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A LINEAR OPERATOR AND SOME APPLICATIONS OF FIRST ORDER DIFFERENTIAL SUBORDINATIONS

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1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z: |z| < 1\}$.

For functions $f_j(z) \in \mathcal{A}_p$ $(j=1,2)$ defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k ,$$

we define the convolution (or Hadamard product) $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k . \tag{1.2}$$

With the convolution above, we define

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \qquad (f(z) \in \mathcal{A}_p)$$

$$= \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!} , \tag{1.3}$$

where n is any integer greater than -p.

For a function $f(z) \in \mathcal{A}_p$, we define the generalized Libera integral operator $J_{\nu,p}$ by

$$J_{\nu,p}(f(z)) = \frac{\nu + p}{z^\nu} \int_0^z t^{\nu-1} f(t)\,dt, \qquad \nu > -p. \tag{1.4}$$

For $p = 1$ and $\nu \in \mathbb{N}$, the operator $J_{\nu,1}$ was introduced by Bernardi [1]. Inparticular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some interesting results for the operator $J_{\nu,p}$ were showed by Saitoh [12] and Saitoh et al. [13].

Now, let the function $\phi_p(a,c)$ be defined by

$$\phi_p(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \qquad (z \in \mathbb{U}), \tag{1.5}$$

for $c \neq 0,-1,-2,\cdots$, where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1) \cdots (a+n-1), & \text{if } n \in \mathbb{N} \end{cases} \tag{1.6}$$

Also, we define a linear operator $L_p(a,c)$ on $\mathcal{A}_p$ by

$$L_p(a,c;z)f(z) = \phi_p(a,c;z)*f(z) \tag{1.7}$$

for $f(z) \in \mathcal{A}_p$ and $c \neq 0,-1,-2,\cdots$.

The operator $L_1(a,c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions.

REMARKS. (1) For $f(z) \in \mathcal{A}_1 = \mathcal{A}$,

$$L_1(n+1,1;z)f(z) = D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

is Ruscheweyh derivative of $f(z)$ ([8]).

(2) For $f(z) \in \mathcal{A}_p$,

$$L_p(n+p,1;z)f(z) = D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$$

is Ruscheweyh derivative introduced by Goel and Sohi [3].

(3) For $f(z) \varepsilon A_p$,

$$L_p(\nu+p,\nu+p+1;z)f(z) = J_{\nu,p}(f(z))$$

is the generalized Libera integral operator ([12],[13]).

(4) $\phi_1(a,c;z)$ is an incomplete beta function, related to the Gauss hypergeometric functions by

$$\phi_1(a,c;z) = z\cdot{}_2F_1(1,a;c;z) .$$

It follows from (1.7) that

$$z(L_p(a,c;z)f(z))' = aL_p(a+1,c;z)f(z)-(a-p)L_p(a,c;z)f(z) \quad (1.8)$$

Let the function f(z) and g(z) be analytic in U. Then the f(z) is said to be subordinate to g(z) if there exists a function w(z) analytic in U, with $w(0)=0$ and $|w(z)|<1$ $(z\varepsilon U)$, such that $f(z) = g(w(z))$ $(z\varepsilon U)$. We denote this subordination by $f(z)\prec g(z)$.

2. MAIN RESULTS

To establish our main results, we need the following lemmas.

LEMMA 1 [6]. Let h(z) be convex and B(z) be analytic in U with $\mathrm{Re}B(z)\geq 0$. If p(z) is analytic in U and $p(0)=h(0)$, then

$$p(z) + B(z)zp'(z)\prec h(z) \text{ implies } p(z)\prec h(z) \quad (z\varepsilon U).$$

LEMMA 2 [9]. Let $\gamma=0$, $\mathrm{Re}\gamma\geq 0$, and let h(z) be convex. If p(z) is analytic in U with $p(0)=h(0)$, then

$$p(z)+\frac{1}{\gamma}zp'(z)\prec h(z) \text{ implies } p(z)\prec\frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}dt \quad (z\varepsilon U).$$

THEOREM 1. Let λ, a be a real number with $\lambda \geqq 0$, $a>0$. Let $h(z)$ be a convex function with $h(0)=1$, $c \neq 0,-1,-2,\cdots$ and $g(z) \varepsilon A_p$ satisfies

$$\operatorname{Re}\left\{\frac{L_p(a+1,c;z)g(z)}{L_p(a,c;z)g(z)}\right\} > 0 \tag{2.1}$$

If $f(z) \varepsilon A_p$ satisfies

$$(1-\lambda)\left\{\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)}\right\} + \lambda\left\{\frac{L_p(a+1,c;z)f(z)}{L_p(a+1,c;z)g(z)}\right\} \prec h(z) \tag{2.2}$$

then we have

$$\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)} \prec h(z) \qquad (z\varepsilon U). \tag{2.3}$$

PROOF. Put

$$H(z)=(1-\lambda)\left\{\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)}\right\} + \lambda\left\{\frac{L_p(a+1,c;z)f(z)}{L_p(a+1,c;z)g(z)}\right\}$$

From assumption, $h(z)$ is convex with $h(0)=1$ and $f(z) \varepsilon A_p$ satisfies $H(z) \prec h(z)$ $(z\varepsilon U)$, where $\lambda \geqq 0$ and $g(z)\varepsilon A_p$ satisfies (2.1).

Set $B(z)= \frac{\lambda}{a} \cdot \frac{L_p(a,c;z)g(z)}{L_p(a+1,c;z)g(z)}$

According to (2.1), we have $\operatorname{Re}B(z) \geqq 0$. Define $p(z)$ by

$$p(z) = \frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)} \tag{2.4}$$

We can see that $p(z)$ is analytic in U and $p(0)=1$. Logarithmic differentiating (2.4) and using (1.8), we can get

$$\frac{L_p(a+1,c;z)f(z)}{L_p(a+1,c;z)g(z)} = p(z) + \frac{1}{\lambda} B(z) \cdot zp'(z) \qquad (z\varepsilon U). \tag{2.5}$$

Then (2.2) can be written

$$p(z) + B(z)zp'(z) \prec h(z).$$

By Lemma 1, we have

$$p(z) \prec h(z) .$$

Thus, we have

$$\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)} \prec h(z) .$$

This completes the proof of Theorem 1.

Putting a = n+p, c=1 in Theorem 1, we have

COROLLARY 1. Let λ be a real number with $\lambda \geqq 0$. Let h(z) be a convex function with h(0)=1 and $g(z) \varepsilon A_p$ satisfies

$$\operatorname{Re}\left\{\frac{D^{n+p}g(z)}{D^{n+p-1}g(z)}\right\} > 0 .$$

If $f(z) \varepsilon A_p$ satisfies

$$(1-\lambda)\left\{\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right\} + \lambda\left\{\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right\} \prec h(z) ,$$

then we have

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \prec h(z) \qquad (z \varepsilon U).$$

Making a = ν+p, c = ν+p+1 in Theorem 1, we can show

COROLLARY 2. Let λ be a real number with $\lambda \geqq 0$. Let h(z) be a convex function with h(0)=1 and $g(z) \varepsilon A_p$ satisfies

$$\operatorname{Re}\left\{\frac{g(z)}{J_{\nu,p}(g(z))}\right\} > 0 .$$

If $f(z) \varepsilon A_p$ satisfies

$$(1-\lambda)\left\{\frac{J_{\nu,p}(f(z))}{J_{\nu,p}(g(z))}\right\} + \lambda\frac{f(z)}{g(z)} \prec h(z) ,$$

then we have

$$\frac{J_{\nu,p}(f(z))}{J_{\nu,p}(g(z))} \prec h(z) .$$

Next, we prove

THEOREM 2. Let $f(z)\in A_p$ and $h(z)$ be a convex function with $h(0)=1$. Then for any complex number λ with $\mathrm{Re}\lambda\geqq 0$ $(\lambda=0)$, $a>0$,

$$(1-\lambda)\left\{\frac{L_p(a,c;z)f(z)}{z^p}\right\}+\lambda\left\{\frac{L_p(a+1,c;z)f(z)}{z^p}\right\}\prec h(z)\quad (z\in U)\tag{2.6}$$

implies

$$\frac{L_p(a,c;z)f(z)}{z^p}\prec\frac{a}{\lambda z^{a/\lambda}}\int_0^z h(t)t^{a/\lambda-1}dt\prec h(z)\quad (z\in U)\tag{2.7}$$

The result is sharp.

PROOF. Choosing $g(z)=z^p$ and $B(z)=\frac{\lambda}{a}$, and use Lemma 2, Theorem 2 follows from Theorem 1.

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