}^{op}] &= [3]\kappa^{(i,j}\kappa^{k,l)}\Big(\kappa^{m,n,o,p} + [4]\kappa^{(m}\kappa^{n,o,p)} + [6]\kappa^{(m}\kappa^{n}\kappa^{o,p)}\\
&\quad + [2]\kappa^{m,(o}\kappa^{p),n} - \kappa^{m,n}\kappa^{o}\kappa^{p} - \kappa^{m}\kappa^{n}\kappa^{o,p}\Big)\\
&\quad + [12]\kappa^{(i,j}(\kappa^{k,m,n} + 2\kappa^{k,m}\kappa^{n})(\kappa^{l,o,p} + 2\kappa^{l,o}\kappa^{p)})\\
&\quad + O(N^{-1}).
\end{aligned}
$$

While the second and fourth cumulants require considerable efforts to be obtained, the first and the third are relatively easy to be computed.

Proposition 4.2. The first and third cumulants are given by

$$
\left.
\begin{aligned}
\kappa_T^{\alpha} &= -\frac{1}{\sqrt{N}}\frac{1}{(\sigma^\alpha)^3}(h_i^\alpha h_{jk}^\alpha h_l^\alpha \kappa^{k,l}\kappa^{i,j} + \frac{1}{2}\kappa_A^{\alpha,\alpha,\alpha}) + O(N^{-3/2}),\\
\kappa_T^{\alpha,\beta,\gamma} &= \frac{1}{\sqrt{N}}\Big\{\frac{1}{\sigma^\alpha\sigma^\beta\sigma^\gamma}\kappa_A^{\alpha,\beta,\gamma} + \frac{\overset{[3]}{(\alpha,\beta,\gamma)}}{\sigma^\alpha\sigma^\beta}h_i^\alpha h_j^\beta\Big[\frac{1}{(\sigma^\gamma)^2}h_{kl}^\gamma\kappa^{i,k}\kappa^{j,l}\\
&\quad - \frac{1}{(\sigma^\gamma)^3}h_k^\gamma h_{lm}^\gamma h_n^\gamma\kappa^{m,n}(\kappa^{i,k}\kappa^{j,l} + \kappa^{i,l}\kappa^{j,k}\\
&\quad - \frac{1}{2(\sigma^\gamma)^3}h_k^\gamma h_l^\gamma h_m^\gamma(\kappa^{i,k}\kappa^{j,l,m} + \kappa^{j,k}\kappa^{i,l,m})\Big]\Big\} + O(N^{-3/2}).
\end{aligned}
\right\}
\qquad (4.3)
$$

Proposition 4.3. The second cumulant of T is given by

$$
\begin{aligned}
\kappa_T^{\alpha,\beta} &= \frac{1}{\sigma^\alpha\sigma^\beta}\kappa_A^{\alpha,\beta} \\
&+ \frac{1}{N}\mathop{}\limits_{(\alpha,\beta)}^{[2]}\Big\{\frac{1}{2\sigma^\alpha(\sigma^\beta)^2}h_i^\alpha h_{jk}^\beta\kappa^{i,j,k} - \frac{1}{2\sigma^\alpha(\sigma^\beta)^3}\kappa_A^{\alpha,\beta,\beta,\beta} \\
&- \frac{1}{\sigma^\alpha(\sigma^\beta)^3}h_i^\alpha h_j^\beta h_k^\beta h_{lm}^\beta(2\kappa^{k,m}\kappa^{i,j,l} + \kappa^{i,m}\kappa^{j,k,l} + \frac{1}{2}\kappa^{i,j}\kappa^{k,l,m}) \\
&- \frac{1}{2\sigma^\alpha(\sigma^\beta)^3}h_i^\alpha h_{jk}^\beta h_{lm}^\beta h_n^\beta(3\kappa^{i,j}\kappa^{k,l}\kappa^{m,n} + \kappa^{k,m}\kappa^{i,n}\kappa^{j,l}) \\
&- \frac{1}{2\sigma^\alpha(\sigma^\beta)^3}h_i^\alpha h_{jk}^\beta h_l^\beta h_m^\beta\kappa^{i,j}\kappa^{k,l,m} - \frac{1}{2\sigma^\alpha\sigma^\beta}h_i^\alpha h_{jk}^\beta\kappa^{i,j,k} \\
&+ \frac{3}{2\sigma^\alpha(\sigma^\beta)^5}h_i^\alpha h_j^\beta h_{km}^\beta h_{ln}^\beta h_o^\beta h_p^\beta\kappa^{m,o}\kappa^{n,p}[3]\kappa^{(i,j}\kappa^{k,l)} \\
&+ \frac{3}{2\sigma^\alpha(\sigma^\beta)^5}h_i^\alpha h_j^\beta h_{kn}^\beta h_l^\beta h_m^\beta h_o^\beta\kappa^{n,o}[3]\kappa^{(i,j}\kappa^{k),l,m} \\
&+ \frac{3}{8\sigma^\alpha(\sigma^\beta)^5}\kappa_A^{\alpha,\beta}\kappa_A^{\beta,\beta,\beta,\beta} + \frac{3}{4\sigma^\alpha\sigma^\beta}\kappa_A^{\alpha,\beta} + \frac{3}{4\sigma^\alpha(\sigma^\beta)^5}\kappa_A^{\alpha,\beta,\beta}\kappa_A^{\beta,\beta,\beta} \qquad (4.4) \\
&- \frac{1}{2\sigma^\alpha(\sigma^\beta)^3}h_i^\alpha h_j^\beta h_{k,l,m}^\beta h_n^\beta\kappa^{m,n}[3]\kappa^{(i,j}\kappa^{k,l)}\Big\} \\
&+ \frac{1}{N}\Big\{\frac{1}{2(\sigma^\alpha\sigma^\beta)^2}h_{ij}^\alpha h_{i'j'}^\beta\kappa^{i,i'}\kappa^{j,j'} - \frac{\substack{[2]\\(\alpha\beta)}}{2(\sigma^\alpha)^2(\sigma^\beta)^3}h_{ij}^\alpha h_{i'}^\beta h_{j'}^\beta h_{k'}^\beta\kappa^{i,i'}\kappa^{j,j',k'} \\
&- \frac{\substack{[2]\\(\alpha\beta)}}{(\sigma^\alpha)^2(\sigma^\beta)^3}h_{ij}^\alpha h_{i'}^\beta h_{j'k'}^\beta h_{l'}^\beta\kappa^{i,i'}\kappa^{j,j'}\kappa^{k',l'} \\
&+ \frac{1}{(\sigma^\alpha\sigma^\beta)^3}h_i^\alpha h_{jk}^\alpha h_l^\alpha h_{i'}^\beta h_{j'k'}^\beta h_{l'}^\beta\kappa^{k,l}\kappa^{k',l'}(\kappa^{i,i'}\kappa^{j,j'} + \kappa^{j,i'}\kappa^{i,j'}) \\
&+ \frac{\substack{[2]\\(\alpha\beta)}}{2(\sigma^\alpha\sigma^\beta)^3}h_i^\alpha h_{jk}^\alpha h_l^\alpha h_{i'}^\beta h_{j'}^\beta h_{k'}^\beta\kappa^{k,l}(\kappa^{i,i'}\kappa^{j,j',k'} + \kappa^{j,i'}\kappa^{i,j',k'}) \\
&+ \frac{1}{4(\sigma^\alpha\sigma^\beta)^3}(\kappa_A^{\alpha,\beta}\kappa_A^{\alpha,\alpha,\beta,\beta} + 2(\kappa_A^{\alpha,\beta})^3 + \kappa_A^{\alpha,\alpha,\alpha}\kappa_A^{\beta,\beta,\beta})\Big\} + O(N^{-2}).
\end{aligned}
$$

To save space, we refrain from writing explicitly the fourth cumulant, just note that

$$\kappa_N^{\alpha,\beta,\gamma,\delta} = \frac{1}{N}C_0^{\alpha\beta\gamma\delta} + O(N^{-2}). \tag{4.5}$$

4.3. Edgeworth expansions

Continuing to use notation (3.7), but with the coefficients $C_i^{\alpha_1\cdots\alpha_r}$ defined by (4.3), (4.4) and (4.5), for approximating both the c.d.f. and density function of T, we have

Theorem 4.1. Denote $g^{\alpha\beta} = (\kappa_A^{\alpha\beta}/(\sigma^\alpha\sigma^\beta))$. Under certain regularity conditions, the probability density function $f(t)$ of T can be expanded as

$$f(t) = \phi(y; (g^{\alpha\beta}))\Big(1 + \frac{1}{\sqrt{N}}q_1(t) + \frac{1}{N}q_2(t)\Big) + O(N^{-3/2}), \tag{4.6}$$

where

$$\left.\begin{aligned} q_1(t) =& C_{1;\alpha}H^{\alpha}(t) + \frac{1}{3!}C_{0;\alpha\beta\gamma}H^{\alpha\beta\gamma}(t),\\ q_2(t) =& \frac{1}{2}(C_{1;\alpha}C_{1;\beta} + C_{1;\alpha\beta})H^{\alpha\beta}(t) + \frac{1}{4!}(C_{0;\alpha\beta\gamma\delta} + \sum_{(\alpha,\beta,\gamma,\delta)}^{[4]} C_{1;\alpha}C_{0;\beta\gamma\delta})H^{\alpha\beta\gamma\delta}(t)\\ &+ \frac{1}{6!}\sum_{(\alpha,\beta,\gamma,\delta,\epsilon,\lambda)}^{[10]} C_{0;\alpha\beta\gamma}C_{0;\delta\epsilon\lambda}H^{\alpha\beta\gamma\delta\epsilon\lambda}(t), \end{aligned}\right\} \tag{4.7}$$

with the coefficients $C_{i;\alpha_1\cdots\alpha_r}$ being defined by (4.3), (4.4) and (4.5).

Theorem 4.2. Under certain regularity conditions, the cumulative distribution function of T,$F(t)$, admits expansion

$$F(t) = \Phi(t; g^{\alpha,\beta}) + \frac{1}{\sqrt{N}}Q_1(t) + \frac{1}{N}Q_2(t) + O(N^{-3/2}), \tag{4.8}$$

where

$$\left.\begin{aligned} Q_1(t) =& (C_{1;\alpha}(-D)^{\alpha} + \frac{1}{3!}C_{0;\alpha\beta\gamma}(-D)^{\alpha\beta\gamma}\Phi(t; g^{\alpha,\beta}),\\ Q_2(t) =& \Big[\frac{1}{2}(C_{1;\alpha\beta} + C_{1;\alpha}C_{1;\beta})(-D)^{\alpha\beta} + \frac{1}{4!}(C_{0;\alpha\beta\gamma\delta} + \sum_{(\alpha,\beta,\gamma,\delta)}^{[4]} C_{1;\alpha}C_{0;\beta,\gamma,\delta})(-D)^{\alpha\beta\gamma\delta}\\ &+ \frac{1}{6!}\sum_{(\alpha,\beta,\gamma,\delta,\epsilon,\lambda)}^{[10]} C_{0;\alpha\beta\gamma}C_{0;\delta\epsilon\lambda}(-D)^{\alpha\beta\gamma\delta\epsilon\lambda}\Big]\Phi(t; g^{\alpha,\beta}). \end{aligned}\right\} \tag{4.9}$$

4.4. Univariate Case

In this subsection, we considered the special case of univariate T, which is much simpler and more imporant for applications.

Proposition 4.5. The r-th cumulant of one-dimensional T has the form

$$\kappa_N^r = N^{-\frac{r-2}{2}}(C_0^r + \frac{1}{N}C_1^r + \cdots) \qquad (r \geq 1), \tag{4.10}$$

with the first four being given by

$$
\begin{aligned}
\kappa_N^1 =& -\frac{1}{\sqrt{N}}\frac{1}{\sigma_A^3}\left(h_i h_{jk} h_l \kappa^{i,j}\kappa^{k,l} + \frac{1}{2}\kappa_{A,3}\right) + O(N^{-3/2}),\\
\kappa_N^2 =& 1 + \frac{1}{N}\Big\{-\frac{1}{\sigma_A^2}h_i h_{jk}\kappa^{i,j,k} - \frac{4}{\sigma_A^4}h_i h_j h_k h_{lm}\kappa^{i,j,l}\kappa^{k,m}\\
& - \frac{2}{\sigma_A^4}h_i h_{jk} h_{lm} h_n \kappa^{i,j}\kappa^{k,l}\kappa^{m,n} - \frac{1}{2\sigma_A^2}h_{ij}h_{kl}\kappa^{i,k}\kappa^{j,l}\\
& + \frac{7}{\sigma_A^6}\left(h_i h_{km} h_o \kappa^{i,k}\kappa^{m,o}\right)^2 + \frac{7}{\sigma_A^6}h_i h_{kn} h_o \kappa^{n,o}\kappa^{i,k}\kappa_{A,3} + 2 + \frac{7}{4\sigma_A^6}\kappa_{A,3}^2\\
& - \frac{1}{\sigma_A^2}h_{ijk}h_l\kappa^{i,j}\kappa^{k,l} - \frac{2}{\sigma_A^4}h_i h_j h_{klm} h_n \kappa^{i,k}\kappa^{j,l}\kappa^{m,n}\Big\} + O(N^{-3/2}), \qquad (4.11)\\
\kappa_N^3 =& -\frac{1}{\sqrt{N}\sigma_A^3}\{2\kappa_{A,3} + 3h_i h_j h_{kl}\kappa^{i,k}\kappa^{j,l}\} + O(N^{-3/2}),\\
\kappa_N^4 =& \frac{1}{N}\Big\{-\frac{2}{\sigma_A^4}\kappa_{A,4} - \frac{24}{\sigma_A^4}h_i h_j h_k h_{lm}\kappa^{i,l}\kappa^{j,k,m} + \frac{42}{\sigma_A^6}h_i h_j h_{kl}\kappa^{i,k}\kappa^{j,l}\kappa_{A,3}\\
& + 6 + \frac{12}{\sigma_A^6}\kappa_{A,3}^2 + \frac{36}{\sigma_A^6}\left(h_i h_j h_{kl}\kappa^{i,k}\kappa^{j,l}\right)^2 - \frac{12}{\sigma_A^4}h_i h_{jk} h_{lm} h_n \kappa^{i,j}\kappa^{k,l}\kappa^{m,n} -\\
& \frac{8}{\sigma_A^4}h_i h_j h_k h_{lmn}\kappa^{i,l}\kappa^{j,m}\kappa^{k,n}\Big\} + O(N^{-2}),
\end{aligned}
$$

where $\sigma_A^2 = h_i h_j \kappa^{i,j}$, $\sigma_{A,3} = h_i h_j h_k \kappa^{i,j,k}$, $\sigma_{A,4} = h_i h_j h_k h_l \kappa^{i,j,k,l}$.

Theorem 4.3. Define $\phi(t)$ to be the standard normal density, H_r the r-th Hermite polynomial w.r.t. $\phi(t)$. Under certain regularity conditions, the prabability density function of one-dimensional T can be expanded as

$$f(t) = \phi(t)\Big(1 + \frac{1}{\sqrt{N}}q_1(t) + \frac{1}{N}q_2(t)\Big) + O(N^{-3/2}), \qquad (4.12)$$

where

$$
\left.
\begin{aligned}
q_1(t) =& C_1^1 H(t) + \frac{1}{3!}C_0^3 H_3(t),\\
q_2(t) =& \frac{1}{2}[C_1^2 + (C_1^1)^2]H_2(t) + \frac{1}{4!}(C_0^4 + 4C_1^1 C_0^3)H_3(t)\\
& + \frac{1}{72}(C_0^3)^2 H_6(t).
\end{aligned}
\right\} \qquad (4.13)
$$

Theorem 4.4. Denote $\Phi(t)$ to be the standard normal cumulative distribution function, then the cumulative distribution function of one-dimensional T can be expanded as

$$F(t) = \Phi(t) + \frac{1}{\sqrt{N}}Q_1(t)\phi(t) + \frac{1}{N}Q_2(t)\phi(t) + O(N^{-3/2}), \qquad (4.14)$$

where

$$\left.\begin{aligned} Q_1(t) =& - (C_1^1 + \frac{1}{6}C_0^3 H_2), \\ Q_2(t) =& - \Big[\frac{1}{2}(C_1^2 + (C_1^1)^2)H_1(t) + \frac{1}{24}(C_0^4 + 4C_1^1 C_0^3)H_3(t) \\ &+ \frac{1}{72}(C_0^3)^2 H_5(t)\Big]. \end{aligned}\right\} \tag{4.15}$$

Theorem 4.5. Denote w_α and z_α to be the α-th quantile w.r.t. the true distribution of T and standard normal distribution, then, with the same regulality conditions requried for valid Edgeworth expansion of the corresponding cumulative distribution function, we have

$$w_\alpha = z_\alpha + \frac{1}{\sqrt{N}}G_1(z_\alpha) + \frac{1}{N}G_2(z_\alpha) + O(N^{-3/2}), \tag{4.16}$$

where

$$\left.\begin{aligned} G_1(t) =& - Q_1(t), \\ G_2(t) =& Q_1(t)\frac{\partial}{\partial t}Q_1(t) - \frac{1}{2}tQ_1^2(t) - Q_2(t). \end{aligned}\right\} \tag{4.17}$$

5. Jackknife-t Confidence Intervals

5.1. Introduction

This section concerns confidence interval estimation for one-dimensional parameter θ, based on the jackknife-t statistic, which is discussed in Subsection 4.4..

For $0 < \alpha < 1$, we define the αth quantile of T by

$$w_\alpha = \inf\{t : P[T \le t] \ge \alpha\}, \tag{5.1}$$

for which the third order Cornish-Fisher expansion is given by Theorem 4.5. For a fixed nominal level α, the confidence interval

$$I = (-\infty, \hat{\theta}_J - \frac{1}{\sqrt{N}}\widehat{Sd}_J w_{1-\alpha}] \tag{5.2}$$

is ideal in the sense that it coveres the unknown parameter θ with parabability α. I of (5.2) is useless since $w_{1-\alpha}$ is unknown. To construct approximate confidence intervals is to choose appropriate estimators of $w_{1-\alpha}$. Define

$$\left.\begin{aligned} \widehat{w}_{1,\alpha} =& z_\alpha, \\ \widehat{w}_{2,\alpha} =& z_\alpha + \frac{1}{\sqrt{N}}\widehat{G}_1(z_\alpha), \\ \widehat{w}_{3,\alpha} =& z_\alpha + \frac{1}{\sqrt{N}}\widehat{G}_1(z_\alpha) + \frac{1}{N}\widehat{G}_2(z_\alpha), \end{aligned}\right\} \tag{5.3}$$

where $\Phi(z_\alpha)=\alpha$, $\widehat{G}_1$ and $\widehat{G}_2$ are sample versions of G_1 and G_2 defined by (4.17). Approximate confidence intervals based on the coarse normal approximation, the second and third Cornish-Fisher expansions,(5.3), are defined to be

$$\widehat{I}_{i,N}=(-\infty,\widehat{\theta}-\frac{1}{\sqrt{N}}\widehat{Sd}_J\widehat{w}_{i,1-\alpha}], \tag{5.4}$$

where $i=1,2,3$. Coverage probability of each interval is defined to be

$$\alpha_{i,N}=P[\theta\in\widehat{I}_{i,N}], \tag{5.5}$$

which is intended to be close to the nominal level α. We give, in next section, explicit formulae up to third order of these coverage probabilities. Two-sided confidence intervals and correctness of intervals are also discussed.

5.2. Asymptotic theory

5.2.1. One-sided confidence intervals

We begin with the discussion of the approximate confidence interval $\widehat{I}_{3,N}$ based on third order Cornish-Fisher expansion. By definition

$$\begin{aligned}\alpha_{3,N}=&P[\theta\in\widehat{I}_{3,N}]\\ =&P[\theta\le\widehat{\theta}_J-\frac{1}{\sqrt{N}}\widehat{Sd}_J\widehat{w}_{3,1-\alpha}]\\ =&P[T\ge\widehat{w}_{3,1-\alpha}]\\ =&1-P[T<\widehat{w}_{3,1-\alpha}]\\ =&1-P[T<z_{1-\alpha}+\frac{1}{\sqrt{N}}\widehat{G}_1(z_{1-\alpha})+\frac{1}{N}\widehat{G}_2(z_{1-\alpha})]\\ =&1-P[T-\frac{1}{N}\widetilde{G}_1(z_{1-\alpha})<z_{1-\alpha}+\frac{1}{\sqrt{N}}G_1(z_{1-\alpha})+\frac{1}{N}G_2(z_{1-\alpha})]+O(N^{-3/2}),\end{aligned} \tag{5.6}$$

where

$$\widetilde{G}_1(t)=\sqrt{N}(\widehat{G}_1(t)-G_1(t)). \tag{5.7}$$

Expanding (5.6) requires Edgeworth expansion for cumlative distribution function of $S_N=T-\widetilde{G}_1/N$, which requres the knowledge of the first four cumulants of S_N.

Proposition 5.1. Denote the first four cumulants of S_N by $\kappa_i(S_N)(i=1,2,3,4)$, then

$$\begin{aligned}\kappa_1(S_N)=&\kappa_N^1+O(N^{-3/2}),\\ \kappa_2(S_N)=&\kappa_N^2+\frac{1}{N}a(t)+O(N^{-2}),\\ \kappa_3(S_N)=&\kappa_N^3+O(N^{-3/2}),\\ \kappa_4(S_N)=&\kappa_N^4+O(N^{-2}),\end{aligned} \tag{5.8}$$

where $\kappa_N^i (i = 1, 2, 3, 4)$ are defined by (4.11),

$$\begin{aligned} a(t) =& \frac{2}{\sigma_A^4}[(2 + H_2(t))h_i h_j h_{kl} h_m \kappa^{i,j,k} \kappa^{l,m} \\ &+ (\frac{1}{2} + \frac{1}{3}H_2(t))\kappa_{A,4}] + \frac{3}{\sigma_A^3}\kappa_{A,3}G_1(t), \end{aligned} \tag{5.9}$$

and $G_1(t)$ defined by (4.17).

Proposition 5.2. Define $S_N = T - \widetilde{G}_1/N$. The c.d.f. of S_N can be expanded as

$$P[S_N \le s] = \Phi(s) + \frac{1}{\sqrt{N}} q_1(s)\phi(s) + \frac{1}{N} q_2(s)\phi(s) + O(N^{-3/2}), \tag{5.10}$$

where $q_1(s) = Q_1(s)$, $q_2(s) = Q_2(s) - sa(s)/2$, and Q_1, Q_2 being defined by (4.15).

Theorem 5.1 Under certain regularity conditions, the following holds uniformly for $\epsilon < \alpha < 1 - \epsilon\, (0 < \epsilon < 1/2)$,

$$\alpha_{3,N} = \alpha + \frac{1}{2N} z_{1-\alpha} a(z_{1-\alpha})\phi(z_{1-\alpha}) + O(N^{-3/2}), \tag{5.12}$$

where $a(t)$ is given by (5.9).

By similar arguments for coverage probabilities of $\widehat{I}_{1,N}$ and $\widehat{I}_{2,N}$, we have next theorem.

Theorem 5.2. The coverage probabilities $\alpha_{1,N}$ and $\alpha_{2,N}$ corresponding to the approximate confidence intervals $\widehat{I}_{1,N}$ and $\widehat{I}_{2,N}$, based on the Normal and second order Cornish-Fisher expansion, admits following expansions, uniformly for $\epsilon < \alpha < 1-\epsilon\, (0 < \epsilon < 1/2)$,

$$\begin{aligned} \alpha_{1,N} =& \alpha - (\frac{1}{\sqrt{N}} Q_1(z_{1-\alpha}) + \frac{1}{N} Q_2(z_{1-\alpha}))\phi(z_{1-\alpha}) + O(N^{-3/2}), \\ \alpha_{2,N} =& \alpha + \frac{1}{N}(G_2(z_{1-\alpha}) + z_{1-\alpha} a(z_{1-\alpha})/2)\phi(z_{1-\alpha}) + O(N^{-3/2}), \end{aligned} \tag{5.13}$$

where Q_1 and Q_2 are defined by (4.15) and G_2 by (4.17).

Theorem 5.1 and Theorem 5.2 tell that both of the confidence intervals based on the Cornish-Fisher expansions have coverage error(the difference between coverage probability and nominal level) of order $O(N^{-1})$, improve considerably the Normal interval, which has coverage error of order $O(N^{-1/2})$. As shall be seen in next section, in the case of coefficient of variation, numerically $\alpha_{3,N}$ usually improves $\alpha_{2,N}$ although they have coverage error of the same asmptotic order.

5.2.2. Two-sided confidence intervals

For nominal level, $0 < \alpha < 1$, two-sided equal-tailed ideal confidence interval is defined as

$$I_{eq} = [\widehat{\theta}_J - \frac{1}{\sqrt{N}}\widehat{S}d_J w_{\frac{1+\alpha}{2}}, \widehat{\theta}_J - \frac{1}{\sqrt{N}}\widehat{S}d_J w_{\frac{1-\alpha}{2}}], \tag{5.14}$$

which covers the true parameter with probability α. Approximate confidence intervals of the same type based on normal approximation, the second and third order Cornish-Fisher expansions are defined by

$$\widehat{I}_{eq,i} = [\widehat{\theta}_J - \frac{1}{\sqrt{N}}\widehat{S}d_J \widehat{w}_{i,\frac{1+\alpha}{2}}, \widehat{\theta}_J - \frac{1}{\sqrt{N}}\widehat{S}d_J \widehat{w}_{i,\frac{1-\alpha}{2}}], \tag{5.15}$$

where $i = 1, 2, 3$, $\widehat{w}_{1,\frac{1\pm\alpha}{2}} = z_{\frac{1\pm\alpha}{2}}$ and

$$\begin{aligned}
\Phi(z_{\frac{1\pm\alpha}{2}}) =& \frac{1 \pm \alpha}{2}, \\
\widehat{w}_{2,\frac{1\pm\alpha}{2}} =& z_{\frac{1\pm\alpha}{2}} + \frac{1}{\sqrt{N}}\widehat{G}_1(z_{\frac{1\pm\alpha}{2}}), \\
\widehat{w}_{3,\frac{1\pm\alpha}{2}} =& z_{\frac{1\pm\alpha}{2}} + \frac{1}{\sqrt{N}}\widehat{G}_1(z_{\frac{1\pm\alpha}{2}}) + \frac{1}{N}\widehat{G}_2(z_{\frac{1\pm\alpha}{2}}).
\end{aligned} \tag{5.16}$$

Let $\alpha_{eq,i}(i = 1, 2, 3)$ be the coverage probabilty corresponding to $\widehat{I}_{eq,i}(i = 1, 2, 3)$.. By similar arguments in last section, we have

Theorem 5.3. Under certain regurality conditions, the probabilities $\alpha_{eq,i}(i = 1, 2, 3)$ admits expansions

$$\begin{aligned}
\alpha_{eq,1} =& \alpha - \frac{2}{N}Q_2(z_{\frac{1+\alpha}{2}})\phi(z_{\frac{1\pm\alpha}{2}}) + O(N^{-3/2}), \\
\alpha_{eq,2} =& \alpha + \frac{2}{N}(G_2(z_{\frac{1+\alpha}{2}}) + \frac{1}{2}z_{\frac{1+\alpha}{2}}a(z_{\frac{1+\alpha}{2}})\phi(z_{\frac{1+\alpha}{2}}) + O(N^{-3/2}), \\
\alpha_{eq,3} =& \alpha + \frac{1}{N}z_{\frac{1+\alpha}{2}}a(z_{\frac{1+\alpha}{2}})\phi(z_{\frac{1+\alpha}{2}}) + O(N^{-3/2}),
\end{aligned} \tag{5.17}$$

where Q_2, G_2, a are the same as in last subsection.

One of the salient features of two-sided equal-tailed confidence interval is that the basic normal theory based interval has coverage error of order $O(N^{-1})$, improves from $O(N^{-1/2})$ for its one-sided counterpart. The improvement is obviously due to the parity properties of both asymptotic expansions and the way of constructing confidence intervals.

5.2.3. Correctness

Although not as useful as the concept of coverage probability(or equivalently, coverage error), correctness may serve as another criterion for measuring accuracy of confidence intervals. Correctness, by definition, is the difference between the end points of an approximate confidence interval and the end points of the ideal confidence interval of the same type. An approximate confidence interval is said to be first or second order correct if this difference is of order $O_p(N^{-1})$ or $O_p(N^{-3/2})$. By the fact that the sample versions $\widehat{G}_1$ and $\widehat{G}_2$ are distant $O_p(1/\sqrt{N})$ away from G_1 and G_2, their population versions, respectively, we immediately have the following theorem.

Theorem 5.4 While $\widehat{I}_{1,N}((5.4))$ is first order correct for $I((5.2))$ and $\widehat{I}_{eq,1}((5.15))$ first order correct for $I_{eq}((5.14))$, all the other approximate confidence intervals defined by (5.4) and (5.15) are second order correct for their respective ideal ones.

Gererally, correct confidence intervals of order higher than 2 is beyond our reach. This is because to get an interval being third order correct requires to estimate the quantile w_α of T with accuracy $O_p(N^{-1})$ or better, which is usually impossible.

6. An Example

Inference for coefficient of variation, one of the ratio-type parameter, will be discussed briefly, see Wang, Taguri and Ouchi(1994) for details. Now, let $W, W_1, \cdots, W_N$ be iid random variables from $N(\mu, \sigma^2)$, Normal distribution with non-zero mean μ and variance σ^2. Coefficient of variation is defined by $CV = \sigma/\mu$, see Example 1 of Section 2. We discuss in this section two issues: (1) approximation for the c.d.f. of jackknife-t coefficient of variation by Normal and Edgeworth approximations; (2)confidence interval estimation for coefficient of variation based on the jackknife-t quantity.

6.1. Estimation of c.d.f.

Figures 6.1. to 6.4. show the goodness of fit by Normal approximation and the second and third order Edgeworth approximations, (4.14), for the true c.d.f. of T. Since Edgeworth expansion, (4.14), is defined by (4.15), which depends on the unknown parameter, thus is useless for estimating the true c.d.f. We define instead

$$\widehat{Ed}_2 = \Phi(t) + \frac{1}{\sqrt{N}}\widehat{Q}_1(t) \tag{6.1}$$

and

$$\widehat{Ed}_3 = \Phi(t) + \frac{1}{\sqrt{N}}\widehat{Q}_1(t) + \frac{1}{N}\widehat{Q}_2(t), \qquad (6.2),$$

where $\widehat{Q}_1$ and $\widehat{Q}_2$ are sample versions of Q_1 and Q_2,respectively. Displayed in each figure are the true distribution, $\Phi(t)$, $\widehat{Ed}_2$ and $\widehat{Ed}_3$.

Fix $\mu = 1$, so $CV = \sigma$. In many practical applications, CV ranges from 1/2 to 2. In our numerical experiment, we fix $CV = 0.8$ and vary the sample size to take $N = 5, 10, 15$ and 20. The following is a summary of the main features of c.d.f. approximations.

Figure 6.1: Comparision of the Normal, the second and third order Edgeworth approximation in estimating the true c.d.f. of the jackknife-t coefficient of variation from a Normal population with unit mean and coefficient of variation 0.8, N=5

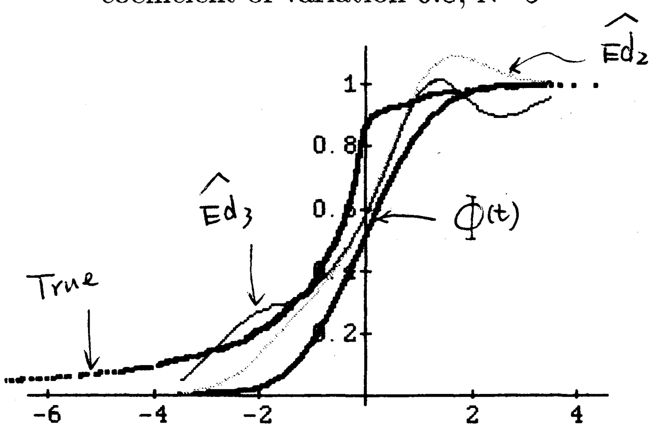


Figure 6.2: Comparision of the Normal, the second and third order Edgeworth approximation in estimating the true c.d.f. of the jackknife-t coefficient of variation from a Normal population with unit mean and coefficient of variation 0.8, N=10

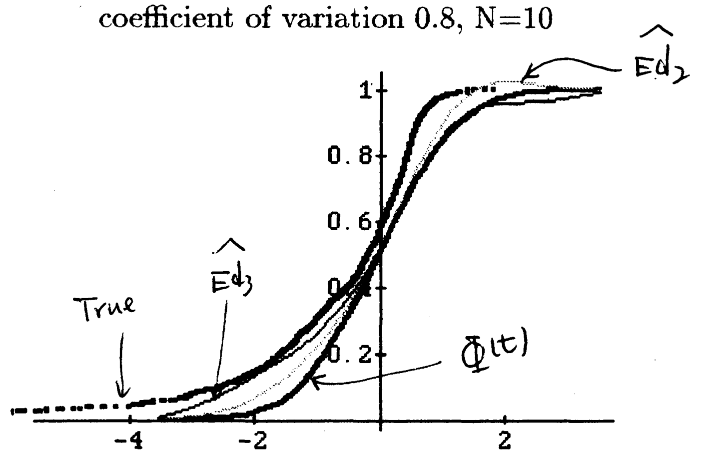


Figure 6.3: Comparision of the Normal, the second and third order Edgeworth approximation in estimating the true c.d.f. of the jackknife-t coefficient of variation from a Normal population with unit mean and coefficient of variation 0.8, N=15

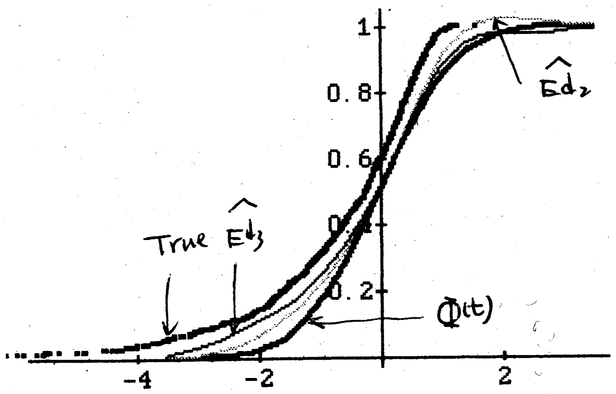
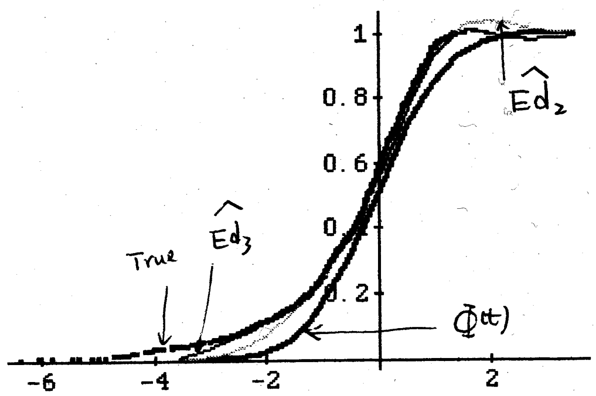


Figure 6.4: Comparision of the Normal, the second and third order Edgeworth approximation in estimating the true c.d.f. of the jackknife-t coefficient of variation from a Normal population with unit mean and coefficient of variation 0.8, N=20



(a) All in all, $\widehat{Ed}_3$ improves $\widehat{Ed}_2$, which in turn improves $\Phi(t)$, with improvements especially remarkable for the left tails;

(b) Although Edgeworth approximations still improve the basic Normal approximation, around the origin, as can be expected, the three approximations are relatively undistinguishable;

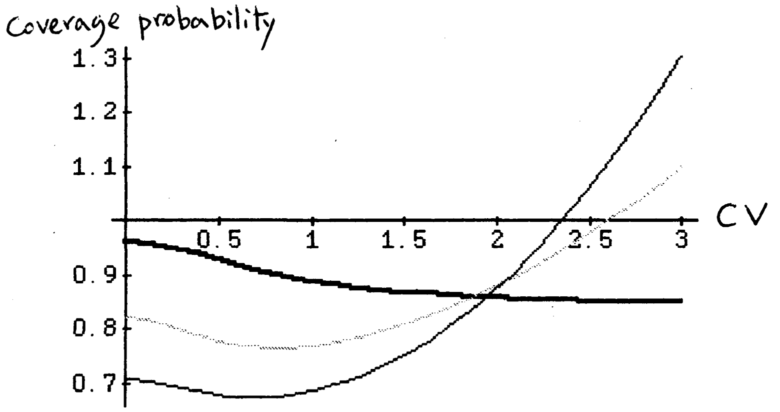
(c) Estimation of the right tails is relatively complicated, with $\widehat{Ed}_3$ approaching the true from below and $\widehat{Ed}_2$ from above. $\widehat{Ed}_3$ is seen to underestimate the true distribution even more seriously than $\Phi(t)$ for right tails. With sample size gradually increasing, both $\widehat{Ed}_2$ and $\widehat{Ed}_3$ become correspondingly trustworthy;

(d) Relative better estimation for the left tails probably is because the true distribution has relative heavy left tails, which is reflected in the Edgeworth approximations, but ignored by Normal approximation. For those with heavy right tails, estimation of the right tail might be better than that of left tails.

6.2. Jackknife-t confidence intervals

Now we turn to the issue of constructing approximate confidence intervals. Figure 6.5. compares the asymptotic coverage probabilities up to order $O(N^{-1})$ for left-sided intervals, defined by (5.4). Here we examine the effect of coefficient of variation upon the three asymptotic coverage probabilities, with a truely small sample size 5. Denote $\alpha^*_{i,N}$ to be the versions of (5.5) with terms of order higher than $O(N^{-1})$ discarded. Figure 6.5. shows that $\alpha^*_{3,N}$, $\alpha^*_{2,N}$, $\alpha^*_{1,N}$ dominates one another given CV is kept below 2.0, which are quite in agreement with our true simulated coverage probabilties. Asymptotic, as well as the true coverage probabilities, are sensitive to the fluctuation of CV. Asymptotically, $\alpha^*_{3,N}$, $\alpha^*_{2,N}$, $\alpha^*_{1,N}$ dominates one another in reverse order when CV exceeds about 2.0, which, however, is not entirely true for the true coverage probabilities.

Figure 6.5. Comparisions of asymptotic coverage probabilities of approximate confidence intervals based on Normal, second and third order Cornish-Fisher expansions,with nominal level $\alpha = 90\%$



Next, we compare the true coverage probabilities of the left-sided confidence intervals, defined by (5.5). In each case, we make 1,000 time trials, and keep other conditions unchanged as in Subsection 6.1.

Table 6.1. Comparisions of true coverage probabilities of left-sided confidence intervals based on Normal, the second and third order Cornish-Fisher expansions, with nominal coverage $\alpha = 90\%$ and true $CV = 0.8$

N	$\alpha_{1,N}$	$\alpha_{2,N}$	$\alpha_{3,N}$
5	.724	.794	.853
10	.787	.854	.893
15	.775	.839	.871
20	.796	.856	.881

Table 6.2. Comparisions of true coverage probabilities of left-sided confidence intervals based on Normal, the second and third order Cornish-Fisher expansions, with nominal coverage $\alpha = 90\%$ and true $CV = 1.5$

N	$\alpha_{1,N}$	$\alpha_{2,N}$	$\alpha_{3,N}$
5	.643	.736	.738
10	.710	.816	.818
15	.768	.858	.865
20	.758	.849	.864

(1) In either case, the following holds

$$\alpha_{1,N} < \alpha_{2,N} < \alpha_{3,N} < \alpha, \tag{6.1}$$

which strongly implies the advantage of doing higher order asymptotic expansions, especially in small-sample situations. (6.1) also tells us the important fact that third order approximation exels its second order counterpart, which we failed to learn in Section 5, where the two share the same asymptotic speed for approaching the nominal level.

(2) In either case, $\alpha_{2,N}$ and $\alpha_{3,N}$ improves $\alpha_{1,N}$ considerably, especially when N is small. For example, when $N = 5$, $CV = 1.5$, $\alpha_{2,N} = .736$ and $\alpha_{3,N} = .738$, compared sharply with $\alpha_{1,N} = .643$. In this specific case, doing even the second order asymptotic expansion achieves the accuracy of Normal theory based inference with sample size being have to be raised 4 times from 5 to $20(\alpha = .758$ when $N = 20)$.

(3) The differences between $\alpha_{2,N}$ and $\alpha_{3,N}$ seem to be rather sensitive to CV. While CV is relatively large, $\alpha_{3,N}$ improves $\alpha_{2,N}$, but moderately. The improvements seem to be significant when CV becomes smaller.

7. Discussions

(1) Jackknife-t variates in inferences for the ratio-type parameters, typically have distributions with rather heavy right or left tails. Incorporation of this piece of information to make more accurate statistical inferences based on asymptotic expansions with respect to correspondingly skewed kernel in stead of the symmetric Normal deserves a serious study.

(2) Alternative ways are needed for estimation problem when higher order asymptotic approximations are worse than lower order approximations. Bootstrap may be one of the partial solusions.

(3) The equally important aspect of statistical inference, namely, testing of hypothese, requires further study.

(4) Ignored is the comparision with grouped jackknife and other methods, including the bootstrap, study of which is certainly important and interesting.

(5) Investigations of other specific applications are important and necessary.

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