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Representations and Pseudo-representations

(Abstract)

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(I) Representations over local rings ([C])

Let G be an abstract group and R a local ring with maximal ideal m and residue field F. We define a *d-dimensional representation of G over R* as usual, i.e. as an homomorphism :

$$\rho : G \longrightarrow GL_d(R);$$

two such representations are called *equivalent* if one is conjugate of the other by some $M \in GL_d(R)$. The *residual representation* $\bar{\rho} : G \to GL_d(F)$ is obtained by reducing modulo m.

Our first result is the following :

THEOREM 1. — *Suppose ρ and ρ' are two d-dimensional representations of G over R. Assume* :

(a) $\forall g \in G,\ \ trace\ \rho(g) = trace\ \rho'(g)$,

(b) $\bar{\rho}$ *is absolutely irreducible*;

then ρ and ρ' are equivalent.

My paper [C] also contains some "Schur-type" result, which allows, under suitable hypothesis, to realize a representation over a subring where the trace takes its values. As a consequence, we give a construction of *Galois representations* associated to some modular forms defined over rings. This kind of results can now be viewed as corollaries of a theorem of Louise Nyssen on pseudo-representations, which I will explain in the next paragraph.

(II) Pseudo-representations

Pseudo-representations were first introduced in dimension 2 by Andrew Wiles, as a sort of substitute for representations; they played a crucial role in the construction, using congruences between modular forms, of some ℓ-adic Galois representations ([W]). Taylor ([T]) generalized them to any dimension.

A pseudo-representation of dimension d of a group is a function on this group which satisfies the formal properties of the trace of a representation : two of those properties are obvious, and the third one reflects a certain polynomial identity on matrix rings ([P]). More precisely :

DEFINITION. — *Let G be a group and R a (commutative) ring. A d-dimensional pseudo-representation of G over R is a map $T : G \longrightarrow R$ which satisfies :*

(a) $T(1) = d$,

(b) $\forall x, y \in G,\ \ T(xy) = T(yx)$,

(c) $\forall x_1, \dots, x_{d+1} \in G,\ \ \sum_{\sigma \in \mathfrak{S}_{d+1}} \varepsilon(\sigma) T_\sigma(x_1, \dots, x_{d+1}) = 0$,

where $\varepsilon(\sigma)$ denotes the signature of σ, and where T_σ is defined as follows : if σ is decomposed into a product of disjoint cycles (including fixed points viewed as 1-cycles) :

$$\sigma = \left(i_1^1 i_1^2 \cdots i_1^{k_1}\right) \dots \left(i_m^1 \cdots i_m^{k_m}\right)$$

$$T_\sigma(x_1, \dots, x_{d+1}) = T\left(x_{i_1^1} \cdots x_{i_1^{k_1}}\right) \dots T\left(x_{i_m^1} \cdots x_{i_m^{k_m}}\right)$$

(this makes unambiguous sense thanks to (b)*).*

The trace of any representation is a pseudo-representation, and according to [T] the converse is also true over an algebraically closed field of characteristic 0. Because theorem 1 asserts that we have a good theory for those representations over local rings which reduce to absolutely irreducible representations, it seems reasonable to compare both notions in this context :

THEOREM 2 [N]. — *Let T be a d-dimensional pseudo-representation of a group G over an henselian separated local ring R. We assume that its reduction $\overline{T}$ modulo the maximal ideal is the trace of some absolutely irreducible d-dimensional representation over the residue field. Then T itself is the trace of a d-dimensional representation of G over R (well-defined up to equivalence according to theorem* 1*).*

Note : A recent preprint of K. Saito ([S]) contains related results in the case of 2-dimensional representations.

(III) References

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