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On the dynamics of bimodal real cubic maps

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1 Definition

1.1 Parameter space of bimodal real cubic maps

A real cubic map f from the real line $\mathbf{R}$ to itself is called *bimodal* if it has two distinct real critical points $c_L < c_R$. By the real affine conjugation, f can be normalized as follows

$$\begin{aligned} f_{a,b}(x) &= x^3 - 3a^2x + b \ (a > 0,\ b \geq 0) \\ &= -x^3 + 3a^2 + b \ (a < 0,\ b \leq 0) \end{aligned}$$

Therefore the two disjoint regions $\mathbf{M} := \mathbf{M}^+ \cup \mathbf{M}^-$

$$\begin{aligned} \mathbf{M}^+ &:= \{(a,b) \in \mathbf{R}^2 | a > 0,\ b \geq 0\} \\ \mathbf{M}^- &:= \{(a,b) \in \mathbf{R}^2 | a < 0,\ b \leq 0\} \end{aligned}$$

can be considered as the parameter space of bimodal real cubic maps. We identify the point $(a,b) \in \mathbf{M}$ and the corresponding map $f_{a,b}$.

1.2 hyperbolic component and its center

Bimodal cubic map is called *hyperbolic* if its critical points c_L and c_R converge to attractive cycles respectively. The set of hyperbolic maps is open subset of the parameter space and its connected components are called *hyperbolic components*. Each hyperbolic component is simply connected and contains the unique map f_0 (the *center* of this component) whose post critical set $\{f_0^n(c_m)\}(m \in \{L,R\}, n \in \mathbf{N}\}$ is a finite set ([Mc],[Mil1],[Mil2],[R]).

1.3 kneading sequences

For a bimodal real cubic map f and its critical points $c_L < c_R$, we put

$$\begin{aligned} I_1 &:= \{x \in \mathbf{R} | x < c_L\} \\ I_2 &:= \{x \in \mathbf{R} | c_L < x < c_R\} \\ I_3 &:= \{x \in \mathbf{R} | c_R < x\} \end{aligned}$$

The kneading sequences $k_L(f)$ and $k_R(f)$ of f consists of sequences of letters I_1, c_L, I_2, c_R, I_3

$$\begin{aligned} k_L(f) &= (a_1, a_2, \cdots) \in \prod\{I_1, c_L, I_2, c_R, I_3\} \\ k_R(f) &= (b_1, b_2, \cdots) \in \prod\{I_1, c_L, I_2, c_R, I_3\} \end{aligned}$$

and its j-th component a_j is defined by

$$\begin{aligned} a_j &= c_m \ (m = L, R) \ if \ f^j(c_L) = c_m \\ &= I_n \ (n = 1, 2, 3) \ if \ f^j(c_L) \in I_n \end{aligned}$$

b_j is also defined by the similar rule. For $f, g \in \mathbf{M}^+$(resp.$f, g \in \mathbf{M}^-$), we define the order of kneading sequences $k_m(f) < k_m(g)$ $(m = L, R)$ as follows; first we define the order of letters by $I_1 < c_L < I_2 < c_R < I_3$. If $f, g \in \mathbf{M}^+$, then we put $sign(I_1) = sign(I_3) = +1$ and $sign(I_2) = -1$. For $k_m(f) = (a_1, a_2, \cdots)$, $k_m(g) = (b_1, b_2, \cdots)$ $(m = L, R)$, we put $k_m(f) < k_m(g)$ if $a_1 = b_1, a_2 = b_2, \cdots, a_{k-1} = b_{k-1}$ consist of the letters I_1, I_2 or I_3 and $a_k \neq b_k$ moreover

$$\begin{aligned} (\prod_{i=1}^{k-1} sign(a_i))a_k < (\prod_{i=1}^{k-1} sign(b_i))b_k \quad if \quad m = L \\ (\prod_{i=1}^{k-1} sign(a_i))a_k > (\prod_{i=1}^{k-1} sign(b_i))b_k \quad if \quad m = R \end{aligned}$$

For the case $f, g \in \mathbf{M}^-$ we put $sign(I_1) = sign(I_3) = -1$ and $sign(I_2) = +1$, and the opposite sign of the abave two inequalities.

1.4 bone curves

Let π be the finite sequence of letters I_1, c_L, I_2, c_R, I_3 whose last letter is c_L or c_R. We define the *bone curve* by

$$bone_L(\pi) \ := \ \{f \in \mathbf{M} | \ k_L(f)|_{|\pi|} = \pi\}$$

$$bone_R(\pi) \quad := \quad \{f \in \mathbf{M} |\ k_R(f)|_{|\pi|} = \pi\}$$

where $|\pi|$ is the length of the finite sequence π and $k_L(f)|_{|\pi|}$ is the first $|\pi|$ letters of $k_L(f)$.

1.5 topological entropy

We write the n times copositions of the bimodal real cubic map f by f^n. The *lap number* is the number of the up-downs of the graph of f^n. Then the following limit $s(f)$ exists and called the *growth number of f*

$$s(f) := \lim_{n\to\infty} \sqrt[n]{l(f^n)}$$

We define the *topological entropy h(f) of f* by

$$h(f) := \log s(f)$$

$1 \leq l(f^n) \leq 3^n$ means $0 \leq h(f) \leq \log 3$. As the function on the parameter space, $h(f)$ is continuous with respect to f ([M-T]). For any $n \in \mathbf{N}$, we can calculate $l(f^n)$ hence $h(f)$ from the kneading sequences of f. For example the next claim tells that if we know which kneading sequences are bigger for two points on the bone curve, we can determine which topological entropies are bigger.

Claim 1.1 *([B],[B-K])*
For $f, g \in bone_L(\pi)$ (resp. $bone_R(\pi)$), $k_R(f) \leq k_R(g)$ (resp. $k_L(f) \leq k_L(g)$) means $h(f) \leq h(g)$.

2 Monotonicity of the topological entropy along bone curve

Next result is an affirmative answer to the problem of [N-N] P.180.

Theorem 2.1 *The topological entropy $h(f)$ is monotone along any arc in the bone curve.*

By the numerical experiments, we conjecture that non empty bone curve is an arc. To show this theorem, we prepare the following three claims.

Claim 2.1 *(Combinatorial rigidity for centers [D-G-M-T],[D-H])*
For centers $f, g \in \mathbf{M}$, $k_m(f) = k_m(g)$ $(m = L, R)$ means $f = g$.

Claim 2.2 *(The shape of the bone curves in hyperbolic components [Mil 2])*
We assume that the intersection $C := B \cap W$ of a bone curve B and a hyperbolic component W is non empty. Then C is a real analytic arc and contains the center f_0 of W in its interior. f_0 makes C into two subarcs C_+ and C_- and kneading sequences are constant on these two subarcs. Moreover for any $f_+ \in C_+$ and $f_- \in C_-$

$$k_m(f_-) < k_m(f_0) < k_m(f_+)$$

where $m = R$ if $B = bone_L(\pi)$ and $m = L$ if $B = bone_R(\pi)$.

Claim 2.3 *("The lack of intermidiate value theorem" [M-T])*
In the situation of Claim 3, if the map $g \in B$ satisfies $k_m(f_-) \leq k_m(g) < k_m(f_0)$, then $k_m(f_-) = k_m(g)$. Similarly if $k_m(f_0) < k_m(g) \leq k_m(f_+)$, then $k_m(g) = k_m(f_+)$.

(The proof of the theorem)
We assume $B = bone_L(\pi)$ and let γ be an arc in B whose endpoints are f and g (we may assume that $k_R(f) \leq k_R(g)$).

Because of Clain 1.1, it is sufficient to show that for any $h \in \gamma$, $k_R(f) \leq k_R(h) \leq k_R(g)$. From the definition of the order of kneading sequences, it is also enough to show that $k_R(f)|_n \leq k_R(h)|_n \leq k_R(g)|_n$ for all $n \in \mathbf{N}$. We prove this by induction on $n \in \mathbf{N}$. The fact that $k_R(h)$ can only changes at centers along γ and Claim 2.1 and 2.2 tells that our claim is true for the case of n=1. Now we assume that our claim is true until n=i. Then there exist finite number of centers $f_1, f_2, \cdots, f_s$ such that they decompose γ into open subarcs $J_1, J_2, \cdots, J_{s+1}$ which satisfy 1) for any $h_1, h_2 \in J_t$, $k_R(h_1)|_i = k_R(h_2)|_i$, 2) for any $h_t \in J_t$ and $h_{t+1} \in J_{t+1}$, $k_R(h_t)|_i < k_R(h_{t+1})|_i$. Then as the case of n=1, we can see that $k_R(h)|_{i+1}$ is also monotone on each subarc J_t. Moreover this monotonicity also holds at the boundary point of two subarcs by Claim 2.3. Therefore our claim is also true for n=i+1 and by induction we finish the proof. Similar argument also holds for $B = bone_R(\pi)$.

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