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# A CHARACTERISTIC CAUCHY PROBLEM OF NON-LERAY TYPE IN THE COMPLEX DOMAIN

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## §0. Introduction

We consider a Cauchy problem in the complex domain. It is assumed to be a *characteristic* problem in the sense that the characteristic points form a submanifold $T$ (of codimension 1) of the initial hypersurface $S$.

Since Leray, the studies on this subject dealt with the cases where the solution is singular on a characteristic hypersurface *tangent* to $S$ along $T$. See [L], [G-K-L], [H], [D], [O-Y] and [Y].

In the present paper, we consider a totally different situation: all the characteristic hypersurfaces issuing from $T$ are *transversal* with $S$.

First we give two examples to show that in this kind of characteristic Cauchy problem, the solution can be singular on the above-mentioned characteristic hypersurfaces even when all the Cauchy data are regular. Next, we consider a (ramified) Cauchy problem for a certain class of operators including the examples. We perform a singular change of coordinates and reduce our problem to results of Wagschal.

## §1. Examples with holomorphic data

In a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z$, let us consider Cauchy problems for the operators $Q_1$ and $Q_2$ defined by

$$Q_1 = (xD_t + tD_x)D_t, \quad Q_2 = Q_1 - xt^2D_z^2.$$

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We are going to solve, for $j = 1$ or $2$,

$$\begin{cases} Q_j u(t,x,z) = 0 \\ u|_S = -\frac{\pi i}{2}x^2 \\ D_t u|_S = ix \\ S = \{t = 0\} \end{cases}$$

On the initial hypersurface $S$, the characteristic points form a submanifold $T = \{t = x = 0\}$. The hypersurfaces $\{x = 0\}$, $\{x = t\}$ and $\{x = -t\}$ are characteristic hypersurfaces issuing from $T$. They are transversal with $S$. Although the data are holomorphic in a neighborhood of the origin, the solution $u$ is singular on the three characteristic hypersurfaces. In fact, we have

$$u = \frac{x^2}{2}\{\frac{t}{x}\sqrt{(\frac{t}{x})^2 - 1} - \log(\frac{t}{x} + \sqrt{(\frac{t}{x})^2 - 1})\}.$$

Since we are dealing with a multi-valued function, we have to clarify the definition of the restriction on $S$. Its precise meaning is that we choose a point $p$ of $S$ and that the initial condition is satisfied by the germ of $u$ at $p$.

We will prove for a class of operators including $Q_1$ and $Q_2$ that *the singular support of the solution is contained in this kind of characteristic hypersurfaces* when the data are arbitrary holomorphic functions. As a matter of fact, we can generalize this result to the case of ramified data.

## §2. Main result

In a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_y \times \mathbb{C}_z^n$, let us consider a second order operator $P(t, y, z; D_t, D_y, D_z)$ with holomorphic coefficients whose principal symbol $\sigma(P)$ is factorized into the form

$$\sigma(P)(t,y,z;\tau,\eta,\zeta) = \prod_{i=0,1}(\tau - \lambda_i(t,y,z;\eta,\zeta)),$$

where $\tau$, $\eta$ and $\zeta = (\zeta_1, \dots, \zeta_n)$ are the dual variables of $t$, $y$ and $z$ respectively.

We assume the following two conditions (1) and (2).

$$(1) \qquad \begin{cases} \lambda_0(t,0,z;1,0,\dots,0) = 0 \\ \lambda_1(t,y,z;1,0,\dots,0) = -qt^{q-1}, \\ q \text{ is an integer} \geq 2. \end{cases}$$

(2) For $i = 0, 1$, the function $(\eta, \zeta) \mapsto \lambda_i(t, y, z; \eta, \zeta)$ is linear.

The most simple example is

$$\lambda_0 = 0 \text{ or } y\eta, \quad \lambda_1 = -qt^{q-1}\eta.$$

Now we consider, in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n$, an operator $Q$ with holomorphic coefficients defined by

$$Q(t, x, z; D_t, D_x, D_z) = x^{2q-1} P(t, x^q, z; D_t, \frac{1}{qx^{q-1}} D_x, D_z).$$

Sometimes the exponent $2q - 1$ is larger than necessary to erase negative powers of $x$. For example, if

$$P(t, y, z; D_t, D_y, D_z) = P(t, y; D_t, D_y) = (D_t + qt^{q-1} D_y) D_t,$$

then

$$x^{q-1} P(t, x^q; \frac{1}{qx^{q-1}} D_x) = (x^{q-1} D_t + t^{q-1} D_x) D_t.$$

When $q = 2$, this is nothing but $Q_1$ which we studied before.

For the purpose of formulating a Cauchy problem, put $S = \{t = 0\} \subset \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n$, which is the initial hypersurface. It is easy to see that $T = \{t = x = 0\}$ is formed by the characteristic points of $Q$ on $S$. By the condition (1), the hypersurfaces

$$K_j = \{x = \exp(j \frac{2\pi i}{q}) \cdot t\} \ (j = 0, \ldots, q - 1), \quad K_q = \{x = 0\}$$

are characteristic hypersurfaces of $Q$ issuing from $T$.

We then consider a ramified characteristic Cauchy problem in an open connected neighborhood $\Omega$ of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n$:

$$(\mathrm{CP}) \begin{cases} Q(t, x, z; D_t, D_x, D_z) u(t, x, z) = 0, \\ D_t^h u(t, x, z)|_S = w_h(x, z), \ h = 0, 1. \end{cases}$$

Here we assume that there exists a point $p \in \Omega \cap (S \backslash T)$ such that for $h = 0, 1$, the function $w_h$ is holomorphic in a neighborhood (relative to $S$) of the point $p$ and can be analytically continued along all the paths from $p$ in $\Omega \cap (S \backslash T)$ (that is, $w_h$ is holomorphic in the universal covering space of $\Omega \cap (S \backslash T)$).

Since $p \notin T$, the usual Cauchy-Kowalevski theorem is valid there. (CP) admits a unique holomorphic solution $u$ in a neighborhood of the point $p$.

We are going to prove the

**Theorem 1.**

*There exists an open connected neighborhood $\Omega'$ of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n$ such that the solution $u$ of (CP) can be analytically continued to the universal covering space of $\Omega' \setminus \cup_{j=0}^{q} K_j$.*

*Of course this conclusion holds true when all the data are regular.*

*Proof.*

Put $x = y^{1/q}$. Then $D_y = \dfrac{1}{qx^{q-1}} D_x$. Therefore

$$Q(t, x, z; D_t, D_x, D_z) = y^{\frac{2q-1}{q}} P(t, y, z; D_t, D_y, D_z).$$

We reduce (CP) to the following *noncharacteristic* ramified Cauchy problem, which has been solved by Wagschal in [W2].

$$\text{(CP')} \begin{cases} P(t, y, z; D_t, D_y, D_z) u(t, y^{1/q}, z) = 0, \\ D_t^h u(t, y^{1/q}, z)|_{t=0} = w_h(y^{1/q}, z), \quad h = 0, 1. \end{cases}$$

The function $w_h(y^{1/q}, z)$ is holomorphic in the universal covering space of $\{(y, z) \in \mathbb{C} \times \mathbb{C}^n; 0 < |y| \ll 1, |z| \ll 1\}$ . ($a \ll 1$ means that $a \geq 0$ is sufficiently small).

Let $p' \in (\{0\} \times \mathbb{C}_y \times \mathbb{C}_z^n) \setminus \{y = 0\}$ be the point corresponding to $p$. Then (CP') admits a unique holomorphic solution $u(t, y^{1/q}, z)$ near $p'$. According to [W2], $u(t, y^{1/q}, z)$ can be analytically continued to the universal covering space of

$$\{(t, y, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n; |(t, y, z)| \ll 1\} \setminus (\{y = 0\} \cup \{y = t^q\}).$$

We finish the proof by coming back to the $(t, x, z)$-space. □

**Example.**

We saw before that $Q_1$ was not quite the same as $Q$, but this does not cause any difficulty. The equation $Q_1 u = 0$ is equivalent to $x^2 Q_1 u = 0$. The operator $x^2 Q_1$ is nothing but $Q$.

This example suggests that the choice of the exponent of $x$ in the definition of $Q$ is not essential.

**Remark 1.**

For convenience, put $y = z_0, \eta = \zeta_0$. Then, by virtue of Remarque 3.1 of [W2], (2) can be replaced by the following condition:

(3) There exists an integer $k$, $0 \leq k \leq n$, such that for $i = 0, 1$, the function $\lambda_i(t, z_0, z; \zeta_0, \dots, \zeta_k, 0, \dots, 0)$ is linear in $(\zeta_0, \dots, \zeta_k)$ and does not depend on the variables $(z_{k+1}, \dots, z_n)$.

This enables us to treat $Q_2$. In fact, when $n = 1$, $q = 2$, put

$$P(t, y, z; D_t, D_y, D_z) = D_t^2 + 2tD_tD_y - t^2D_z^2.$$

Then

$$\begin{aligned}
\sigma(P) &= \tau^2 + 2t\tau\eta - t^2\zeta^2 \\
&= (\tau + t\eta)^2 - t^2(\eta^2 + \zeta^2) \\
&= \{\tau + t(\eta + \sqrt{\eta^2 + \zeta^2})\}\{\tau + t(\eta - \sqrt{\eta^2 + \zeta^2})\}, \\
Q_2 &= xP(t, x^2, z; D_t, \frac{1}{2x}D_x, D_z).
\end{aligned}$$

**Remark 2.**

A singular change of coordinates was useful in some papers mentioned in the introduction ([L], [D], [O-Y] and [Y]). One introduces a new variable $w$ by setting $w = (t - x^l)^{1/l}$ for some positive integer $l$. In the present paper, we have performed a different kind of singular change of coordinates.

## §3. Inhomogeneous problem

If we choose a special class of $P$, we can treat an inhomogeneous problem. Assume that

$$\sigma(P)(t, y, z; \tau, \eta, \zeta) = \tau(\tau + qt^{q-1}\eta).$$

We employ the same notation as in §2. Let us consider:

$$(\mathrm{CP}^i)\begin{cases} Q(t, x, z; D_t, D_x, D_z)u(t, x, z) = v(t, x, z), \\ D_t^h u(t, x, z)|_S = w_h(x, z), \ h = 0, 1. \end{cases}$$

Here we assume that the function $v$ is holomorphic in a neighborhood of $p$ and can be analytically continued along all the paths from $p$ in $\Omega \setminus \cup_{j=0}^q K_j$ (that is, $v$ is holomorphic in the universal covering space of $\Omega \setminus \cup_{j=0}^q K_j$). Then we have

**Theorem 2.**

*There exists an open connected neighborhood $\Omega'$ of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n$ such that the solution $u$ of $(CP^i)$ can be analytically continued to the universal covering space of $\Omega' \setminus \cup_{j=0}^q K_j$.*

*Of course this conclusion holds true when all the data are regular.*

*Proof.*

We have to solve

$$\begin{cases} P(t,y,z;D_t,D_y,D_z)u(t,y^{1/q},z) = y^{-\frac{2q-1}{q}} v(t,y^{1/q},z), \\ P = D_t(D_t + qt^{q-1}D_y) + \text{lower}, \\ D_t^h u(t,y^{1/q},z)|_{t=0} = w_h(y^{1/q},z), \quad h = 0,1. \end{cases}$$

Since $v(t,x,z)$ is holomorphic in the universal covering space of $\Omega \setminus \cup_{j=0}^q K_j$, the function $y^{-\frac{2q-1}{q}} v(t,y^{1/q},z)$ is holomorphic in the universal covering space of

$$\{(t,y,z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n; |(t,y,z)| \ll 1\} \setminus (\{y=0\} \cup \{y = t^q\}).$$

This noncharacteristic inhomogeneous problem has been solved in [W1]. □

## §4. Geometry

What distinguishes the present study from conventional ones is the absence of singularities on a hypersurface tangent to the initial hypersurface S. It is explained by the following

**Proposition.**

*Under the assumption (1), there is no characteristic hypersurface of $Q$ that is tangent to $S$ along $T$.*

*Proof.*

We have

$$\begin{aligned} \sigma(Q)(t,x,z;\tau,\xi,\zeta) &= x^{2q-1} \prod_{i=0,1} \{\tau - \lambda_i(t,x^q,z;\frac{1}{qx^{q-1}}\xi,\zeta)\} \\ &= x \prod_{i=0,1} \{x^{q-1}\tau - \lambda_i(t,x^q,z;\frac{1}{q}\xi, x^{q-1}\zeta)\}. \end{aligned}$$

It is easy to see that $S$ itself is not a characteristic hypersurface. A hypersurface $\neq S$ which is tangent to $S$ along $T$ has an expression of the form:

$$\varphi = t + x^N \psi(x, z) = 0, \quad N \geq 2$$

where $\psi$ is a holomorphic function with $\psi(0, z) \not\equiv 0$.

We have

$$\begin{aligned}
&\sigma(Q)(t, x, z; \operatorname{grad} \varphi) \\
=&x \prod_{i=0,1} [x^{q-1} - \lambda_i(t, x^q, z; \frac{1}{q} N x^{N-1} \psi(x, z) + \frac{1}{q} x^N D_x \psi, x^{N+q-1} D_z \psi)].
\end{aligned}$$

For a generic $z$ we have $\psi(0, z) \neq 0$. We fix such a $z$. Obviously $\psi(x, z) \neq 0$ holds if $|x| \ll 1$. Then it follows that

$$\begin{aligned}
&\sigma(Q)(t, x, z; \operatorname{grad} \varphi) \\
=&x \prod_{i=0,1} [x^{q-1} - \frac{1}{q} N x^{N-1} \psi \lambda_i(t, x^q, z; 1 + \frac{x}{N\psi} D_x \psi, \frac{q x^q}{N\psi} D_z \psi)] \\
=&x \prod_{i=0,1} [x^{q-1} - \frac{1}{q} N x^{N-1} \psi (1 + \frac{x}{N\psi} D_x \psi) \lambda_i(t, x^q, z; 1, (1 + \frac{x}{N\psi} D_x \psi)^{-1} \frac{q x^q}{N\psi} D_z \psi)].
\end{aligned}$$

The assumption (1) implies that as $x$ tends to zero

$$\begin{aligned}
\lambda_0(t, x^q, z; 1, (1 + \frac{x}{N\psi} D_x \psi)^{-1} \frac{q x^q}{N\psi} D_z \psi) &= O(x^q) \\
\lambda_1(t, x^q, z; 1, (1 + \frac{x}{N\psi} D_x \psi)^{-1} \frac{q x^q}{N\psi} D_z \psi) &= -q t^{q-1} + O(x^q).
\end{aligned}$$

Therefore by restricting them on the hypersurface $\{\varphi = 0\}$, we obtain

$$\begin{aligned}
&\lambda_0(t, x^q, z; 1, (1 + \frac{x}{N\psi} D_x \psi)^{-1} \frac{q x^q}{N\psi} D_z \psi)|_{\varphi=0} = O(x^q) \\
&\lambda_1(t, x^q, z; 1, (1 + \frac{x}{N\psi} D_x \psi)^{-1} \frac{q x^q}{N\psi} D_z \psi)|_{\varphi=0} = -q(-x^N \psi)^{q-1} + O(x^q) = O(x^q).
\end{aligned}$$

Hence $\sigma(Q)|_{\varphi=0}$ is different from zero if $0 < |x| \ll 1$. Thus $\{\varphi = 0\}$ is not a characteristic hypersurface. □

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