Uniform convexity, uniform non-squareness and von Neumann-Jordan constant for Banach spaces

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Introduction

In connection with the famous work [7] of Jordan and von Neumann concerning inner products Clarkson [2] introduced the von Neumann-Jordan (NJ-) constant for Banach spaces X. Despite its fundamental nature very little is known on the NJ-constant by now. This note is a résumé of some recent results of the authors [12, 15] on the NJ-constant especially concerning some geometrical properties of Banach spaces such as uniform convexity, uniform non-squareness, and also super-reflexivity.

1. Definitions and prelimainary results

The von Neumann-Jordan constant for a Banach space X ([2]), we denote it by $C_{NJ}(X)$, is defined to be the smallest constant C for which

(1. 1) $$\frac{1}{C} \leq \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

hold for all x, y ∈ X with (x, y) ≠ (0, 0). (Note that the left and right-hand side inequalities in (1. 1) are equivalent; indeed, put $x + y = u$, $x - y = v$). The following facts are easily seen:

A. Proposition. (i) $C_{NJ}(X) = 2^{2/t-1}$, $1 \leq t \leq 2$, if and only if

$$\| A : l_2^2(X) \to l_2^2(X) \| = 2^{1/t},$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $l_2^2(X)$ denotes the X-valued l_2^2-space; and hence

(ii) $C_{NJ}(X') = C_{NJ}(X)$, where X' is the dual space of X. (This was observed for L_p in Clarkson [2].)

Let us recall some classical and recent results in [7], [2], and [10], [9] (see also [11]), where the NJ-constant is calculated for some concrete Banach spaces:

B. Theorem. (i) $1 \leq C_{NJ}(X) \leq 2$ for any Banach space X; and $C_{NJ}(X) = 1$ if and only if X is a Hilbert space (Jordan and von Neumann [7]).

(ii) Let $1 \leq p \leq \infty$. Then, $C_{NJ}(L_p) = 2^{2/t-1}$, where $t = \min\{p, p'\}$, $1/p + 1/p' = 1$ (Clarkson [2]; see also [10]).

(iii) Let $1 \leq p, q \leq \infty$. Then, for $L_p(L_q)$ (L_q-valued L_p-space on arbitrary measure spaces), $C_{NJ}(L_p(L_q)) = 2^{2/t-1}$, where $t = \min\{p, q, p', q'\}$, $1/p + 1/p' = 1/q + 1/q' = 1$; and for the Sobolev space $W_p^k(\Omega)$, $C_{NJ}(W_p^k(\Omega)) = 2^{2/t-1}$, where $t = \min\{p, p'\}$ (Kato and Miyazaki [10]).

(iv) For $E = C_c(K)$ resp. $C_b(K)$ (the spaces of continuous functions on a locally compact Hausdorff space K which have compact support resp. are bounded), $C_{NJ}(E) = 2$ (Kato and Miyazaki [9]).

The above results (iii) (in particular, (ii)) can be obtained in a more simple way than [10] by using arguments in Hashimoto, Kato and Takahashi [8; Corollary 3.6; see also Theorem 3.2].

Let us recall some definitions. A Banach space X is called:

(i) *strictly convex* if $\| (x + y)/2 \| < 1$ whenever $\| x \| = \| y \| = 1$, $x \neq y$,

(ii) *uniformly convex* provided for each ε ($0 < \varepsilon < 2$) there exists a $\delta > 0$ such that $\| (x + y)/2 \| < 1 - \delta$ whenever $\| x - y \| \geq \varepsilon$, $\| x \| = \| y \| = 1$,

(iii) $(2, \varepsilon)$-*convex*, $\varepsilon > 0$, (cf. [13]) provided $\min\{ \| x + y \|, \| x - y \| \} \leq 2(1 - \varepsilon)$ whenever $\| x \| = \| y \| = 1$,

(iv) *uniformly non-square* ([5]; cf. [1], [3]) if there is a $\delta > 0$ such that there do not exist x and y in the closed unit ball of X for which $\| (x + y)/2 \| > 1 - \delta$ and $\| (x - y)/2 \| > 1 - \delta$. (Note that uniform non-squareness is equivalent to $(2, \varepsilon)$-convexity.)

A Banach space Y is said to be *finitely representable* in X if for any $\lambda > 1$ and for each finite-dimensional subspace F of Y, there is an isomorphism T of F into X for which

$$\lambda^{-1} \| x \| \leq \| Tx \| \leq \lambda \| x \| \quad \text{for all } x \in F.$$

X is said to be *super-reflexive* ([6]; cf. [1], [3], [13], [14]) if no non-reflexive Banach space is finitely representable in X.

Super-reflexive spaces are characterized as those uniformly con-vexifiable:

C. Theorem (Enflo [4]; cf. [1], [3], [14]). A Banach space X is super-reflexive if and only if X admits an equivalent uniformly convex norm.

2. Uniform convexity, super-reflexivity and von Neumann-Jordan constant

We bigin with the following proposition which will give effective examples later.

2.1 Proposition. Let $\lambda > 1$. Let $X_{2,\lambda}$ be the space l_2 equipped with the norm $\| x \|_{2,\lambda} := \max\{ \| x \|_2, \lambda \| x \|_\infty \}$. Then:

(i) $X_{2,\lambda}$ is isomorphic to a Hilbert space and

$$C_{NJ}(X_{2,\lambda}) = \min\{\lambda^2, 2\}.$$

(ii) $X_{2,\lambda}$ is not strictly convex for any $\lambda > 1$.

Proof (sketch). (i) Since

$$\| x \|_2 \leqq \| x \|_{2,\lambda} \leqq \lambda \| x \|_2 \quad \text{for all } x \in X_{2,\lambda},$$

we have

$$\| x+y \|_{2,\lambda}^2 + \| x-y \|_{2,\lambda}^2 \leqq \lambda^2 (\| x+y \|_2^2 + \| x-y \|_2^2)$$

$$= \lambda^2 2(\| x \|_2^2 + \| y \|_2^2)$$

$$\leqq \lambda^2 2(\| x \|_{2,\lambda}^2 + \| y \|_{2,\lambda}^2),$$

or $C_{NJ}(X_{2,\lambda}) \leqq \lambda^2$. By considering $x = (1/\lambda, 1/\lambda, 0, \ldots)$ and $y = (1/\lambda, -1/\lambda, 0, \ldots) \in X_{2,\lambda}$, we have $C_{NJ}(X_{2,\lambda}) = \min\{\lambda^2, 2\}$.

(ii) To see that $X_{2, \lambda}$ is not strictly convex, take an α satisfying $(1/\lambda)^2 + \alpha^2 \leqq 1$ and $0 < \alpha \leqq 1/\lambda$; then put $x = (1/\lambda, 0, 0, \ldots)$ and $y = (1/\lambda, \alpha, 0, \ldots)$.

Now, in the following two theorems we see that uniform convexity is nearly characterized by the condition $C_{NJ}(X) < 2$.

2. 2 Theorem. (i) If X is uniformly convex, then $C_{NJ}(X) < 2$; the converse is not true; indeed,

(ii) For any $\varepsilon > 0$ there exists a Banach space X (isomorphic to a Hilbert space) with $C_{NJ}(X) < 1 + \varepsilon$ which is not even strictly convex.

Proof (sketch). (i) Let X be uniformly convex. Let ε be any positive number with $0 < \varepsilon < 2^{1/2}$. Then, there exists a $\delta > 0$ such that $\| x \| \leqq 1$, $\| y \| \leqq 1$ and $\| x - y \| \geqq \varepsilon$ imply

$$(2.1) \qquad \| (x + y)/2 \|^2 \leqq (1 - \delta)\{ (\| x \|^2 + \| y \|^2)/2 \}$$

(cf. [1], p. 190). Now, let x and y be any elements in X with $\| x \|^2 + \| y \|^2 = 1$. We first assume that $\| x - y \| \geqq \varepsilon$. Then, using (2. 1), we have

$$(2.2) \qquad \| x + y \|^2 + \| x - y \|^2 \leqq 2(2 - \delta).$$

Next, if $\| x - y \| \leqq \varepsilon$, we have

$$(2.3) \qquad \| x + y \|^2 + \| x - y \|^2 \leqq 2(\| x \|^2 + \| y \|^2) + \varepsilon^2$$

$$\leqq 2(1 + \varepsilon^2/2).$$

Consequently, by (2. 2) and (2. 3) we obtain

$$\frac{\| x + y \|^2 + \| x - y \|^2}{2(\| x \|^2 + \| y \|^2)} \leq 1 + \max\{1 - \delta, \varepsilon^2/2\},$$

or $C_{NJ}(E) < 2$.

(ii) By Proposition 2. 1, for the spaces $X_{2, \lambda}$ ($\lambda > 1$) we have $C_{NJ}(X_{2, \lambda}) \to 1$ as $\lambda \to 1$, whereas $X_{2, \lambda}$ is not strictly convex.

Although the condition $C_{NJ}(X) < 2$ does not even imply strict convexity for X, it assures the existence of an equivalent norm on X for which X becomes uniformly convex (cf. Theorem C):

2. 3 Theorem. Let $C_{NJ}(X) < 2$. Then, X is super-reflexive; the converse is not true.

Proof. Assume $C := C_{NJ}(X) < 2$. Let x and y be any elements in X with $\| x \| = \| y \| = 1$. Then,

$$\min_{\varepsilon_i = \pm 1} \| \varepsilon_1 x + \varepsilon_2 y \| \leq \left\{ \frac{1}{2}(\| x + y \|^2 + \| x - y \|^2) \right\}^{1/2}$$

$$\leq C^{1/2}(\| x \|^2 + \| y \|^2)^{1/2}$$

$$= (2C)^{1/2},$$

that is, X is (2, ε)-convex with some ε, or equivalently uniformly non-square, which implies that X is super-reflexive (James [6]; see also [1], [13]).

For the latter assertion, consider the space $X_{2, \sqrt{2}}$. Indeed, $X_{2, \sqrt{2}}$ is isomorphic to a Hilbert space and hence super-reflexive, whereas $C_{NJ}(X_{2, \sqrt{2}}) = 2$ by Proposition 2. 1. (l_1^n and l_∞^n are also

such examples.)

2. 4 **Definition.** Let $\widetilde{C}_{NJ}(X)$ be the infimum of all NJ-constants for equivalent norms of X.

Theorems 2. 2 and 2. 3 assert that super-reflexivity is characterized by the condition $\widetilde{C}_{NJ}(X) < 2$.

2. 5 **Theorem.** The following are equivalent:

(i) $\widetilde{C}_{NJ}(X) < 2$.

(ii) X is super-reflexive.

(iii) X admits an equivalent uniformly convex norm.

(iv) X admits an equivalent uniformly non-square norm.

(v) X admits an equivalent uniformly smooth norm (cf. [1]).

(vi) X is J-convex (cf. [1]).

For some further conditions quivalent to super-reflexivity, we refer the reader to [1], [3] and [14].

2. 6 **Corollary.** $\widetilde{C}_{NJ}(X) = 2$ if and only if X is not super-reflexive.

3. **Uniform non-squareness and von Neumann-Jordan constant**

Very recently the authors [15] proved some homogeneous characterizations of uniformly non-square spaces, one of which is similar to a well-known characterization of uniformly convex spaces (we have used it in the proof of Theorem 2. 2!):

3. 1 Theorem. Let $1 < p < \infty$. For a Banach space X the following are equivalent:

(i) X is uniformly non-square.

(ii) There exist some ε and δ ($0 < \varepsilon, \delta < 1$) such that $\| x - y \| \geqq 2(1 - \varepsilon)$, $\| x \| \leqq 1$, $\| y \| \leqq 1$ implies

$$\left\| \frac{x + y}{2} \right\|^p \leqq (1 - \delta) \frac{\| x \|^p + \| y \|^p}{2}.$$

(iii) There exists some δ ($0 < \delta < 2$) such that for any x, y in X,

$$\left\| \frac{x + y}{2} \right\|^p + \left\| \frac{x - y}{2} \right\|^p \leqq (2 - \delta) \frac{\| x \|^p + \| y \|^p}{2}.$$

(iv) $\| A : l_p^2(X) \rightarrow l_p^2(X) \| < 2$.

We omit the proof, which will appear elsewhere.

Owing to Theorem 3. 1 a precise characterization of Banach spaces with NJ-constant less than two is obtained (cf. Theorems 2. 2 and 2. 3):

3. 2 Theorem. The following are equivalent:

(i) $C_{NJ}(X) < 2$.

(ii) X is uniformly non-square.

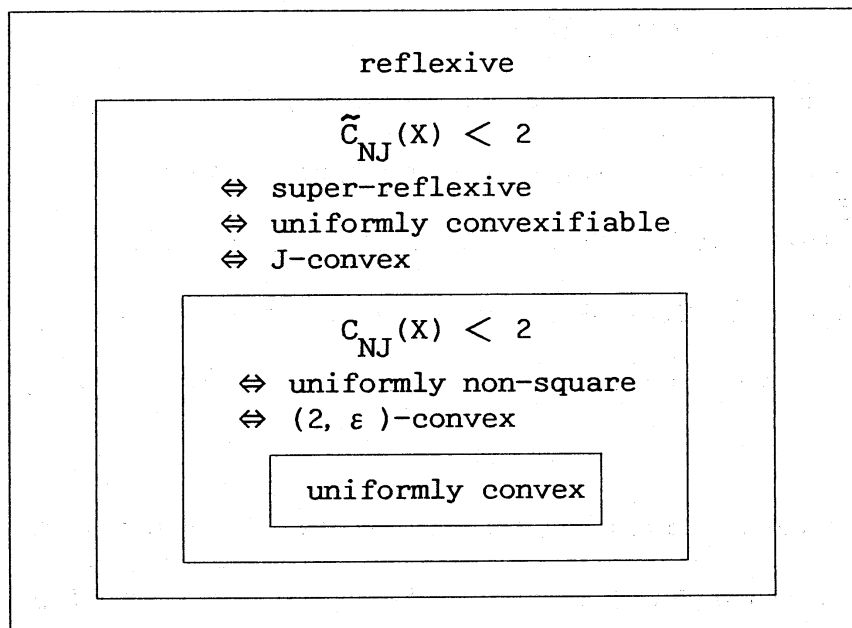(iii) X is (2, ε)-convex for some $\varepsilon > 0$.

To see this, merely recall Proposition A.

3. 3 Corollary. $C_{NJ}(X) = 2$ if and only if X is uniformly square.

3. 4 Note. Further investigation on the NJ-constant is made in [12] especially for the spaces having NJ-constant $2^{2/p-1}$, $1 \leqq p \leqq 2$

(the same value of that of L_p-spaces; see Proposition B).

Our results stated in this note are summarized as follows:

reflexive

$\widetilde{C}_{NJ}(X) < 2$

$\Leftrightarrow$ super-reflexive
$\Leftrightarrow$ uniformly convexifiable
$\Leftrightarrow$ J-convex

$C_{NJ}(X) < 2$

$\Leftrightarrow$ uniformly non-square
$\Leftrightarrow$ $(2, \varepsilon)$-convex

uniformly convex

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