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Estimates of Convergence Rates for Approximate Solutions of Stochastic Differential Equations

Shuya KANAGAWA

Department of Mathematics, Faculty of Liberal Arts & Education,
Yamanashi University, 4-4-37, Takeda, Kôfu 400, JAPAN,
e-mail: sgk02122@niftyserve.or.jp

Abstract

We estimate the error of the Euler-Maruyama type approximate solutions for Itô's stochastic differential equations using the K-M-T inequality. The obtained result can be applied to Monte Carlo simulations of stochastic differential equations.

Key words

Euler-Maruyama scheme, Stocahstic differential equation, Monte Carlo simulation, Pseudo-random numbers, K-M-T inequality

1. INTRODUCTION AND RESULTS

Let $\{B(t), 0\le t\le 1\}$ be an r-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Consider Itô's stochastic differential equation for a d-dimensional continuous process $\{X(t), 0\le t\le 1\}$ $(d\ge 1)$:

$$(1.1)\qquad \begin{cases} dX(t)=\sigma(t,X(t))dB(t)+b(t,X(t))dt, \quad 0\le t\le 1 \\ X(0)=X_0, \end{cases}$$

where $\sigma(t,x)$ is a Borel measurable function $(t,x)\in[0,1]\times\mathbf{R}^d$

$\to \mathbf{R}^d \otimes \mathbf{R}^r$ and $b(t,x)$ is a Borel measurable function $(t,x) \in [0,1]\times\mathbf{R}^d \to \mathbf{R}^d$. If $\sigma(t,x)$ and $b(t,x)$ satisfy the Lipschitz condition, then their exists a unique solution of (1.1). To prove the existence of the unique solution of (1.1) Maruyama [8] constructed an Euler type approximate solution $Z_n := \{Z_n(t),\ 0\le t\le 1\}$ defined by

$$Z_n(t) := X_0 + \int_0^t \sigma_n(u)dB(u) + \int_0^t b_n(u)du, \quad 0\le t\le 1,$$

where

$$\sigma_n(t) := \sigma_n\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \le t \le (k+1)/n, \quad k=0,\cdots,n-1,$$

$$b_n(t) := b_n\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \le t \le (k+1)/n, \quad k=0,\cdots,n-1,$$

$$x_k := X_0 + \sum_{j=1}^{k} \sigma\left(\frac{j-1}{n}, x_{j-1}\right)\eta_j + \sum_{j=1}^{k} b\left(\frac{j-1}{n}, x_{j-1}\right)\Big/ n, \quad k=0,1,\cdots,n,$$

$$\eta_k := B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right), \quad k=1,\cdots,n..$$

Applying the approximate solution $Z_n := \{Z_n(t), 0\le t\le 1\}$ to Monte Carlo simulations on digital computers we must discretize it as following. Let $X_n := \{X_n(t), 0\le t\le 1\}$ be a stochastic process in $D[0,1]$ defined by

$$\begin{cases} X_n(t) := x_k, & k/n \le t < (k+1)/n, \quad k=0.\cdots,n-1 \\ X_n(1) := x_n, & \end{cases}$$

where $\{\eta_k\}$ are i.i.d. random variables with the r-dimensional normal distribution $N(0,1/n)$ which are realized by pseudo-normal random numbers. As for the error estimation for X_n and Z_n, Ghiman-Skorohod [1] and Kanagawa [2]-[4] showed the rate of convergence of them to the real solution X of (1.1) in L^p-mean for some $p \geq 2$. Ogawa [7] estimated the error of Euler-Maruyama approximate solutions of some nonlinear diffusion processes governed by Itô's stochastic differential equation. For more details of other types of approximate solutions, see also Kloeden-Platen [5].

Since pseudo-uniform random numbers are generated by some algebraic algorithms, it is obvious that the distribution of them is not the real uniform distribution. Thus pseudo-normal random numbers, which are generated from pseudo-uniform random numbers by some methods, e.g. the Box-Muller method, do not obey the real normal distribution, too. Furthermore, as for the speed of computation by digital computers, approximate solutions constructed from pseudo-uniform random numbers have advantage over the construction from pseudo-normal random numbers. Therefore we shall estimate the convergence rate of the approximate solutions to the real solution of (1.1) when the distribution of $\{\eta_k\}$ is not the normal distribution. In [3] the rate of convergence is considered assuming the existence of the third absolute moment for $\{\eta_k\}$. Let $\{\xi_k\}$ be r-dimensional i.i.d. random variables which do not always obey the normal distribution and define an approximate solution $\{Y_n(t), 0 \leq t \leq 1\}$ by

$$\begin{cases} Y_n(t) := y_k, & k/n \leq t < (k+1)/n, \quad k=0,\cdots,n-1 \\ Y_n(1) := y_n, \end{cases}$$

here

$$y_k := X_0 + \sum_{j=1}^{k} \sigma\left(\frac{j-1}{n}, y_{j-1}\right)\xi_j / \sqrt{n} + \sum_{j=1}^{k} b\left(\frac{j-1}{n}, y_{j-1}\right)\Big/ n, \quad k=0,1,\cdots,n.$$

Theorem A. *([3]) Let* $\{\xi_k, k \geq 1\}$ *be* r-dimensional *i.i.d. random variables with*

$$\text{(1.2)} \qquad \mathrm{E}(\xi_1) = 0, \mathrm{E}\left(|\xi_1|^2\right) = 1 \text{ and } \mathrm{E}\left(|\xi_1|^3\right) < \infty.$$

Suppose that for any $0 \le s, t \le 1$ *and* $x, y \in \mathbf{R}^d$

$$(1.3)\qquad |\sigma(t,x)-\sigma(s,y)|^2+|b(t,x)-b(s,y)|^2 \le K_1\left(|x-y|^2+|t-s|^2\right),$$

$$(1.4)\qquad |\sigma(t,x)|^2+|b(s,y)|^2 \le K_2,$$

where K_1 *and* K_2 *are some positive constants independent of* s, t, x *and* y. *Then we can redefine* $\{X(t), 0\le t\le 1\}$ *and* $\{Y_n(t), 0\le t\le 1\}$ *on a common probability space such that for any* $p \ge 2$ *and* $\varepsilon > (2+\delta)^2/2(3+\delta)$,

$$(1.5)\qquad \mathrm{E}\left(\max_{0\le t\le 1}|X(t)-Y_n(t)|^p\right)=o\left(n^{-p/6}(\log n)^{\varepsilon}\right)\quad as\, n\to\infty,$$

where the power of n *cannot be improved by better one.*

The aim of this paper is to improve the above result under the Cramér condition which is satisfied several types of distributions, e.g. the uniform distribution.

Theorem 1. *Let* $\{\xi_k, k\ge 1\}$ *be* r-dimensional *i.i.d. random variables with zero mean and finite variance. Moreover let* $\{\xi_k, k\ge 1\}$ *satisfy the Cramér condition;*

$$(1.6)\qquad E(\exp(s|\xi_1|))<\infty \text{ in a neighborhood of } s=0.$$

Assume $\sigma(t,x)$ *and* $b(t,x)$ *satisfy (1.3) and (1.4). Then we can redefine* $\{X(t), 0\le t\le 1\}$ *and* $\{Y_n(t), 0\le t\le 1\}$ *on a common probability space such that for any* $p \ge 2$ *and* $\varepsilon > p$,

$$(1.7)\qquad \mathrm{E}\left(\max_{0\le t\le 1}|X(t)-Y_n(t)|^p\right)=o\left(n^{-p/4}(\log n)^{\varepsilon}\right)\quad as\, n\to\infty.$$

2. PRELIMINARIES

Before proving the theorem we define two random processes $\{\overline{X}_n(t),\ 0\le t\le 1\}$ and $\{\overline{Y}_n(t),\ 0\le t\le 1\}$ as follows. Let $\{\zeta_k,\ 1\le k\le M\}$ and $\{\eta_k,\ 1\le k\le M\}$ be random variables defined by

$$\zeta_k := \sum_{i=(k-1)[n^{1/2}]+1}^{k[n^{1/2}]} \frac{\xi_i}{\sqrt{n}},\ 1\le k\le M-1,\quad \zeta_M := \sum_{i=(M-1)[n^{1/2}]+1}^{n} \frac{\xi_i}{\sqrt{n}}$$

$$\eta_k := B\left(k[n^{1/2}]/n\right)-B\left(((k-1)[n^{1/2}]+1)/n\right), 1\le k\le M-1,$$

and

$$\eta_M := B(1)-B\left(((M-1)[n^{1/2}]+1)/n\right),$$

where $M := [n/[n^{1/2}]]+1$. Define $\{\overline{X}_n(t),\ 0\le t\le 1\}$ and $\{\overline{Y}_n(t),\ 0\le t\le 1\}$ by

$$\begin{cases} \overline{X}_n(t) := u_k, & ((k-1)[n^{1/2}]+1)/n\le t< k[n^{1/2}]/n, \quad 1\le k\le M-1 \\ \overline{X}_n(1) := u_M, & \end{cases}$$

$$\begin{cases} \overline{Y}_n(t) := v_k, & ((k-1)[n^{1/2}]+1)/n\le t< k[n^{1/2}]/n, \quad 1\le k\le M-1 \\ \overline{Y}_n(1) := v_M, & \end{cases}$$

where

$$u_k := X_0+\sum_{j=1}^{k}\sigma\left((j-1)[n^{1/2}]/n, u_{j-1}\right)\eta_j+\sum_{j=1}^{k} b\left((j-1)[n^{1/2}]/n, u_{j-1}\right)[n^{1/2}]/n, 1\le k\le M,$$

$$v_k := X_0+\sum_{j=1}^{k}\sigma\left((j-1)[n^{1/2}]/n, v_{j-1}\right)\zeta_j+\sum_{j=1}^{k} b\left((j-1)[n^{1/2}]/n, v_{j-1}\right)[n^{1/2}]/n, 1\le k\le M.$$

The following result, which is called the K-M-T inequality obtained by Komlós-Major-Tusnády [6], plays an important roll to estimate $E\left(|\eta_1-\zeta_1|^2\right)$.

Lemma 1. *Given the condition (1.6), there exist a Brownian motion* $\{B(t), 0\le t\le 1\}$ *and* $\{\xi_k, k\ge 1\}$ *on a common probability space such that for all* *real x and every n we have*

$$(2.1)\qquad P\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k}\xi_k - B(k)\right|\ge C\log n + x\right\} < Ke^{-\lambda x},$$

where C, K, λ are positive constants depending only on the distribution function of ξ_1.

Using the K-M-T inequality we have the next lemma.

Lemma 2. *Without changing distributions of* $\{\xi_k, 1\le k\le n\}$ *and* $\{\zeta_k, 1\le k\le M\}$, *we can redefine them on a richer probability space with a Brownian motion* $\{B(t), 0\le t\le 1\}$ *with the increments* $\{\eta_k, 1\le k\le M\}$ *such that for each* $1\le k\le M$ *and for any* $\varepsilon > 2$

$$(2.2)\qquad E\left(|\zeta_k-\eta_k|^2\right) = o\left(n^{-1}(\log n)^{\varepsilon}\right)\ \ as\ \ n\to\infty,$$

$$(2.3)\qquad \{\eta_1,\cdots,\eta_k,\xi_1,\cdots,\xi_k\}\ \text{is independent of}\ \{\eta_{k+1},\cdots,\eta_M,\xi_{k+1},\cdots,\xi_M\}.$$

Proof. By (2.1) we have

$$nE\left(|\zeta_1-\eta_1|^2\right) = \int_{\left\{\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B\left([n^{1/n}]/n\right)\right|\ge C\log n+\frac{4}{\lambda}\log n\right\}}\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B([n^{1/n}]/n)\right|^2 dP$$

$$+\int_{\left\{\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B\left([n^{1/n}]/n\right)\right|< C\log n+\frac{4}{\lambda}\log n\right\}}\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B([n^{1/n}]/n)\right|^2 dP$$

$$\le \sum_{k=0}^{\infty}\int_{\left\{C\log n+\frac{4}{\lambda}\log n\le\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B\left([n^{1/n}]/n\right)\right|\ge C\log n+\frac{4}{\lambda}\log n+k\right\}}\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B([n^{1/n}]/n)\right|^2 dP$$

$$+\left(C\log n+\frac{4}{\lambda}\log n\right)^2 \boldsymbol{P}\left\{\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B([n^{1/n}]/n)\right| < C\log n+\frac{4}{\lambda}\log n\right\}$$

$$\leq \sum_{k=0}^{\infty}\left(C\log n+\frac{4}{\lambda}\log n+k\right)^2 \boldsymbol{P}\left\{\left|\sum_{i=1}^{[n^{1/n}]}\xi_k - B([n^{1/n}]/n)\right| \geq C\log n+\frac{4}{\lambda}\log n+k\right\}$$

$$+\left(C\log n+\frac{4}{\lambda}\log n\right)^2$$

$$\leq K\sum_{k=0}^{\infty}\left(C\log n+\frac{4}{\lambda}\log n+k\right)^2 \exp\left\{-\lambda\left(\frac{4}{\lambda}\log n+k\right)\right\}+\left(C\log n+\frac{4}{\lambda}\log n\right)^2$$

$$= O\left((\log n)^2\right).$$

Moreover $\boldsymbol{E}\left(\left|\zeta_k-\eta_k\right|^2\right), k\geq 2$, are treated similarly. On the other hand, taking independent copies of (ζ_1,η_1), (2.3) can be shown easily. ❑

3. PROOF OF THEOREM 1

For simplicity we treat the case $d=r=1$. The multidimensional case can be proved similarly. In what follows K's are different positive constants independent of n in different equations. From the definitions of $X_n(t)$ and $Y_n(t)$

$$\text{(3.1)}\quad \boldsymbol{E}\left(\max_{0\leq u\leq t}\left|X(u)-Y_n(u)\right|^p\right)^{1/p} \leq \boldsymbol{E}\left(\max_{0\leq u\leq t}\left|X(u)-\overline{X}_n(u)\right|^p\right)^{1/p}$$

$$+\boldsymbol{E}\left(\max_{0\leq u\leq t}\left|\overline{X}_n(u)-\overline{Y}_n(u)\right|^p\right)^{1/p}+\boldsymbol{E}\left(\max_{0\leq u\leq t}\left|\overline{Y}_n(u)-Y_n(u)\right|^p\right)^{1/p}$$

$$=: I_1^{1/p}+I_2^{1/p}+I_3^{1/p}.$$

We first estimate I_2. For $\frac{k[n^{1/2}]}{n} \le t \le \frac{(k+1)[n^{1/2}]}{n}$, $1 \le k \le M$, we have from (1.3)

$$(3.2) \qquad I_2 \le \boldsymbol{E}\left(\max_{1\le i\le k} |u_i - v_i|^p \right)$$

$$\le K\boldsymbol{E}\left(\max_{1\le i\le k} \left| \sum_{j=1}^{i} \sigma((j-1)[n^{1/2}]/n, u_{j-1})\eta_j - \sum_{j=1}^{i} \sigma((j-1)[n^{1/2}]/n, v_{j-1})\zeta_j \right|^p \right)$$

$$+K\boldsymbol{E}\left(\max_{1\le i\le k} \left| \sum_{j=1}^{i} b((j-1)[n^{1/2}]/n, u_{j-1})[n^{1/2}]/n - \sum_{j=1}^{i} b((j-1)[n^{1/2}]/n, v_{j-1})[n^{1/2}]/n \right|^p \right)$$

$$\le K\boldsymbol{E}\left(\max_{1\le i\le k} \left| \sum_{j=1}^{i} \sigma((j-1)[n^{1/2}]/n, u_{j-1})(\eta_j - \zeta_j) \right|^p \right)$$

$$+K\boldsymbol{E}\left(\max_{1\le i\le k} \left| \sum_{j=1}^{i} (\sigma((j-1)[n^{1/2}]/n, u_{j-1}) - \sigma((j-1)[n^{1/2}]/n, v_{j-1}))\zeta_j \right|^p \right)$$

$$+K\boldsymbol{E}\left(\max_{1\le i\le k} \left| \sum_{j=1}^{i} (b((j-1)[n^{1/2}]/n, u_{j-1}) - b((j-1)[n^{1/2}]/n, v_{j-1}))[n^{1/2}]/n \right|^p \right)$$

$$=: I_{21} + I_{22} + I_{23}.$$

Put

$$S_k := \sum_{j=1}^{k} \sigma((j-1)[n^{1/2}]/n, u_{j-1})(\eta_j - \zeta_j), \quad 1 \le k \le M.$$

Since (2.3) implies that $\{S_k, k \ge 1\}$ is a $\mathcal{F}_k := \sigma\{\eta_1, \cdots, \eta_k, \xi_1, \cdots, \xi_k\}$-adapted martingale, using Doob's inequality and (1.4), we have

$$(3.3)\qquad I_{21}=K\,\boldsymbol{E}\left(\max_{1\le i\le k}|S_i|^p\right)$$

$$\le K\left\{\boldsymbol{E}\left(\sum_{j=1}^{k}\boldsymbol{E}\left(\left|\sigma((j-1)[n^{1/2}]/n,u_{j-1})(\eta_j-\zeta_j)\right|^2\,\middle|\,\mathcal{F}_{j-1}\right)\right)\right\}^{p/2}$$

$$\le K\left\{\sum_{j=1}^{k}\boldsymbol{E}\left(|\eta_j-\zeta_j|^2\right)\right\}^{p/2}.$$

Therefore we obtain from (2.2) and (3.3) that

$$(3.4)\qquad I_{22}\le K\,t^{p/2}n^{-p/4}(\log n)^p.$$

On the other hand by Doob's inequality it is easy to see that

$$(3.5)\qquad I_{21}\le K\int_0^t \boldsymbol{E}\left(\max_{0\le u\le s}|\overline{X}_n(u)-\overline{Y}_n(u)|^p\right)ds,$$

$$(3.6)\qquad I_{23}\le K\int_0^t \boldsymbol{E}\left(\max_{0\le u\le s}|\overline{X}_n(u)-\overline{Y}_n(u)|^p\right)ds.$$

From (3.1)-(3.6), for any $0\le t\le 1$

$$\boldsymbol{E}\left(\max_{0\le u\le t}|\overline{X}_n(u)-\overline{Y}_n(u)|^p\right)\le K\,t^{p/2}n^{-p/4}(\log n)^p+K\int_0^t\boldsymbol{E}\left(\max_{0\le u\le s}|\overline{X}_n(u)-\overline{Y}_n(u)|^p\right)ds.$$

Thus by Gronwall's lemma

$$(3.7)\qquad I_2=o\left(n^{-p/4}(\log n)^{\varepsilon}\right),$$

as $n\to\infty$, for any $\varepsilon>p$. As for I_1 and I_3, from Lemmas 5 and 6 in [3], we have as $n\to\infty$

(3.8) $$I_1 = O(n^{-p/4}) \text{ and } I_3 = O(n^{-p/4}).$$

From (3.1), (3.7) and (3.8) we conclude the proof of the theorem. ❑

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