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テータ級数のトレースの計算へのWeil表現の応用

京都大理D1 久米貴浩 (Takahiro Kume)*

1. はじめに

標題の講演は、講演者が作成中のプレプリント（英文）にもとづいている。そこで、このプレプリントの原稿に手を加えたものを講演のレジュメとして以下に掲げる。その主な内容は

- 問題の説明 [§2]
- 定理Vと定理L（このふたつが主結果）の記述 [§4.2]
- 定理 L_p（定理Lの局所化）の記述 [§5.2]
- 定理 L_p から主結果が従うことの証明 [§5.4]

である（定理 L_p 自体の証明は紙数の都合で割愛した）。

この講演ならびにプレプリントの内容について、京都大学の吉田敬之教授から多くの有益な助言を頂いた。講演者は吉田教授に対して心からの感謝の意をあらわしたい。

2. Motivation

Using the Weil representation, we calculate a trace of a theta series associated with a lattice of a certain quadratic space over $\mathbb{Q}$ in order to examine the relation of the space of Siegel modular forms and the space of such theta series.

Let us explain our problem in more detail. For simplicity, suppose n is an even positive integer. Let S be a rational symmetric matrices of size $2n$ such that the determinant of S is a square of nonzero rational numbers. Let $V = (\mathbb{Q}^{2n}, Q)$ be a positive definite regular quadratic space of rank $2n$ over $\mathbb{Q}$ represented by S.

*e–mail: kume@kusm.kyoto–u.ac.jp

For every integral lattice L of V, we define the theta series ϑ_L by the following formula:

$$\vartheta_L(\mathfrak{z}) = \sum_{\mathfrak{z} \in L^m} \exp\left(2\pi\sqrt{-1}\mathrm{tr}\left({}^t xSx\right)\right)$$

where $\mathfrak{z}$ is an element of the Siegel upper half space of degree m. If the level of L devides a positive integer N, the theta series ϑ_L belongs to the space $\mathcal{M}_m(n, N)$ of Siegel modular forms of weight n, degree m and level N. Let $\theta_m(V, N)$ be the subspace of $\mathcal{M}_m(n, N)$ spanned by such theta series.

For any positive divisor N' of N we obtain the follwing inclusion:

$$\theta_m(V, N') \subset \theta_m(V, N) \cap \mathcal{M}_m(n, N').$$

But it is not obvious whether or not the equality holds in the above formula.

Problem V. *Is the following equality*

$$\theta_m(V, N') = \theta_m(V, N) \cap \mathcal{M}_m(n, N')$$

true?

To attack Problem V, we can use the global trace operator $T^{(n)}_{N,N'}$ which is defined as follows. For every $\varphi \in \mathcal{M}_m(n, N)$ put

$$T^{(n)}_{N,N'}(\varphi)(\mathfrak{z}) = \sum_{\gamma} \det(c\mathfrak{z} + d)^{-1} \varphi\left((a\mathfrak{z} + b)(c\mathfrak{z} + d)^{-1}\right)$$

($\mathfrak{z} \in H_m$) where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ runs over a complete set of representative of $\Gamma_0^{(m)}(N) \backslash \Gamma_0^{(m)}(N')$.
Then we have $T^{(n)}_{N,N'}(\varphi) \in \mathcal{M}_m(n, N')$.

If we get

$$T^{(n)}_{N,N'}\left(\theta_m(V, N')\right) \subset \theta_m(V, N'),$$

we solve Problem V. Thus we reduce Problem V to Problem L stated below:

Problem L. *Does the global trace $T^{(n)}_{N,N'}(\vartheta_L)$ belongs to $\mathcal{M}_m(n, N')$ for every integral lattice L of V of level N?*

These two problems are discussed in several papers [3, 4, 1] . The authors of these papers reduce Problem L to the case when degree m is greater than $2n$ by establishing relative commutation relations of the global trace operators and the Siegel's ϕ-operators.

In contrast to their global method, we try to transform Problem L to a p-adic analogue by means of the global and local Weil representation.

3. Notation

The following notation will be used throuout of this paper: Let m, n be positive integers. For an associative ring A with identity element we denote by $A^\times$ the group of all invertible elements and by $\mathrm{Mat}_{m,n}(A)$ the module of all $m \times n$ -matrices with all entries in A; we put $A^m = \mathrm{Mat}_{m,1}(A)$, $\mathrm{Mat}_m(A) = \mathrm{Mat}_{m,m}(A)$ for simplicity. The identity and zero elements of the ring $\mathrm{Mat}_m(A)$ are denoted by 1_m and 0_m (when m needs to be stressed). The transpose of a matrix g is denoted by ${}^t g$. We denote by $\mathrm{tr}(x)$ the trace of a square matrix x. Let J be an A-submodule of A. We denote by $\mathrm{Sym}_m(J)$ the module of all $m \times m$-symmetric matrix with all entries in J. If all entries of a matrix $g \in \mathrm{Mat}_{m,n}(A)$ belongs to J, we write $g \equiv 0 \mod J$. We put

$$\mathbb{T} = \left\{ \zeta \in \mathbb{C} \,\middle|\, |\zeta| = 1 \right\},$$
$$\mathbf{e}(c) = \exp(2\pi\sqrt{-1}c) \qquad (c \in \mathbb{C}).$$

For any set E, $|E|$ means the cardinality of E. The characteristic function of a subset E' of E is denoted by $\mathrm{I}_{E'}$. For every locally compact Hausdorff group X, we denote by $\mathcal{S}(X)$ the space of Schwartz-Bruhat functions on X.

Let ∞ and $\mathbf{h}$ be the infinite place and the set of all finite places of $\mathbb{Q}$, respectively. We identify the latter set $\mathbf{h}$ with the set of all rational primes. We denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ at v for any place v of $\mathbb{Q}$. Let $\mathfrak{G}$ be an algebraic group defined over $\mathbb{Q}$. For any field k containing $\mathbb{Q}$, we denote by $\mathfrak{G}_k$ the group of k-rational points of $\mathfrak{G}$ and abbreviate $\mathfrak{G}_{\mathbb{Q}_v}$ to $\mathfrak{G}_v$ for each place v of $\mathbb{Q}$. We define the adelization $\mathfrak{G}_{\mathbb{A}}$ of $\mathfrak{G}$ and view $\mathfrak{G}_{\mathbb{Q}}$ and $\mathfrak{G}_v$ as subgroups of $\mathfrak{G}_{\mathbb{A}}$ as usual. We then denote by $\mathfrak{G}_\infty$ and $\mathfrak{G}_{\mathbf{h}}$ the infinite and the finite part of $\mathfrak{G}_{\mathbf{A}}$, respectively. For $g \in \mathfrak{G}_{\mathbf{A}}$, we denote by g_v, g_∞, and $g_{\mathbf{h}}$ its projections to $\mathfrak{G}_v$, $\mathfrak{G}_\infty$, $\mathfrak{G}_{\mathbf{h}}$.

We denote by $G^{(m)}$ the symplectic group of genus m. For a commutative ring R with identity element, we assume that the group of all R-ratoinal points of $G_R^{(m)}$ of $G^{(m)}$ is given explicitly by

$$G_R^{(m)} = \left\{ g \in \mathrm{GL}_{2m}(R) \,\middle|\, {}^t g \begin{bmatrix} 0_m & 1_m \\ -1_m & 0_m \end{bmatrix} g = \begin{bmatrix} 0_m & 1_m \\ -1_m & 0_m \end{bmatrix} \right\}.$$

We usually denote every element g of $G_R^{(m)}$ as $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $m \times m$-matrices a, b, c, d. Let $\mathcal{H}_m$ be the Siegel upper half space of genus m. We define a action of $G_\infty^{(m)}$ on $\mathcal{H}_m$ and the factors of automorphy $j(\cdot,\cdot)$ as follows:

$$g\mathfrak{z} = (a\mathfrak{z} + b)(c\mathfrak{z} + d)^{-1},$$
$$j(g, \mathfrak{z}) = (c\mathfrak{z} + d),$$

where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\infty^{(m)}$ and $\mathfrak{z} \in \mathcal{H}_m$. For any positive integer N, we define a congruence subgroup $\Gamma_0^{(m)}(N)$ by

$$\Gamma_0^{(m)}(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_{2m}(\mathbb{Z}) \cap G_\infty^{(m)} \,\middle|\, c \equiv 0 \mod N \right\}.$$

Let (n, N) be a pair of positive integers such that N is arbitrary if n is even or N divides 4 if n is odd. For such a pair (n, N) we define the action of $\Gamma_0^{(m)}(N)$ on the space of all holomorphic functions on $\mathcal{H}_m$ as follows :

$$(f\|_n\gamma)(\mathfrak{z}) = \chi_n(\gamma) j(\gamma, \mathfrak{z})^{-n} f(\gamma\mathfrak{z})$$

where f is a holomorphic function on $\mathcal{H}_m$, $\gamma \in \Gamma_0^{(m)}(N)$, $\mathfrak{z} \in \mathcal{H}_m$, and χ_n is the character of $\Gamma_0^{(m)}(N)$ given by

$$\chi_n(\gamma) = (\det a, (-1)^n)_{\mathbb{Q}_2}$$

$\left(\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0^{(m)}(N)\right)$. We denote by $\mathcal{M}_m(n, N)$ the space of all holomorphic functions on $\mathcal{H}_m$ satisfying the condition

$$f\|_n\gamma = f$$

for any element γ of $\Gamma_0^{(m)}(N)$.

Let F be a local field with the characteristic of $F \neq 2$. We denote by $(\cdot,\cdot)_F$ the Hilbert symbol of index 2 over F. Let $V = (V, Q)$ be a regular quadratic space of rank n over F. We denote by B the nondegenerated bilinear form associated with the quadratic form Q given by $B(x,y) = Q(x+y) - Q(x) - Q(y)$ $(x, y \in V)$. For some basis $\{e_i\}$ of V and for some $n \times n$-regular symmetric matrix S, we have $Q(\sum_i x_i e_i) = {}^t x S x$ for any $x = [x_i] \in F^n$). We put

$$\det V = \det S \mod (F^\times)^2 \in F^\times \big/ (F^\times)^2$$

It is independent of the choice of $\{e_i\}$ and S. The Hasse symbol of V is denoted by $\varepsilon_F(V)$. For some $a_i \in F^\times (i = 1, \ldots, n)$ and some basis $\{f_i\}$ of V, we get

$$Q\left(\sum_i x_i f_i\right) = \sum_i a_i x_i^2$$

for any $x = [x_i] \in F^n$; in this case we obtain

$$\varepsilon_F(V) = \prod_{1 \le i < j \le n} (a_i, a_j)_F.$$

Furthermore assume F is non-archimedean. Let R, ϖ and q be the maximal compact subring, a prime and the module of F. For every lattice L of V, put

$$L^\vee = \left\{ x \in V \;\middle|\; B(x,y) \in R \quad (\forall y \in L) \right\}.$$

Then, $L^\vee$ is also a lattice of V. If L is integral $(Q(L) \subset R)$ the R-module generated by $Q(L^\vee)$ in F can be written as $\varpi^{-l}R$ for some non negative integer l. This number is denoted by $\mathrm{lev}_V(L)$. Represent the quadratic form Q as a symmetric matrix S' by taking some R-basis of L. We denote by $\det L$ the element $\det S' \mod (R^\times)^2$ of $F^\times/(R^\times)^2$. It is independent of the choice of R-basis and S'.

Let $V = (V, Q)$ be a regular quadratic space of rank n over $\mathbb{Q}$. For each place v of $\mathbb{Q}$, we denote by $V_v = (V_v, Q)$ the scalar extention of V over $\mathbb{Q}_v$ as quadratic space. We put $\varepsilon_v(V) = \varepsilon_{\mathbb{Q}_v}(V)$. For every $p \in \mathbf{h}$ and every lattice M of V_p, we put $\mathrm{lev}_p(M) = \mathrm{lev}_{V_p}(M)$. For every $p \in \mathbf{h}$ and every lattice L of V, we put $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This module L_p becomes a lattice of V_p. If L is an integral lattice of V $(Q(L) \subset \mathbb{Z})$, L_p is also an integral lattice of V_p for every $p \in \mathbf{h}$ and $\mathrm{lev}_p(L_p) = 0$ for almost all $p \in \mathbf{h}$. We denote by level(L) the integer $\prod_{p\in\mathbf{h}} p^{\mathrm{lev}_p}$. We define $\det V$ and $\det L$ as similarly as in the local field case.

4. Main results

4.1. Preliminaries. Let N, m, n be positive integers. Fix a nonsingular symmetric matrix $S \in \mathrm{Sym}_{2n}(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathbb{Q})$. We obtain a regular quadratic space $V = (\mathbb{Q}^{2n}, Q)$ of rank $2n$ over $\mathbb{Q}$ by $Q(x) = {}^t xSx$ $(x \in \mathbb{Q}^{2n})$. Let B be the nondegenerated bilinear form associated with the quadratic form Q determined by $B(x, y) = Q(x + y) - Q(x) - Q(y)$ $(x, y \in V)$. Let X be the vector-space direct sum of m-copies of V. We identify this vector space X with $\mathrm{Mat}_{2n,m}(F)$. For any $x \in X$, we write $x = (x_i)$ by column vectors x_i $(1 \le i \le m)$.

For any integral lattece L of V, define the theta series associated with L by the following formula:

$$\vartheta_L(\mathfrak{z}) = \sum_{x \in L^m} \mathbf{e}\left(\mathrm{tr}\left({}^t xSx\mathfrak{z}\right)\right)$$

$(\mathfrak{z} \in \mathcal{H}_m)$. It is well-known that if level(L) divides N then

$$\vartheta_L \in \mathcal{M}_m(n, N) .$$

Let $\Theta_m(V, N)$ be the subspace of $\mathcal{M}_m(n, N)$ generated by

$$\{\vartheta_L | L \text{ is an integral lattice of } V \text{with level}(L) | N\}$$

For N and its positive divisor N', define a subset $P(V;N,N')$ of $\mathbf{h}$ by

$$P(V;N,N') = \left\{p \in \mathbf{h} | N_p > 1,\ N'_p = 0,\ (-\sqrt{-1}^{n\delta_2(p)})\varepsilon_p(V) = -1\right\}$$

where we write the prime factorization of $N,\ N'$ as

$$N = \prod_{p\in\mathbf{h}} p^{N_p},\ N' = \prod_{p\in\mathbf{h}} p^{N'_p},$$

$(0 \leq N_p,\ N'_p \in \mathbb{Z})$ and set

$$\delta_2(p) = \begin{cases} 0 & p \neq 2 \\ 1 & p = 1 \end{cases}$$

for any $p \in \mathbf{h}$.

Since $\det V \equiv 1 \mod (\mathbb{Q}^\times)^2$, we can write

$$4^n \det L = \prod_{p\in\mathbf{h}} p^{2t_p(L)}$$

$(t_p(L) \in \mathbb{Z},\ p \in \mathbf{h})$ for any lattice L of V.

For N and its positive divisor N', define the global trace operator $T^{(n)}_{N,N'}$ by the following formula:

$$T^{(n)}_{N,N'}(f) = \sum_{\gamma\in\Gamma_0^{(m)}(N)\backslash\Gamma_0^{(m)}(N')} f\|_n\gamma \quad (f \in \mathcal{M}_m(n,N)).$$

Then $T^{(n)}_{N,N'}$ is a well-defined $\mathbb{C}$-linear mapping of $\mathcal{M}_m(n,N)$ onto $\mathcal{M}_m(n,N')$.

4.2. Statement of main results. Now we state our main results.

Theorem V. *Let the notation and the assumptions be as above. Write the prime factorization of* $N,\ N'$ *as*

$$N = \prod_{p\in\mathbf{h}} p^{N_p},\ N' = \prod_{p\in\mathbf{h}} p^{N'_p}.$$

Suppose

$$\begin{cases} 2 \leq N_2 = N'_2 & n \text{ odd}, \\ 2 \leq N_2 = N'_2 \text{ or } 1 \geq N_2 \geq N'_2 \geq 0 & n \text{ odd}. \end{cases}$$

Then we have

$$\Theta_m(V,N') = \Theta_m(V,N) \cap \mathcal{M}_m(n,N')$$

if $m \geq n$.

Theorem L. *Let the notation and the assumptions be as above. Suppose* L *is an integral lattice of* V *with* $\mathrm{level}(L) = N$. *Set*

$$t_0(L) = \max\left\{t_p(L) - \left[\frac{N_p - 1}{2}\right] \,\middle|\, p \in P(V;N,N')\right\}$$

(1) *We have*

$$T^{(n)}_{N,N'}(\vartheta_L) \in \Theta_m(V,N')$$

if $m \geq \min\{n, t_o(L)\}$.

(2) *Furthermore suppose* $t_o(L) > 0$. *We obtain*

$$T^{(n)}_{N,N'}(\vartheta_L) = 0$$

if $m \geq \min\{n, t_o(L)\}$.

5. Localization of the global traces

5.1. Preliminaries. Let v be any place of $\mathbb{Q}$.

Define the nontrivial character $\psi_v : \mathbb{Q}_v \to \mathbb{T}$ by

$$\psi_v(x) = \mathbf{e}(x) \qquad \text{if } v = \infty,$$

$$\psi_v(x) = \mathbf{e}(-\mathrm{Fr}(x)) \qquad \text{if } v \in \mathbf{h},$$

where $\mathrm{Fr}(x)$ $(x \in \mathbb{Q}_p)$ is the fractional part of the p-adic expansion of x.

Give a self-duality on X_v by $(x,y) \mapsto \psi_v(\mathrm{tr}(2^t xSy))$.

For any $\varphi \in \mathcal{S}(X_v)$, define the Fourier transform $\varphi^\wedge$ of φ by the following formula:

$$\varphi^\wedge(y) = \int_{X_v} \varphi(x)\psi_v(\mathrm{tr}(2^t xSy))d_v x$$

$(y \in X_v)$. Here d_v is the self-dual Haar measure on X_v with respect to the above self-duality.

Under our assumptions, the Weil constant $\gamma_v(V)$ can be easily determined [2, 5]:

$$(5.1)\qquad \gamma_v(V)=\begin{cases}(\sqrt{-1})^n & \text{if } v=\infty,\\ \varepsilon_v(V) & \text{if } v\in \mathbf{h}\backslash\{2\},\\ (-\sqrt{-1})^n & \text{if } v=2.\end{cases}$$

The extention $\mathbb{Q}_v(\sqrt{(-1)^n\det V})\big/\mathbb{Q}_v$ determines the unique character ω_v of $\mathbb{Q}_v^\times$ by the local class field theory [6]: namely,

$$\omega_v(x)=(x,(-1)^n)_{\mathbb{Q}_v}\qquad (x\in\mathbb{Q}_v^\times).$$

Notice that, for any $p\in\mathbf{h}$,

$$(5.2)\qquad \ker\omega_p\supset\begin{cases}1+4\mathbb{Z}_2 & p=2,\\ \mathbb{Z}_p^\times & p\neq 2.\end{cases}$$

We have the so called local Weil representation π_v of G_v realized on $\mathcal{S}(X_v)$; π_v is characterized by the following three conditions (cf. [7]):

$$(5.3)\qquad \left(\pi_v\left(\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}\right)\varphi\right)(x)=\psi_v(\operatorname{tr}(b^t xSx))\varphi(x),$$

$$(5.4)\qquad \left(\pi_v\left(\begin{bmatrix}a & 0\\ 0 & {}^t a^{-1}\end{bmatrix}\right)\varphi\right)(x)=\omega_v(\det a)|\det a|_v^n\varphi(xa),$$

$$(5.5)\qquad \left(\pi_v\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)\varphi\right)(x)=\gamma_v(V)^m\varphi^\frown(x),$$

for $\varphi\in\mathcal{S}(X_v)$, $x\in X$, $a\in \mathrm{GL}_m(\mathbb{Q}_v)$ and $b\in\mathrm{Sym}_m(\mathbb{Q}_v)$. The mapping

$$G_v\times\mathcal{S}(X_v)\ni(g,\varphi)\mapsto\pi_v(g)\varphi\in\mathcal{S}(X_v)$$

is continuous. For any $p\in\mathbf{h}$ and $\varphi\in\mathcal{S}(X_p)$, the stabilizer of φ in G_p under π_p contains an open compact subgroup of G_p.

The global Weil representation $\pi_{\mathbb{A}}$ of $G_{\mathbb{A}}$ realized on $\mathcal{S}(X_{\mathbb{A}})$ is defined as follows. Let φ be an element of $\mathcal{S}(X_{\mathbb{A}})$ of the form $\varphi=\prod_v\varphi_v$ such that $\varphi_p=\mathrm{I}_{\mathrm{Mat}_{2n,m}(\mathbb{Z}_p)}$ for almost all

$p \in \mathbf{h}$. For any $g = (g_v) \in G_{\mathbb{A}}$, put

$$\pi_{\mathbb{A}}(g)\varphi = \prod_v \pi_v(g_v)\varphi_v.$$

This action of $G_{\mathbb{A}}$ extends by continuity to the representation $\pi_{\mathbb{A}}$ of $G_{\mathbb{A}}$ on $\mathcal{S}(X_{\mathbb{A}})$.

Let v be any place of $\mathbb{Q}$. We give some examples of compact subgroups in G_v and of semi-invarivant vectors under the action of these subgroups.

First, suppose $v = p \in \mathbf{h}$. For any non-zero element λ of $\mathbb{Z}_p$, define an open compact subgroup $D_p(\lambda)$ of G_p by

$$D_p(\lambda) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{\mathbb{Z}_p} \;\middle|\; c \equiv 0 \mod \lambda\mathbb{Z}_p \right\}.$$

If L_p be an integral lattice of V_p with $\mathrm{lev}_p(L_p) = l$, then we can easily see $\ker \omega_p \supset (1 + p^l\mathbb{Z}_p) \cap \mathbb{Z}_p^\times$ (see (5.3),(5.4),(5.5)) and

$$\pi_p(g)\mathrm{I}_{L_p^m} = \begin{cases} \mathrm{I}_{L_p^m} & (l = 0) \\ \omega_p(\det a)\mathrm{I}_{L_p^m} & (l \geq 1) \end{cases} \tag{5.6}$$

for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_p(p^l)$ (see (5.3),(5.4),(5.5), [9]).

Next suppose $v = \infty$. Set $\mathbf{i} = \sqrt{-1} \cdot 1_m \in \mathcal{H}_m$. Let U_∞ be the stabilizer of $\mathbf{i}$ under the standard action of G_∞ on $\mathcal{H}_m$. We can immediately see

$$U_\infty = \left\{ u \in G_\infty \;\middle|\; u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \right\}$$

and that U_∞ is compact. For every $\mathfrak{z} \in \mathcal{H}_m$, we can define an element $\varphi_{\mathfrak{z}}$ of $\mathcal{S}(X_\infty)$ by

$$\varphi_{\mathfrak{z}}(x) = \mathbf{e}(\mathrm{tr}({}^t x S x \mathfrak{z}))$$

$(x \in X_\infty)$. Then we can show (cf. [9])

$$\pi_\infty(u)\varphi_{\mathbf{i}} = \det(A - \sqrt{-1}B)^n \varphi_{\mathbf{i}}$$

for any $u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in U_\infty$. Therefore we obtain

$$\pi_\infty(g_\infty)\varphi_{\mathbf{i}} = j(g_\infty, \mathbf{i})^{-n}\varphi_{g_\infty \mathbf{i}} \tag{5.7}$$

for any $g \in G_\infty$.

5.2. The local trace operators. We define a character χ_n of a compact group $D_2(\lambda)$ ($\lambda \in \mathbb{Z}_2$ if n is even or $\lambda \in 4\mathbb{Z}_2\backslash\{0\}$ if n is odd) by

$$\chi_n(g) = \begin{cases} 1 & \text{if } n \text{ even} \\ (\det a, -1)_{\mathbb{Q}_2} & \text{if } n \text{ odd} \end{cases} \tag{5.8}$$

$\left(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_2(\lambda) \right)$.

Take a non-negative integer l and a finite place $p \in \mathbf{h}$. Let $\mathcal{L}^{(m)}(V_p, l)$ be the subspace of $\mathcal{S}(X_p)$ spanned by the set of all functions of the form I_{M^m} such that M is an integral lattice of V_p with $\mathrm{lev}_p(M) \leq l$. Notice that $\mathcal{L}^{(m)}(V_p, l) = 0$ if $p = 2$, n is odd and $0 \leq l \leq 1$, or if $p \in \mathbf{h}$, $\gamma_p(V_p) = -1$ and $l = 0$ (see (5.6)).

From now on, we assume that

$$\begin{cases} l \geq 0 & \text{if } p \neq 2, \\ l \geq 2 & \text{if } p = 2 \text{ and } n \text{ is odd}, \\ l \geq 0 & \text{if } p = 2 \text{ and } n \text{ is even}. \end{cases} \tag{5.9}$$

Under this assumption, we define a $\mathbb{C}$ -linear map $\tau_{p,l}^{(m)}$ on $\mathcal{S}(X_p)$ by the follwing formula: (see (5.2),(5.6), (5.8),(5.9))

$$(\tau_{p,l}^{(m)}(\varphi))(x) = \begin{cases} \int_{D_p(p^l)} (\pi_p(u)\varphi)(x) d_{p,l}(u) & \text{if } p \neq 2, \\ \int_{D_p(p^l)} \chi_n(u)(\pi_p(u)\varphi)(x) d_{p,l}(u) & \text{if } p = 2, \end{cases} \tag{5.10}$$

($\varphi \in \mathcal{S}(X_p)$, $x \in X_p$) where $d_{p,l}$ is the Haar measure of $D_p(p^l)$ normalized such that $\int_{D_p(p^l)} d_{p,l}(u) = 1$. The integrals on the right-hand side are essentially finite sum, and the map $\tau_{p,l}^{(m)}$ is well-defined. We call this map local trace operator.

By the definition of the local trace operator, we get $\tau_{p,l}^{(m)}(\varphi) = \varphi$ for $\varphi \in \mathcal{L}^{(m)}(V_p, l)$. Let l, l' be non-negative integers satisfying the above assumption and the condition $l \geq l' \geq 0$. The following theorem is a local analogue of Theorem L.

Theorem $\mathbf{L_p}$. *Let the notation and assumption be as above. Take an integral lattice L_p of V_p with* $\mathrm{lev}_p(L_p) = l \geq 1$.

(1) *Suppose $p \neq 2$ and $l > l' > 0$. If $m \geq 1$, we have*

$$\tau_{p,l'}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \in \mathcal{L}^{(m)}(V_p, l').$$

(2) *Suppose that $p \neq 2$ and $l > l' = 0$, or that $p = 2$, $l = 1 > l' = 0$ and n is even.*

(a) *If $\gamma_p(V_p) = 1$ and $m \geq 1$, we have*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \in \mathcal{L}^{(m)}(V_p, 0).$$

(b) *If $\gamma_p(V_p) = -1$ and $m \geq \min\left\{n, t_p(L_p) - \left[\frac{l-1}{2}\right]\right\}$, we have*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) = 0$$

and

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \in \mathcal{L}^{(m)}(V_p, 0).$$

(c) *If $\gamma_p(V_p) = -1$ and $m < \min\left\{n, t_p(L_p) - \left[\frac{l-1}{2}\right]\right\}$, we have*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \neq 0$$

and

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \notin \mathcal{L}^{(m)}(V_p, 0).$$

Since $t_p(M) > t_p(M')$ for any two lattices M, M' of V_p with the condition $M \subsetneq M'$, Theorem $\mathrm{L_p}$ follows immediately from the two lemmas below.

Lemma 5.1. *The notation and the assumptions are as in Theorem* $\mathrm{L_p}$ *. Moreover suppose* $p \neq 2$ *and* $l \geq 2$*. Then, if* $m \geq 1$*, there exist a finite number of integral lattices* $M_i (i \in I)$ *of* V_p *satisfying*

$$l - 2 \leq \mathrm{lev}_p(M_i) \leq l - 1 \ (\forall i \in I)$$

such that

$$\tau_{p,l-1}^{(m)}(\mathrm{I}_{L_p^m}) = \sum_{i \in I} c_i \mathrm{I}_{M_i^m}$$

$(c_i \in \mathbb{C})$.

Lemma 5.2. *The notation and assumptions are as in theorem . Moreover,suppose* $l = 1$.

(1) *Let* $\gamma_p(V_p) = 1$*. Then, if* $m \geq 1$*, we obtain*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^m}) = c \sum_{M} \mathrm{I}_{M^m}$$

$(c \in \mathbb{C}^\times)$*, where the sum is taken over all integral lattices* M *of* V_p *such that* $\mathrm{lev}_p(M) = 0$ *and* $L_p \subset M \subset \check{L}_p$.

(2) *Let* $\gamma_p(V_p) = -1$*. Then, we have*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^m}) = 0 \qquad \text{if } m \geq \min\{n, t_p(L_p)\}$$

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^m}) \neq 0 \qquad \text{if } m < \min\{n, t_p(L_p)\}$$

We omit the proof of these two lemmas in this article.

5.3. Construction of theta series via the Weil representation. Let f be an element of $\mathcal{S}(X_\mathbf{h})$. Put

$$(\varphi_\mathbf{i} \otimes f)(x) = \varphi_\mathbf{i}(x_\infty) f(x_\mathbf{h}) \qquad x = (x_\infty, x_\mathbf{h}) \in X_\mathbb{A},$$

then $\varphi_\mathbf{i} \otimes f \in \mathcal{S}(X_\mathbb{A})$. For each $g \in G_\mathbb{A}$, set

$$\Psi(f; g) = \sum_{x \in X_\mathbb{Q}} \pi_\mathbb{A}(g)(\varphi_\mathbf{i} \otimes f)(x).$$

We can show (cf. [5, 8, 9])

- the series in the right-hand side is absolutely convergent on every compact subset of $G_{\mathbf{A}}$; hence $\Psi(f;\cdot)$ is a cotinuous function on $G_{\mathbf{A}}$;
- this function $\Psi(f;\cdot)$ is left $G_{\mathbf{Q}}$ invariant and right invariant for some open compact subgroup of $G_{\mathbf{h}}$;
- the restriction to G_∞ determines $\Psi(f;\cdot)$ by the strong approximation theorem for $G_{\mathbf{A}}$.

For every $\mathfrak{z} \in \mathcal{H}_m$, take an element g_∞ of G_∞ such that $\mathfrak{z} = g_\infty \mathbf{i}$ and set

$$\vartheta(f;\mathfrak{z}) = j(g_\infty, \mathbf{i})^n \Psi(f;(g_\infty, 1_{\mathbf{h}})). \tag{5.11}$$

Then we have

$$\begin{aligned}\vartheta(f;\mathfrak{z}) &= \sum_{x\in X_{\mathbf{Q}}} (\varphi_{\mathfrak{z}} \otimes f)(x) \\ &= \sum_{x=(x_\infty, x_{\mathbf{h}})\in X_{\mathbf{Q}}} \mathbf{e}(\mathrm{tr}({}^t x_\infty S x_\infty \mathfrak{z})) f(x_{\mathbf{h}}).\end{aligned}$$

Therefore we get a well-defined function $\mathfrak{z} \mapsto \vartheta(f;\mathfrak{z})$ on $\mathcal{H}_m$. Furthermore we can immediately see that

$$\vartheta(f;\gamma_\infty \mathfrak{z}) = j(\gamma_\infty, \mathbf{i})^n \vartheta(\pi_{\mathbf{h}}(\gamma_{\mathbf{h}}^{-1})f;\mathfrak{z}) \tag{5.12}$$

for any $\gamma = (\gamma_\infty, \gamma_{\mathbf{h}}) \in G_{\mathbf{Q}}$ and any $\mathfrak{z} \in \mathcal{H}_m$. Take an integral lattice L of V and put $f_{L^m} = \prod_{p\in\mathbf{h}} \mathrm{I}_{L_p^m}$. We can easily see $f_{L^m} \in \mathcal{S}(X_{\mathbf{h}})$ and

$$\vartheta(f_{L^m};\cdot) = \vartheta_L.$$

5.4. Relation of the global and local traces. Let N be a positive integer and N' its positive divisor. We define an open compact subgroup $D_{\mathbf{h}}(N)$ of $G_{\mathbf{h}}$ by $D_{\mathbf{h}}(N) = \prod_{p\in\mathbf{h}} D_p(N)$ and set

$$D_{\mathbf{Q}}(N) = G_{\mathbf{Q}} \cap G_\infty D_{\mathbf{h}}(N).$$

By a morphism $\gamma \mapsto (\gamma, \gamma, \gamma, \dots)$, we identify $\Gamma_0^{(m)}(N)$ with $D_{\mathbf{Q}}(N)$. Since the strong approximation theorem is hold for G, we can easily see that $G_\infty D_{\mathbf{Q}}(N)$ is dense in $G_\infty D_{\mathbf{h}}(N)$. Thus the image of $D_{\mathbf{Q}}(N)$ under the canonical projection $G_{\mathbf{A}} \ni g = (g_\infty, g_{\mathbf{h}}) \mapsto g_{\mathbf{h}} \in$

$G_{\mathbf{h}}$ is dense in $D_{\mathbf{h}}(N)$. Therefore we can identify a complete set of representatives of $\Gamma_0^{(m)}(N)\backslash\Gamma_0^{(m)}(N')$ with that of $D_{\mathbf{h}}(N)\backslash D_{\mathbf{h}}(N')$.

From now on, assume (see (5.9))

$$\begin{cases} N'|N & \text{if } n \text{ even,} \\ 4|N'|N & \text{if } n \text{ odd.} \end{cases} \tag{5.13}$$

We see χ_n is also a character of $D_{\mathbf{h}}(N)$ via the canonical projection $D_{\mathbf{h}}(N) \to D_2(N)$. Let L be an integral lattice of V with level$(L)|N$. We compute the global trace $T^{(n)}_{N,N'}\vartheta_L$ as follows:

$$(T^{(n)}_{N,N'}\vartheta_L)(g_\infty\mathbf{i}) = \sum_{\xi\in\Gamma_0^{(m)}(N)\backslash\Gamma_0^{(m)}(N')} \chi_n(\xi)j(\xi,g_\infty\mathbf{i})^{-n}\vartheta(f_{L^m};\xi g_\infty\mathbf{i})$$

$$= j(g_\infty,\mathbf{i})^n\sum_\gamma \chi_n(\gamma_{\mathbf{h}}^{-1})\Psi(f_{L^m};(g_\infty,\gamma_{\mathbf{h}}^{-1})) \qquad (see (5.12)) \tag{5.14}$$

$$= \vartheta\left(\sum_u \chi_n(u)\pi_{\mathbf{h}}(u)f_{L^m};(g_\infty,1_{\mathbf{h}})\right), \qquad (see (5.11)) \tag{5.15}$$

where $\gamma = (\gamma_\infty,\gamma_{\mathbf{h}})$ and u extend over $D_{\mathbb{Q}}(N)\backslash D_{\mathbb{Q}}(N')$ in (5.14) and $D_{\mathbf{h}}(N')/D_{\mathbf{h}}(N)$ in (5.15), respectively. Let the prime factorization of N' be $N' = \prod_{p\in\mathbf{h}} p^{l_p(N')}$. Notice that, since $D_p(N) = D_p(N')$ for almost all $p \in \mathbf{h}$, $\prod_{p\in\mathbf{h}}\tau^{(m)}_{p,l_p(N')}(\mathrm{I}_{L_p^m})$ is an element of $\mathcal{S}(X_{\mathbf{h}})$. The last equality (5.15) shows that, up to the multiplication by an nonzero constant, the two functions

$$T^{(n)}_{N,N'}\vartheta_L \quad \text{and} \quad \vartheta\left(\prod_{p\in\mathbf{h}}\tau^{(m)}_{p,l_p(N')}(\mathrm{I}_{L_p^m});\cdot\right)$$

are equal on $\mathcal{H}_m$. Therefore Theorem $\mathrm{L_p}$ implies Theorem L.

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