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Harmonic tori in complex Grassmann manifolds and quaternionic projective spaces

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The problem of constructing harmonic maps of two-spheres into spheres, complex Grassmann manifolds and quaternionic projective spaces is already solved and well understood(see [Ba-W], [B-W], [Ca1], [Ca2],[Wo], [W]). The next problem is to construct and understand the harmonic maps of two-tori in spheres, complex Grassmann manifolds and quaternionic projective spaces.

In contrast with the case of harmonic two-spheres, there is a class of non-conformal harmonic maps for two-tori. For non-conformal harmonic two-tori in compact symmetric space of rank one, a beautiful theory is established by [B-F-P-P], which says that they are obtained by integrating certain commuting Hamiltonian flows. They called the map of this kind a map of *finite type*. However, the geometrically interesting class of harmonic maps is that of conformal ones. The (weakly) conformal harmonic maps are divided into two subclasses, the class of superminimal ones and the class of non-superminimal ones. The former class is well understood (see [Ca1], [Ca2], [Ch], [E-W], [Ba-W]); its members are projections of horizontal holomorphic curves in certain generalized flag manifolds, which are twistor spaces of the underlying symmetric space. The latter class was recently treated by Hitchin[H] in case of S^3 as target and by Ferus-Pedit-Pinkall-Sterling [F-P-P-S] in case of S^4 as target.

Recently, Burstall[B] proved that any non-superminimal harmonic tori in a sphere or a complex projective space is covered by a *primitive* harmonic map of finite type into a certain generalized flag manifold (see [B-P-W] for superconformal harmonic two-tori in a complex projective space, which is a special case of Burstall' theorem stated above).

In this paper, we treat and show some results on harmonic two-tori in complex Grassmann manifolds and quaternionic projective spaces.

1. Preliminaries and the fundamental facts

Let $\mathbf{C}^n$ be an n-dimensional complex number space with the standard Hermitian inner product $<\ ,\ >$ defined by $< v, w >= \sum_{i=1}^{n} v_i \overline{w_i}$, where $v = (v_1, v_2, \cdots, v_n), w = (w_1, w_2, \cdots, w_n)$. Let $G_k(\mathbf{C}^n)$ be the Grassmann manifold of all complex k-dimensional subspaces of $\mathbf{C}^n$ with its standard Kähker structure. Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a smooth map of a Riemann surface. Let $V(\varphi)$ be the pull-back of universal bundle over $G_k(\mathbf{C}^n)$ by φ. Then, $V(\varphi)$ is a subbundle of the trivial bundle $V(\mathbf{C}^n) = M \times \mathbf{C}^n$. We equip $V(\mathbf{C}^n)$ with the standard Hermitian connected structure compatible with the Hermitian fiber metric $<\ ,\ >$. For any subbundle F of $V(\mathbf{C}^n)$, we denote by $F^{\perp}$ the Hermitian orthogonal complement of F in $V(\mathbf{C}^n)$ with respect to $<\ ,\ >$. Then, F anf $F^{\perp}$ are both equipped with the Hermitian connected structures induced from that of $V(\mathbf{C}^n)$. Moreover, F and $F^{\perp}$ both have the Koszul- Malgrange holomorphic structures. Let $A_{\prime}^{F,F^{\perp}}$ be the $(1,0)$-part of the second fundamental form of F in $V(\mathbf{C}^n)$. By taking the image of the second fundamental form, we may define the new subbundle F_1 of $V(\mathbf{C}^n)$, which is defined on M except the singularity subset S. If $A_{\prime}^{F,F^{\perp}}$ is a holomorphic section, S is a discrete set. In this case, the line bundle $[S]$ defined by the divisor S enables us to extend F_1 smoothly over M, which is also a holomorphic subbundle of $F^{\perp}$ and denoted by F_1 again. Set $V_0 = V(\varphi)$. It is known that $A_{\prime}^{V_0,V_0^{\perp}}$ is a $\mathrm{Hom}(V_0, V_0^{\perp})$-valued holomorphic differential if and only if φ is a harmonic map. It is also known that V_1 defines a harmonic map $\varphi_1 : M \longrightarrow G_{k_1}(\mathbf{C}^n)$ with $k_1 \leq k$, where V_1 is isomorphic to the pull-back of the universal bundle over $G_{k_1}(\mathbf{C}^n)$ by φ_1. Now, starting from V_0, we may define the harmonic sequence $V_0 \rightarrow V_1 \rightarrow \cdots V_{r-1} \rightarrow R$, where $V_i = \mathrm{Im} A_{\prime}^{V_{i-1},V_{i-1}^{\perp}}$ for $i = 1, \cdots, r-1$ and $R = V(\mathbf{C}^n) \ominus (\bigoplus_{i=0}^{r-1} V_i)$. This situation assumes that each of $V_0, V_1, \cdots, V_{r-1}$ and R are orthogonal to each other with respect to the Hermitian metric on $V(\mathbf{C}^n)$. In this case, we say that φ has *strong isotropy order* $\geq r$. From the definition of harmonic sequence, it is always true that $r \geq 1$. In the case where we use $(0,1)$-part of the second fundamental form, we denote the corresponding harmonic sequence by $V_0 \leftarrow V_{-1} \leftarrow \cdots \leftarrow V_{-r+1} \leftarrow R'$, where $V_{-i} = \mathrm{Im}A_{''}^{V_{-i+1}, V_{-i+1}^{\perp}}$ for $i = 1, \cdots, r-1$ and $R' = V(\mathbf{C}^n) \ominus (\bigoplus_{i=0}^{r-1} V_{-i})$. It is known that V_i and V_j is orthogonal to each other for $0 < \mid i - j \mid \leq r$ and that each V_i defines a harmonic map of M into $G_s(\mathbf{C}^n)$, where $s = \mathrm{rank}V_i$.

Now, we give the definition of isotropy order of φ : We denote by ∇ the pull-back connection on the pull-back bundle $\varphi^{-1}TG_k(\mathbf{C}^n)$, which is extended by complex linearity to $\varphi^{-1}(TG_k(\mathbf{C}^n))^{\mathbf{C}}$. According as the type decomposition of the complexified cotangent bundle of M, we set $\nabla = \nabla' + \nabla''$.

Definition(cf. [E-W], [Er-W]). (1) φ is said to have *isotropy order* r if r is the largest integer such that the following equation holds :

$$(1.1) \qquad < \nabla'^{\alpha}\varphi, \nabla''^{\beta}\varphi > \equiv 0 \qquad \text{for} \quad 2 \leq \alpha + \beta \leq r \ ,$$

where $\nabla'\varphi = \partial\varphi$, $\nabla''\varphi = \overline{\partial}\varphi$, $\nabla'^{\alpha}\varphi = \nabla'(\nabla'^{\alpha-1}\varphi)$ and $\nabla''^{\beta}\varphi = \nabla''(\nabla''^{\beta-1}\varphi)$. In the case of $r = \infty$, φ is said to be *isotropic.*

(2) φ is said to have *strong isotropy order* r if $V_0 \perp V_i$ for $i = 1, \cdots r$ and V_{r+1} is not perpendicular to V_0 with respect to $< \, , \, >$. In the case of $r = \infty$, φ is said to be *strongly isotropic* or *superminimal.*

If φ has strong isotropy order r, then we define the *first return map* $A_{'}^{FR}$ of φ by

$$A_{'}^{FR} = A_{'}^{V_r, V_0} \circ A_{'}^{V_{r-1}, V_r} \circ \cdots \circ A_{'}^{V_0, V_1} \quad ,$$

where $A_{'}^{V_r, V_0}$ is the composition of the $(1,0)$-part of the second fundamental form $A_{'}^{V_r, V_r^{\perp}}$ and the holomorphic orthogonal projection $V_r^{\perp} \to V_0$. Therefore, $A_{'}^{FR}$ is a holomorphic differential with values in $\mathrm{End}(V_0)$.

Example. If $M = S^2$, i.e. a Riemann sphere, then $A_{'}^{FR}$ is nilpotent. In particular, when the target manifold is a complex projective space $\mathbf{C}P^{n-1}$ (the case of $k = 1$), any harmonic map $S^2 \longrightarrow \mathbf{C}P^{n-1}$ is superminimal. For $k \geq 2$, the procedure called the *forward replacement* or *backward replacement* is useful for classifying harmonic maps of S^2 into $G_k(\mathbf{C}^n)$ (see [B-W], [Wo], [W]). On the other hand, certainly there are some examples where $A_{'}^{FR}$ is not nilpotent. For example, consider a Clifford torus which is total really and minimally immersed in $\mathbf{C}P^2$. Then, its harmonic sequence is periodic with strong isotropy order two.

2. Primitive harmonic maps of finite type

Let G be a compact semisimple Lie group. Let $N = G/K$ be a reductive homogeneous space. We have the reductive decomposition of Lie algebra $\mathcal{G}$ of G as follows :

$$\mathcal{G} = \mathcal{K} + \mathcal{M} \ , \qquad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}$$

where $\mathcal{K}$ is the Lie algebra of K and $\mathcal{M}$ is identified with the tangent space of N at the base point. Suppose that there is an (inner) automorphism $\tau : \mathcal{G} \longrightarrow \mathcal{G}$ of order k with fixed set $\mathcal{K}$. Set $\zeta = \exp(2\pi\sqrt{-1}/k)$. Then, the complexification $\mathcal{G}^{\mathbf{C}}$ of $\mathcal{G}$ is decomposed as

$$\mathcal{G}^{\mathbf{C}} = \sum_{i \in \mathbf{Z}_k} \mathcal{G}_i \tag{2.1}$$

where $\mathcal{G}_i$ is the ζ^i-eigenspace of τ. Moreover, we have

$$\begin{aligned} \mathcal{M}^{\mathbf{C}} &= \sum_{i=1}^{k-1} \mathcal{G}_i \ , \qquad \mathcal{K}^{\mathbf{C}} = \mathcal{G}_0 \ . \\ \overline{\mathcal{G}_i} &= \mathcal{G}_{-i} \ , \qquad [\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j} \ . \end{aligned} \tag{2.2}$$

Let $o \in N$ be the base point. Suppose that τ exponentiates to give an order k automorphism of G, which is also denoted by τ. Define $\hat{\tau} : N \longrightarrow N$ by $\hat{\tau}(g \cdot o) = \tau(g) \cdot o$ for $g \in G$. For $x = g \cdot o$, define $\hat{\tau}_x : N \longrightarrow N$ by $\hat{\tau}_x = g \circ \hat{\tau} \circ g^{-1}$. Then, each $\hat{\tau}_x$ is a diffeomorphism of order k of N with isolated fixed point x and we may use the Killing form of $\mathcal{G}$ to equip N with a biinvariant metric for which each $\hat{\tau}_x$ is an isometry. Therefore, N is a k-symmetric space in the sense of Kowalski[K]. The map $\mathcal{G} \longrightarrow T_xN$ given by $\xi \longmapsto \frac{d}{dt}\big|_{t=0} \exp t\xi \cdot x$ restricts to an isomorphism $Adg\mathcal{M} \longrightarrow T_xN$. We denote the inverse map by $\beta_x : T_xN \longrightarrow Adg\mathcal{M} \subset \mathcal{G}$ and we may regard β as a $\mathcal{G}$-valued 1-form on N, which is called Maurer-Cartan form of N in [B-R]. We define the bundle $[\mathcal{G}_i]$ by $[\mathcal{G}_i]_x = Adg\mathcal{G}_i$. Then, $[\mathcal{G}_i]$ is a subbundle of the trivial bundle $N \times \mathcal{G}$. Note that $[\mathcal{G}_i]_x$ is a ζ^i-eigenspace of $d\hat{\tau}_x$ at x.

Let $\psi : M \longrightarrow N$ be a smooth map.

Definition([B]). ψ is said to be a *primitive* map if $\psi^*\beta(\partial/\partial z)$ is $[\mathcal{G}_1]$-valued.

In case of $k = 2$, N is a symmetric space and $\mathcal{M}^{\mathbf{C}} = \mathcal{G}_1$, hence any ψ is a primitive map. In case of $k = 3$, since $\mathcal{M}^{\mathbf{C}} = \mathcal{G}_1 + \overline{\mathcal{G}_1}$, we may give N an almost complex structure by declaring that $T^{1,0}N \cong [\mathcal{G}_1]$, hence a primitive map is just an (almost) holomorphic map.

Let $F : M \longrightarrow G$ be a (local) lift of $\psi : M \longrightarrow N$ with projection given by $F \longmapsto F \cdot o$. Such F always exists locally and is called a *framing* of ψ. When G is a matrix group, set $\alpha = F^{-1}dF$. Corresponding to the reductive decomposition $\mathcal{G} = \mathcal{K} + \mathcal{M}$, set $\alpha = \alpha_{\mathcal{M}} + \alpha_{\mathcal{K}}$, and $\alpha_{\mathcal{M}} = \alpha'_{\mathcal{M}} + \alpha''_{\mathcal{M}}$ is a decomposition into (1,0)-form and (0,1)-form, respectively. Then, we have

$$\psi^*\beta' = AdF\alpha'_{\mathcal{M}} \tag{2.3}$$

Using the Maurer-Cartan equation for α, we see that a primitive map is a harmonic map if $k > 2$, because $\mathcal{G}_1 \cap \mathcal{G}_{-1} = \{0\}$ holds when $k > 2$, which is an essential part of this observation (see [B-P]).

We fix an Iwasawa decomposition of $\mathcal{K}^{\mathbf{C}}$:

$$\mathcal{K}^{\mathbf{C}} = \mathcal{K} \oplus \mathcal{B} ,$$

where $\mathcal{B}$ is a solvable subalgebra of $\mathcal{K}^{\mathbf{C}}$. Such a decomposition exists since $\mathcal{K}$ is compact so that $\mathcal{K}^{\mathbf{C}}$ is reductive.

Set

$$\wedge\mathcal{G}_\tau^{\mathbf{C}} = \{\xi : S^1 \longrightarrow \mathcal{G}^{\mathbf{C}} \mid \xi(\zeta\lambda) = \tau\xi(\lambda) \text{ for } \lambda \in S^1\}$$

which is an infinite dimensional Lie algebra. We equip it with the Sobolev H^r-topology for some $r > 1/2$. Let $\wedge\mathcal{G}_\tau$ be the real form

$$\wedge\mathcal{G}_\tau = \{\xi \in \wedge\mathcal{G}_\tau^{\mathbf{C}} \mid \xi : S^1 \longrightarrow \mathcal{G}\}$$

and define a complementary subalgebra by

$$\wedge_+\mathcal{G}_\tau^{\mathbf{C}} = \{\xi \in \wedge\mathcal{G}_\tau^{\mathbf{C}} \mid \xi \text{ extends holomorphically to } \xi : D \longrightarrow \mathcal{G}^{\mathbf{C}} \text{ and } \xi(0) \in \mathcal{B}\} ,$$

where D is a unit disc. Any element $\xi \in \wedge\mathcal{G}_\tau$ has a Fourier expansion $\xi = \sum \xi_n \lambda^n$. Define a finite dimensional subspace $\wedge_d$ as follows :

$$\wedge_d = \{\xi \in \wedge\mathcal{G}_\tau \quad | \quad \xi_n = 0 \quad \text{for all} \quad | n |> d\} \ .$$

Let $d \equiv 1 \bmod k$. Then, $\xi_d \in \mathcal{G}_1$ and $\xi_{d-1} \in \mathcal{K}^{\mathbf{C}}$. Let $\mathcal{T}$ be the given maximal torus in $\mathcal{K}$ and $\mathcal{N}$ the nilpotent subalgebra given by the positive root spaces and set $\mathcal{H} = \mathcal{T}^{\mathbf{C}}$. Then, we have

$$\mathcal{K}^{\mathbf{C}} = \mathcal{N} \oplus \mathcal{H} \oplus \overline{\mathcal{N}} \ , \quad \mathcal{B} = (\sqrt{-1}\mathcal{T}) \oplus \mathcal{N} \ .$$

Any element $\eta \in \mathcal{K}^{\mathbf{C}}$ may be written as $\eta = \eta_{\mathcal{N}} + \eta_{\mathcal{H}} + \eta_{\overline{\mathcal{N}}}$. Define a map $r : \mathcal{K}^{\mathbf{C}} \longrightarrow \mathcal{K}^{\mathbf{C}}$ by

$$r(\eta) = \eta_{\overline{\mathcal{N}}} + \frac{1}{2}\eta_{\mathcal{H}}$$

(see Section 2.4 in [B-P]). Now, take a $\xi_0 \in \wedge_d$ and solve the differential equation

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda\xi_d + r(\xi_{d-1})] \ ; \qquad \xi(0) = \xi_0 \ . \tag{2.4}$$

Then, there is a primitive harmonic map $\psi : \mathbf{R}^2 \longrightarrow N$ with framing $F : \mathbf{R}^2 \longrightarrow G$ satisfying $F^{-1}\partial F/\partial z = \xi_d + r(\xi_{d-1})$. Alternatively, define $a : \mathbf{R}^2 \longrightarrow \wedge G_\tau$ $, b : \mathbf{R}^2 \longrightarrow \wedge_+ G_\tau^{\mathbf{C}}$ by

$$\exp(\bar{z}\lambda^{d-1}\xi_0) = a(z)b(z) \quad .$$

Then, $\psi = \pi \circ (a \mid_{\lambda=1})$ is a primitive harmonic map, where $\pi : G \to N$ is the coset projection.

Definition. A primitive harmonic map ψ obtained by solving the equation (2.4) is said to be of *finite type*.

In this case, it is observed that $\xi_d = \alpha'_{\mathcal{M}}(\partial/\partial z)$ takes values in a single $AdK^{\mathbf{C}}$-orbit in $\mathcal{G}_1$. For a primitive harmonic map ψ of two-torus, this condition is almost sufficient to prove that ψ is of finite type. In fact, one needs an additional condition that the orbit is semisimple. The last condition may be replaced by more useful condition, that is

Theorem 2.1 ([B-F-P-P], [B-P], [B]). *Let $\psi : T^2 \longrightarrow N$ be a primitive harmonic map of a two-torus into a k-symmetric space ($k \geq 2$). Suppose that $\psi^*\beta(\partial/\partial z)$ is semisimple on a dense subset of T^2. Then, ψ is of finite type.*

Example. Let $\varphi : T^2 \longrightarrow N$ be a non-conformal harmonic map into a rank one symmetric space of compact type. Then, φ is of finite type (see [B-F-P-P]).

Thus, we work on a question "What kind of harmonic two-tori in complex Grassmann manifolds and quaternionic projective spaces are covered by primitive harmonic maps $\psi : T^2 \longrightarrow G/K$ with $\psi^*\beta'$ being semisimple ?".

Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a non-superminimal harmonic map of a Riemann surface. Suppose that the strong isotropy order of φ is r. Then, we have a harmonic sequence $V_0 \to V_1 \to \cdots \to V_{r-1} \to R$. For notational simplicity, set $V_r = R$. Let k_i be tha rank of V_i for $i = 0, 1, \cdots, r-1$, where $k_0 = k$. Set $G = SU(n)$. Let $N = SU(n)/S(U(k_0) \times \cdots \times U(k_{r-1}) \times U(n - \sum_{i=0}^{r-1} k_i))$ be the flag manifold. Any point x of N may be expressed as $x = (w_0, w_1, \cdots, w_r)$, where w_i is a k_i-plane for $i = 0, 1, \cdots, r-1$ and w_r is a $(n - \sum_{i=0}^{r-1} k_i)$-plane. Let $p : N \longrightarrow G_k(\mathbf{C}^n)$ be the projection map which assigns to the flag its first element ; $p(w_0, w_1, \cdots, w_r) = w_0$.

Fix any point $x = (w_0, w_1, \cdots, w_r) \in N$ and define $Q \in G$ by

$$Q = \zeta^i \qquad \text{on} \quad w_i \quad \text{for} \quad i = 0, \cdots, r$$

where $\zeta = \exp(2\pi\sqrt{-1}/r+1)$. Then, $\tau = AdQ$ is an order $(r+1)$-automorphism of G and the identity component of its fixed set is $S(U(k_0) \times \cdots \times U(k_{r-1}) \times U(n - \sum_{i=0}^{r-1} k_i))$, which we denote by K. Thus, $N = G/K$ becomes an $(r+1)$-symmetric space. We define a map $\psi : M \longrightarrow N = G/K$ by

$$\psi(x) = ((V_0)_x, (V_1)_x, \cdots, (V_r)_x), \qquad \text{for} \quad x \in M \quad .$$

Then, $\varphi = p \circ \psi$. We have the following :

Proposition 2.1 ([U1]). *$A_{\prime}^{V_i, V_i^\perp}$ is $[\mathcal{G}_1]$-valued for $i = 0, \cdots, r$. Moreover, ψ is a primitive harmonic map.*

In fact, take $\psi(x)(x \in M)$ as the base point of N. Then, we see that $AdQ(A_{\prime}^{V_i, V_i^\perp}) = \zeta A_{\prime}^{V_i, V_i^\perp}$. Moreover, since

$$\psi^*\beta(\partial/\partial z) = \sum_{i=0}^{r} A_{\prime}^{V_i, V_i^\perp} \quad ,$$

we find that ψ is a primitive harmonic map. Therefore, it remains to know the answer to the question "When is $\psi^*\beta'$ semisimple ?".

3. Statement of results

Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a non-superminimal harmonic map with strong isotropy order r. Let $\psi : M \longrightarrow N = G/K$ be a primitive harmonic map obtained by lifting φ to an $(r+1)$-symmetric space G/K as in Section 2. We define the first return map $A_\prime^{FR}$ of φ as in Section 1. Then, our idea to settle down the question raised at the end of the previous section is to link the semisimplicity of $A_\prime^{FR}$ with the semisimplicity of $\psi^*\beta'$. In fact, we have a following answer :

Lemma 3.1 ([U1]). *If $A_\prime^{FR}$ is semisimple and invertible, then $\psi^*\beta(\partial/\partial z)$ is semisimple.*

By Theorem 2.1, Proposition 2.1 and Lemma 3.1, we have the following theorem :

Theorem 3.1 ([U1]). *Let $\varphi : T^2 \longrightarrow G_k(\mathbf{C}^n)$ be a harmonic map. If the first return map $A_\prime^{FR}$ for φ is semisimple and invertible on a dense subset of T^2, then φ is covered by a primitive harmonic map of finite type into $SU(n)/S(U(k) \times \cdots \times U(k) \times U(n-rk))$, where r is the strong isotropy order of φ.*

Using Theorem 3.1, we may obtain some answers to the problems of constructing harmonic two-tori in complex Grassmann manifolds and quaternionic projective spaces. Before stating our resuls, we give some definitions :

Definition. Let φ be a harmonic map with harmonic sequence $\{V_i\}$ of the bundles, where $V_0 = V(\varphi)$.

(1) If $\mathrm{rank}V_1 = \mathrm{rank}V_0$, then V_0 is obtained from V_1 by $V_0 = \mathrm{Im}A_{\prime\prime}^{V_1,V_1^\perp}$. In general, if $\mathrm{rank}V_i = \mathrm{rank}V_0$, then V_0 is obtained from V_i by the successive applications of this procedure. In this case, we say that V_0 is obtained from V_i by the *flag transforms.*

(2) If $\mathrm{rank}V_1 < \mathrm{rank}V_0$, then there is a rank s anti-holomorphic subbundle F of $(V_1 \oplus \mathrm{Im}A_{\prime\prime}^{V_1,V_1^\perp})^\perp$, where $s = \mathrm{rank}V_0 - \mathrm{rank}V_1$, such that $V_0 = F \oplus \mathrm{Im}A_{\prime\prime}^{V_1,V_1^\perp}$. Conversely, any harmonic map φ with reducible $A_\prime^{V_0,V_0^\perp}$ is constructed from a harmonic map ψ with $V_1 = V(\psi)$ in this way. In this case, we say that φ is obtained from ψ by the *extension.*

For example, if there is a positive integer k such that $\mathrm{rank}V_i = \mathrm{rank}V_0$ for $i = 1, \cdots, k-1$ and $\mathrm{rank}V_k < \mathrm{rank}V_0$, then V_{k-1} is obtained from V_k by the extension and V_0 is obtained from V_{k-1} by the $(k-1)$-times flag transforms.

Theorem 3.2 ([U1]). *Let $\varphi : T^2 \longrightarrow G_2(\mathbf{C}^4)$ be a weakly conformal non-superminimal harmonic map. Then, either φ is constructed from a harmonic map into $\mathbf{C}P^3$ by extension and flag transforms or φ is of finite type.*

Theorem 3.3 ([U1]). *Let $\varphi : T^2 \longrightarrow G_2(\mathbf{C}^{2n})$ be a harmonic map with strong isotropy order $n-1$. If φ has isotropy order $\geq n$, then either φ is constructed from a harmonic map into $\mathbf{C}P^{2n-1}$ by extension and flag transforms or φ is of finite type.*

Example. Suppose that $d = 1$ and $\xi_0 = \lambda^{-1}\eta_{-1} + \lambda\eta_1$, where $\eta_1 = \begin{pmatrix} \mathbf{0}_2 & J \\ \sqrt{-1}I_2 & \mathbf{0}_2 \end{pmatrix}$ with $J = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. Then, $\eta_{-1} = {}^t\eta_1$ and $[\eta_1, \eta_{-1}] = 0$. Thus, $F = \exp(z\eta_1)\exp(\bar{z}\eta_{-1})$ and $\varphi = \pi \circ F$ is a weakly conformal non-superminimal harmonic map of a square torus into $G_2(\mathbf{C}^4)$.

Next, let $\eta_1 = \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & J \\ \sqrt{-1}I_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \sqrt{-1}I_2 & \mathbf{0}_2 \end{pmatrix}$. Then, $\eta_{-1} = {}^t\eta_1$ and $[\eta_1, \eta_{-1}] = 0$. Thus, $F = \exp(z\eta_1)\exp(\bar{z}\eta_{-1})$ and $\varphi = p \circ \pi \circ F$ is a non-superminimal harmonic map of a square torus into $G_2(\mathbf{C}^6)$ which has strong isotropy order 2 and isotropy order ≥ 3.

For harmonic tori in quaternionic projective space $\mathbf{H}P^n$, we may apply Theorem 3.1 and obtain some results. Let $J : \mathbf{C}^{2n} \longrightarrow \mathbf{C}^{2n}$ be the conjugate linear map given by left multiplication by a unit quaternion, where we use an identification $\mathbf{C}^{2n} \cong \mathbf{H}^n$. We regard $\mathbf{H}P^{n-1}$ as the totally geodesic submanifold of $G_2(\mathbf{C}^{2n})$ as follows :

$$\mathbf{H}P^{n-1} = \{V \in G_2(\mathbf{C}^{2n}) \mid V = JV\},$$

that is, the set of all complex 2-dimensional subspaces of $\mathbf{C}^{2n}$ which are closed under the action J.

Definition. Let $\varphi : M \longrightarrow \mathbf{H}P^{n-1} \subset G_2(\mathbf{C}^{2n})$ be a harmonic map. We say that φ is a *quaternionic pair by the flag transforms* of ψ if there is a harmonic map $\psi : M \longrightarrow \mathbf{C}P^{2n-1}$

and an integer k such that

$$V(\varphi) = V_k(\psi) \oplus JV_k(\psi)\,.$$

Now, our results are the following :

Theorem 3.4 ([U2]). *Let $\varphi : T^2 \longrightarrow \mathbf{H}P^2 \subset G_2(\mathbf{C}^6)$ be a non-superminimal harmonic map of a two-torus. Then, either φ is a quaternionic pair by the flag transforms of a harmonic map into $\mathbf{C}P^5$, or φ is covered by a primitive harmonic map of finite type into $\mathbf{H}P^2$ or $Sp(3)/Sp(1)\times U(2)$ according as the isotropy order of φ is one or two, respectively, where $U(2)$ is embedded in $Sp(2)$ by $A \to \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$.*

Theorem 3.5 ([U2]). *Let $\varphi : T^2 \longrightarrow \mathbf{H}P^3 \subset G_2(\mathbf{C}^8)$ be a non-superminimal harmonic map of a two-torus. If the strong isotropy order of φ is odd, then φ is covered by a primitive harmonic map of finite type into $\mathbf{H}P^3$ or $Sp(4)/Sp(1) \times Sp(1) \times U(2)$ according as the strong isotropy order of φ is one or three. If the strong isotropy order of φ is even, then φ is obtained by either of the following methods :*

(1) If $\det(A^{FR}) \neq 0$, then φ is covered by a primitive harmonic map of finite type into $Sp(4)/Sp(1) \times Sp(1) \times U(2)$.

(2) If $\det(A^{FR}) \equiv 0$, then either φ is a quaternionic pair by the flag transforms of a harmonic map into $\mathbf{C}P^7$, or φ is obtained from $\varphi_1 : T^2 \longrightarrow G_2(\mathbf{C}^8)$, which has strong isotropy order 3 and satisfies $V_{-1}(\varphi_1) = JV(\varphi_1)$, by the backward replacement. Moreover, φ_1 is obtained by either of the following methods : (2-1) φ_1 is covered by a primitive harmonic map of finite type into $SU(8)/S(U(2)\times U(2)\times U(2)\times U(2))$, (2-2) φ_1 is obtained by the forward replacement from some φ_2, which is quaternionic and has strong isotropy order 3 and is covered by a primitive harmonic map into $Sp(4)/Sp(1)\times Sp(1)\times U(2)$, (2-3) φ_1 is obtained from a harmonic map into $\mathbf{C}P^7$ by the extension and flag transforms.

Remark. Any non-superminimal harmonic map $\varphi : T^2 \longrightarrow \mathbf{H}P^{n-1}$ with odd isotropy order has the first return map of the following form : $A_l^{FR} = aI_2$ with a non-zero on a dense subset of T^2. Hence, Theorem 3.1 implies that φ is of finite type.

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