数理解析研究所講究録
第 881 巻 1994 年 165-170

Some Remarks on a Distortion Theorem

Yoshimi UENOYAMA (Wakayama Univ.) (和歌山大教育・上野山好美)

Abstract. The object of the present paper is to derive the boundary value of $\left|\frac{z}{f(z)}-1\right|$ for the class of starlike functions of order α in the open unit disk.

Let A denote the class of functions of the form

(1) $$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$. Let S denote the class of normalized analytic and univalent functions in U. We denote by $S^*(\alpha)$ and C the subclasses of S of starlike functions of order α and of close-to-convex functions, respectively. In particular for $f(z)\in C$, a function $f(z)$ is said to be close-to-convex if it satisfies

$Re\, f'(z)>0\ (z\in U)$.

In [1], Causey, Krzyz and Merkes obtained the following

Theorem A. If f is in S and $|z|=r<1$, then

(2) $$\left|\frac{z}{f(z)}-1\right|\leq\left\{A^2(t_0)-2A(t_0)\cos\psi(t_0)+1\right\}^{\frac{1}{2}}$$

where

(3) $$A(t)=A_r(t)=(1-r^2)\left(\frac{1+r}{1-r}\right)^{\cos t}, \quad \psi(t)=\psi_r(t)=\sin t \log\frac{1+r}{1-r}$$

and $t_0=t_0(r)$ is a suitable zero of the function

(4) $$D_r(t)=\sin(t+\psi_r(t)) - A_r(t)\sin t.$$

For each $r\in(0,1)$ there is a function in S such that the equality holds in (2).

The proof of Theorem A was shown by using Lemma B.

Lemma B[3]. For each z, $|z|=r<1$, the region $\{\log\frac{f(z)}{z} : f\in S\}$ is the disk

(5) $$\{\zeta : \left|\zeta+\log(1-r^2)\right|\leq\log\frac{1+r}{1-r}\}.$$

This lemma was discovered by Grunsky in 1932.

For analytic functions $g(z)$ and $h(z)$ in U with $g(0)=h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $\omega(z)$ so that $\omega(0)=0$, $|\omega(z)|<1$ $(z\in U)$ and $g(z)=h(\omega(z))$. We denote this subordination by

$$g(z) \prec h(z).$$

Lemma C[4]. Let $f(z)\in A$, and let $g(z)\in A$ be convex in U. If $f(z)\prec g(z)$, then

(6) $$\int_0^z\frac{f(t)}{t}dt \prec \int_0^z\frac{g(t)}{t}dt.$$

We have the following lemma on $S^*(\alpha)$.

Lemma. For each z, $|z|=r<1$, the region $\{\log\frac{f(z)}{z} : f \in S^*(\alpha)\}$ is the disk

(7) $$\left\{\zeta ; \left|\zeta+(1-\alpha)\log(1-r^2)\right| \leqq (1-\alpha)\log\frac{1+r}{1-r}\right\}.$$

where $0 \leqq \alpha < 1$.

Proof. We define the function $p(z)$ by

(8) $$p(z)=\frac{zf'(z)}{f(z)}.$$

Then $p(z)$ is regular in $\mathbf{U}$ with $p(0)=1$ and $Re\, p(z)>\alpha$ in $\mathbf{U}$. Hence

(9) $$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z}.$$

It follows from (9) that

$$\frac{zf'(z)}{f(z)} -1 \prec \frac{2(1-\alpha)z}{1-z}.$$

Since $\frac{2(1-\alpha)z}{1-z}$ is convex in $\mathbf{U}$, an application of lemma C gives that

(10) $$\int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)dt \prec 2(1-\alpha)\int_0^z \frac{1}{1-t}dt$$

which is equivalent to

(11) $$\log\frac{f(z)}{z} \prec -2(1-\alpha)\log(1-z).$$

Hence,

(12) $$\frac{1}{1-\alpha}\log\frac{f(z)}{z} \prec -2\log(1-z).$$

The region $\{-2\log(1-z)\,;\, z\in U\}$ is contained in the closed disk with center $w_0=-\log(1-r^2)$ and radius $R=\log\frac{1+r}{1-r}$. Therefore by properties for subordinations, we have

(13)
$$\left|\frac{1}{1-\alpha}\log\frac{f(z)}{z}+\log(1-r^2)\right|\leqq\log\frac{1+r}{1-r}.$$

It follows from (13) and $0\leqq\alpha<1$ that

$$\left|\log\frac{f(z)}{z}+(1-\alpha)\log(1-r^2)\right|\leqq(1-\alpha)\log\frac{1+r}{1-r}.$$

This completes the proof of the lemma. From this lemma, we can obtain a result which is similar to theorem A for all $f\in S^*(\alpha)$.

Indeed, the boundary of the range of $\frac{z}{f(z)}$, for $f\in S^*(\alpha)$, $|z|=r$, can be by (7)

(14)
$$-\log\frac{f(z)}{z}-(1-\alpha)\log(1-r^2)=e^{it}(1-\alpha)\log\frac{1+r}{1-r}.$$

From (14), it holds that

$$\log\frac{z}{f(z)}=(1-\alpha)\log\frac{1+r}{1-r}(\cos t+i\sin t)+(1-\alpha)\log(1-r^2)$$

$$=(1-\alpha)\log(1-r^2)\left(\frac{1+r}{1-r}\right)^{\cos t}+i(1-\alpha)\sin t\ \log\left(\frac{1+r}{1-r}\right)$$

$$=(1-\alpha)\log(1-r^2)\left(\frac{1+r}{1-r}\right)^{\cos t}+(1-\alpha)\log\ e^{i\sin t\log\frac{1+r}{1-r}}$$

$$=(1-\alpha)\left\{\log(1-r^2)\left(\frac{1+r}{1-r}\right)^{\cos t}\cdot e^{i\sin t\log\frac{1+r}{1-r}}\right\}.$$

Consequently,

(15) $$\frac{z}{f(z)} = \{(1-r^2)(\frac{1+r}{1-r})^{\cos t}\}^{1-\alpha} \cdot e^{i(1-\alpha)\sin t \log\frac{1+r}{1-r}}$$

The boundary of the range of $\frac{z}{f(z)}$ can be paramentrized as

(16) $$\frac{z}{f(z)} = A(t,\alpha)(\cos\psi(t,\alpha)+i\sin\psi(t,\alpha)),$$

where

(17) $$A(t,\alpha) = \{(1-r^2)(\frac{1+r}{1-r})^{\cos t}\}^{1-\alpha}, \qquad \psi(t,\alpha) = (1-\alpha)\sin\ t\cdot\log\frac{1+r}{1-r}.$$

Furthemore, by a simple computation, we obtain the following

Theorem. If f is in $S^*(\alpha)$ and $|z|=r<1$, then

(18) $$\left|\frac{z}{f(z)}-1\right| \leqq \{A^2(t,\alpha)-2A(t,\alpha)\cos\psi(t,\alpha)+1\}^{\frac{1}{2}},$$

where $A(t,\alpha), \psi(t,\alpha)$ are defined by (17).

Putting $\alpha=0$, $t=0$ in theorem, we have

Corollary. If f is in S^* and $|z|=r<1$, then

(19) $$\left|\frac{z}{f(z)}-1\right| \leqq 2r+r^2.$$

Equality holds in (19) if and only if $f(z)=\frac{z}{(1+z)^2}$.

In[2], P.Pawlowski obtained the following theorem.

Theorem D. Let $|z|=r<1$ and $f\in C$. Then

(20) $$\left|\frac{z}{f(z)}-1\right| \leqq 2r+r^2.$$

As stated above, We have the same result for S^* and C. Consequently, the expression on the left in (20) is sharp.

References

[1] W.M.Causey, J.G.Krzyz and E.P.Merkes, Some remarks on a distortion lemma, Ann. Univ. Mariae, Curie-sklodowska, 45(1991),15-19.

[2] P.Pawlowski, A distortion theorem for close-to-convex functions, Ann. Univ. Mariae, Curie-sklodowska, 45(1991),113-115.

[3] P.L.Duren, Univalent Functions, Springer Verlag, 1983, 323-326.

[4] K.Sakaguchi, A property of convex functions and an application to criteria for univalence, Bull. Nara. Univ. Educ. 22(1973), 1-5.

Department of Mathematics
Wakayama University
Sakaedani, Wakayama 640
Japan