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Decay Results for Solutions to the Magneto-Hydrodynamics equations

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In this paper we describe the long-time behavior of the solutions to the viscous Magneto-Hydrodynamic equations. We will only mention the results obtained. The details of the proofs and corresponding references can be found in [3].

We study the Magneto-Hydrodynamic equations

$$
\text{(MHD)} \qquad \begin{aligned}
\frac{\partial}{\partial t}u + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla p &= \Delta u + f, \\
\frac{\partial}{\partial t}B + (u\cdot\nabla)B - (B\cdot\nabla)u &= \Delta B, \\
\nabla\cdot u = 0, \quad \nabla\cdot B &= 0,
\end{aligned}
$$

supplemented with the initial conditions

$$
u(x,0) = u_0(x), \quad B(x,0) = B_0(x),
$$

for $x \in \mathbf{R}^n$, $2 \le n \le 4$, $t \ge 0$. We assume that the forcing function f is divergence free; i.e., that $\nabla\cdot f(t) = 0$ for all $t \ge 0$. We show that solutions to the MHD equations, unlike solutions to the underlying heat equation, will

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generically decay algebraically at a slow rate of $(t+1)^{-\alpha}$,where $\alpha = n/2$ or $n/2-1$, with n equal to the spatial dimension. The idea is that the nonlinear terms will produce slow modes which will launch long waves preventing a too rapid decay. In other words, even if the data is highly oscillatory, the convection term will produce some mixing of the modes which will introduce long waves slowing down the decay.

More precisely, we show that weak solutions to the MHD equations, subject to large initial data outside a class of functions with total radially equidistributed energy, decay algebraically (rather than exponentially). In particular we prove that, for such solutions, the total energy (kinetic plus magnetic) and the magnetic energy have slowly decaying algebraic lower bounds. Moreover, we show in which cases the lower bounds are valid for the kinetic energy alone or the magnetic energy alone. Thus, our results reinforce mathematically the observation made by Chandrasekhar [1] that "the magnetic field in systems of large linear dimensions can endure for relatively long periods of time".

We study upper and lower bounds. For the upper bounds we improve the results obtained by [2]. We show that

THEOREM [A]. *Let the initial datum* $(u_0, B_0) \in H \times H$ *and assume* $f \in L^1(0,\infty; L^2(\mathbf{R}^n)^n) \cap A_{n/4+1} \cap B_4 \cap C_{(n+3)/2}$.

a) *If* $n = 2$ *and not all components of* $\hat{u}_0(0)$ *or* $\hat{B}_0(0)$ *are zero (in the sense defined at the beginning of Section 2), then*

$$\|u(t) - u_0(t)\|_2^2 + \|B(t) - B_0(t)\|_2^2 \le C_{D_0}(t+1)^{-n/2-1/2};$$

b) *if* $n \ge 3$, *or if* $n = 2$ *and* $(u_0, B_0) \in [H \cap L^1(\mathbf{R}^n)^n]^2$, *then*

$$\|u(t) - u_0(t)\|_2^2 + \|B(t) - B_0(t)\|_2^2 \le C_{D_1}(t+1)^{-n/2-1}.$$

For the lower bounds we show

THEOREM [B]. *Let* $(u_0, B_0) \in [W_2 \cap H]^2$, $f \in L^1(0,\infty; L^2(\mathbf{R}^n)^n) \cap A_{n/4+1} \cap B_4 \cap C_{(n+3)/2}$, *and let* $(u(x,t), B(x,t))$ *be a weak solution of the MHD equations with initial datum* $(u(x,0), B(x,0)) = (u_0(x), B_0(x))$.

a) If $\hat{u}_0(0) \neq 0$ or $\hat{B}_0(0) \neq 0$, *then*

$$C_0(t+1)^{-n/2} \le \|u(\cdot,t)\|_2^2 + \|B(t)\|_2^2 \le C_1(t+1)^{-n/2}.$$

b) If $(u_0, B_0) \in [W_2 \cap H \cap [L^1(\mathbf{R}^n)]^n]^2$ *(so that*

$$\hat{u}_0(0) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} u_0(x)\,dx = 0, \hat{B}_0(0) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} B_0(x)\,dx = 0,$$

by Borchers' Lemma) and $(u, B) \notin \mathcal{M}_0$*,then*

$$C_2(t+1)^{-n/2-1} \le \|u(\cdot,t)\|_2^2 \le C_3(t+1)^{-n/2-1}.$$

where Borchers' lemma was
LEMMA. *Let* $u \in L^1(\mathbf{R}^n)^n \cap H$. *Then*

$$\int_{\mathbf{R}^n} u\,dx = 0.$$

We have used the following notation . $H^m(\mathbf{R}^n)$ denotes the Hilbertian Sobolev space on $\mathbf{R}^n$ of index m, $m \ge 0$; $L^p(\mathbf{R}^n)$ stands for the Lebesgue space equipped with its standard norm $\|\cdot\|_p$, $1 \le p \le \infty$;

$$\mathcal{V} = \{v \in [C_0^\infty(\mathbf{R}^n)]^n \,:\, \nabla \cdot u = 0\}, \quad H = \text{closure of } \mathcal{V} \text{ in } [L^2(\mathbf{R}^n)]^n.$$

$$W_1 = \{v \,:\, \int_{\mathbf{R}^n} |x|^2 |v(x)| dx < \infty\}, \quad W_2 = \{v \,:\, \int_{\mathbf{R}^n} |x||v(x)|^2 dx < \infty\},$$

Moreover, if $\mu, \nu, \sigma \in \mathbf{R}$, we say

a $f \in A_\mu$ if there exists $C \ge 0$ such that

$$\|f(t)\|_2 \le C(t+1)^{-\mu} \quad \text{for} \quad t \ge 0.$$

b $f \in B_\sigma$ if there exists $C \ge 0$ such that

$$|\hat{f}(\xi,t)| \le C|\xi|^\sigma \quad \text{for} \quad t \ge 0, \xi \in \mathbf{R}^n.$$

c $f \in C_\nu$ if there exists $C \ge 0$ such that

$$\|f(t)\|_\infty \le C(t+1)^{-\nu} \quad \text{for} \quad t \ge 0.$$

Finally, gGiven $u = (u_1, ..., u_n)$ and $B = (B_1, ..., B_n)$ in $[L^1(0,\infty; L^2(\mathbf{R}^n)]^n$, let

$$\tilde{\mathcal{A}}_{ij} = \int_0^\infty \int_{\mathbf{R}^n} (u_i u_j - B_i B_j) dx,$$

$$\tilde{\mathcal{C}}_{ij} = \int_0^\infty \int_{\mathbf{R}^n} (u_i B_j - B_i u_j) dx.$$

Then, introducing the matrices $\tilde{\mathcal{A}} = [\tilde{\mathcal{A}}_{ij}]$ and $\tilde{\mathcal{C}} = [\tilde{\mathcal{C}}_{ij}]$, we define

$$\mathcal{M}_0 = \{(u, B) \in [L^2(\mathbf{R}^n)]^{2n} \ : \ \tilde{\mathcal{A}} \text{is scalar and } \tilde{\mathcal{C}} = 0\}.$$

The idea of the proof is to compare the decay of the solutions to MHD with the decay of the solution to the underlying heat system. It is first shown that solutions to the heat equation such that the Fourier transform of the initial data has a zero of order k at the origin, decay at most like $(t+1)^{-k/2-n/4}$; that is, if

$$|\hat{u}_0(\xi)| = C_0|\xi|^k + O(|\xi|^{k+1}),$$

then the solution v of $v_t = \Delta v$, $v(x,0) = u_0(x)$ satisfies

(1) $$C_0(t+1)-n/2-k \leq \|v(\cdot,t)\|_2^2 \leq C_0(t+1)-n/2-k$$

for $t \geq 0$. By an argument similar to Wiegner's for solutions to the Navier-Stokes equations, it is easy to show that if V satisfies $V_t = \Delta V$, $V(x,0) = (u_0(x), B_0(x))$ and $W = V - (u, B($, then

(2) $$\|W(t)\|_2^2 \leq C_W(t+1)^{-n/2-/b}$$

where $\beta = 1/2$ or $\beta = 1$ depending on whether $n = 2$ or $n \geq 3$, respectively. Thus, for the casein which $\int_{\mathbf{R}^n} u_0\, dx \neq 0$; i.e., $\hat{u}_0(0) \neq 0$, it is immediate from (1) and (2) that

$$\|u(t)\|_2^2 + \|B(t\|_2^2 \geq C(t+1)^{-n/2}.$$

When the Fourier transform of the initial data has a zero of higher order the conditions from the theorem are such that a long wave is always present. That is, we can compare the solution to the MHD equations with data at a large time T with the the corresponding solution V_T to the heat system with data at time T. In this case, because of the long wave, it follows that

$$\|V_T(\cdot,t)\|_2^2 \geq C_T(t+1)^{-n/2-1}.$$

Setting $W_T(\cdot,t) = V_T(\cdot,t) - (u(\cdot,t+T), B(\cdot,t+T))$, we'll have our lower bounds if we can show that $C_T \geq C_{W_T}$. Since C_{W_T} will depend on the L^2 norm of the data $(u(T), B(T))$, T sufficiently large, it will follow that C_{W_T} is as small as needed. Combining once more the lower bound of V_T with the upper bound of W_T, we get

$$\|u(t)\|_2^2 + \|B(t\|_2^2 \geq C(t+1)^{-n/2-1}.$$

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