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ON THE CONFLUENT HYPERGEOMETRIC FUNCTIONS IN 2 VARIABLES

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§0. Introduction.

Let $M(r,n)$ be the set of complex matrices, $\lambda=(\lambda_0,\ldots,\lambda_{l-1})$ a partition of n. We conside the action of $GL(r,\mathbb{C})\times H_\lambda$ on $Z_{r,n}:=\{z\in M(r,n): rankz=r\}$ defined by

$$(*)\qquad \begin{aligned} GL(r,\mathbb{C})\times Z_{r,n}\times H_\lambda &\longrightarrow Z_{r,n}\\ (g,z,h) &\longmapsto gzh, \end{aligned}$$

where $H_\lambda=J(\lambda_0)\times\cdots\times J(\lambda_{l-1})\subset GL(n)$ be the associated maximal abelian subgroup with respect to λ, $J(m)=\left\{\sum_{i=0}^{m-1}h_i\Lambda^i\ :\ h_0\in\mathbb{C}^\times,h_1,\ldots,h_{m-1}\in\mathbb{C}\right\}$

$$\Lambda:=\begin{pmatrix} 0 & 1 & & \mathbf{0}\\ & 0 & \ddots & \\ & & \ddots & 1\\ \mathbf{0} & & & 0 \end{pmatrix}$$

Let $\iota: H_\lambda\longrightarrow\prod_i(\mathbb{C}^\times\times\mathbb{C}^{\lambda_i-1})$. For $z\in Z_{r,n}$, the generalized confluent hypergeometric function (CHG function, for short) is defined as

$$(0.1)\qquad \Phi(z;\alpha)=\int_\Delta\chi(\iota^{-1}(tz);\alpha)\cdot\omega$$

where α be an n-tuple of complex numbers satisfying $\sum_{i=0}^{l-1}\alpha_0^{(i)}=-r$, λ the character of the universal covering group of H_λ,and Δ is a *twisted cycle* in the

t-space depending on z and α. The function Φ admits the following symmetries:

$$\Phi(gz;\alpha) = (\det g)^{-1}\Phi(z;\alpha) \qquad g \in GL(r,\mathbb{C}) \tag{0.2}$$

$$\Phi(zh_\lambda;\alpha) = \chi(h_\lambda)\Phi(z;\alpha) \qquad h_\lambda \in H_\lambda. \tag{0.3}$$

$$\Phi(zw_\lambda;\alpha) = \Phi(z;\alpha^t w_\lambda) \qquad w_\lambda \in W_\lambda, \tag{0.4}$$

where W_λ is an analogue of the Weyl group, see [K-K] and Section 1.

The CHG functions Φ on $Z_{2,4}$ and $Z_{2,5}$ for various partitions λ of 4 and 5 were investigated in the papers [K-H-T],[O-K] and [K-K]. It is known that the functions Φ are generalizations of Gauss', Kummer's, Bessel's, Hermite's, Airy's functions and the classical hypergeometric functions of two variables, i.e., $F_1, \phi_1, \phi_2, \phi_3, G_2, \Gamma_1, \Gamma_2$ in Horn's list ([Erd 1]). In this talk, we study the hypergeometric functions of type λ in two variables on the strata of the set $M(3,6)$ of 3×6 complex matrices

§1. Construction of the group W_λ.

Set $\lambda^{(0)} = (1,1,1,1,1,1)$, $\lambda^{(1)} = (2,1,1,1,1)$, $\lambda^{(2)} = (2,2,1,1)$, $\lambda^{(3)} = (2,2,2)$, $\lambda^{(4)} = (3,1,1,1)$, $\lambda^{(5)} = (3,2,1)$, $\lambda^{(6)} = (3,3)$, $\lambda^{(7)} = (4,1,1)$ and $\lambda^{(8)} = (4,2)$. We set

$$P_{\lambda^{(0)}} = \mathfrak{S}_6, \quad P_{\lambda^{(1)}} = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & \mathfrak{S}_4 \end{pmatrix} \right\}$$

$$P_{\lambda^{(2)}} = \left\{ \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & \mathfrak{S}_2 \end{pmatrix}, \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & \mathfrak{S}_2 \end{pmatrix} \right\}$$

$$P_{\lambda^{(3)}} = \left\{ \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ I_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix} \right\}$$

$$P_{\lambda^{(4)}} = \left\{ \begin{pmatrix} I_3 & 0 \\ 0 & \mathfrak{S}_3 \end{pmatrix} \right\}, \quad P_{\lambda^{(5)}} = \left\{ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$P_{\lambda^{(6)}} = \left\{ \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}, \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \right\}$$

$$P_{\lambda^{(7)}} = \left\{ \begin{pmatrix} I_4 & 0 \\ 0 & \mathfrak{S}_2 \end{pmatrix} \right\}, \quad P_{\lambda^8} = \left\{ \begin{pmatrix} I_4 & 0 \\ 0 & I_2 \end{pmatrix} \right\}$$

where I_i is the $i \times i$ identity matrix, $\mathfrak{S}_i$ is the group of $i \times i$ permutation matrices and

$$\left\{\begin{pmatrix} * & 0 \\ 0 & \mathfrak{S}_i \end{pmatrix}\right\} := \left\{\begin{pmatrix} * & 0 \\ 0 & A \end{pmatrix} \;:\; A \in \mathfrak{S}_i\right\}.$$

Then we have the following proposition (see [K-K]).

Proposition 1.1. *For the partitions $\lambda^{(\nu)}$, the Weyl groups $W_{\lambda^{(\nu)}}$ ($\nu = 0, \ldots, 8$) are given by*

$$W_{\lambda^{(\nu)}} = R_{\lambda^{(\nu)}} \rtimes P_{\lambda^{(\nu)}},$$

where

$$\begin{aligned} R_{\lambda^{(0)}} &= I_6 & R_{\lambda^{(1)}} &= \operatorname{diag}(W(2), I_4) \\ R_{\lambda^{(2)}} &= \operatorname{diag}(W(2), W(2), I_2) & R_{\lambda^{(3)}} &= \operatorname{diag}(W(2), W(2), W(2)) \\ R_{\lambda^{(4)}} &= \operatorname{diag}(W(3), I_3) & R_{\lambda^{(5)}} &= \operatorname{diag}(W(3), W(2), 1) \\ R_{\lambda^{(6)}} &= \operatorname{diag}(W(3), W(3)) & R_{\lambda^{(7)}} &= \operatorname{diag}(W(4), I_2) \\ R_{\lambda^{(8)}} &= \operatorname{diag}(W(4), W(2)). \end{aligned}$$

§2. Orbital decomposition of the set of strata.

Set $D(i,j,k) = \det(z_i, z_j, z_k)$ for $z = (z_0, z_1, \ldots, z_5) \in M(3,6)$.

Definition 2.1. *Let λ be a Young diagram of weight 6, $(i,j,k), (i,m,n)$ two subdiagrams of λ, where i,j,k,m,n are mutually distinct. We denote by the symbol $\{(i,j,k),(i,m,n)\}$ the set*

$$\left\{ z \in M(3,6) \;\middle|\; \begin{array}{l} D(i,j,k) = D(i,m,n) = 0, \\ D(p,q,r) \neq 0 \text{ for any} \\ \text{other subdiagram } (p,q,r) \end{array} \right\}$$

and call it a general stratum of type $(3,6)$ associated to λ (for short, a stratum).

Let S_λ denote the set of strata $\{(i,j,k),(i,m,n)\}$ associated to the Young diagram λ. We simply write S for $S_{\lambda^{(0)}}$.

Proposition 2.1. *(1) The Weyl group W acts transitively on S.*
(2) $\#S = 90$.

Proposition 2.2. *Under the action of $P_{\lambda(\nu)}$, the orbital decomposition of $S_{\lambda(\nu)}$ is described as $S_{\lambda(\nu)} = \coprod_i O_{P_{\lambda(\nu)}}(s^i_\nu)$, where*

$$
\begin{aligned}
&s^1_1 = \{(0,1,2),(0,4,5)\}, s^2_1 = \{(4,0,1),(4,2,3)\}, s^3_1 = \{(4,0,5),(4,2,3)\},\\
&s^4_1 = \{(0,2,3),(0,4,5)\}; s^1_2 = \{(0,1,2),(0,4,5)\}, s^2_2 = \{(2,0,1),(2,3,4)\},\\
&s^3_2 = \{(4,0,1),(4,2,3)\}, s^4_2 = \{(0,2,3),(0,4,5)\}, s^5_2 = \{(0,1,4),(0,2,5)\},\\
&s^6_2 = \{(4,0,5),(4,2,3)\}; s^1_3 = \{(0,1,2),(0,4,5)\}, s^2_3 = \{(4,0,1),(4,2,3)\};\\
&s^1_4 = \{(0,1,2),(0,3,4)\}, s^2_4 = \{(0,1,3),(0,4,5)\}, s^3_4 = \{(3,0,1),(3,4,5)\};\\
&s^1_5 = \{(0,1,2),(0,3,4)\}, s^2_5 = \{(0,1,2),(0,3,5)\}, s^3_5 = \{(3,0,1),(3,4,5)\},\\
&s^4_5 = \{(5,0,1),(5,3,4)\}; s^1_6 = \{(0,1,2),(0,3,4)\}; s^1_7 = \{(0,1,2),(0,4,5)\};\\
&s^1_8 = \{(0,1,2),(0,4,5)\}.
\end{aligned}
$$

§3. Normal forms of matrices.

Set $Z^\nu = \bigcup_{s\in S_{\lambda(\nu)}} s$. For simplicity we write Z for Z^0. By the action of $GL(3)\times H_{\lambda(\nu)}$ on Z^ν defined by (*), We first describe how to take the elements of $GL(3)\setminus Z/H$ as the normal forms of $z\in Z$. We fix one stratum $s_0 = \{(4,0,1),(4,2,3)\}\in S$.

Proposition 3.1. *For each $i = 0,\dots,14$, the following assertion holds: For any $z\in s_0$ there exists a unique $(x,y)\in\mathbb{C}^2$ such that*

(1) $f_i(x,y)\neq 0$, (2) $z\in GL(3)\vec{z}_i(x,y)H$,

where $\vec{z}_i = \vec{z}_i(x,y)$ and $f_i = f_i(x,y)$ are given by

$$\vec{z}_0(x,y) = \begin{pmatrix}1&1&0&0&0&y\\0&0&1&1&0&x\\0&1&0&1&1&1\end{pmatrix} \quad \vec{z}_1(x,y) = \begin{pmatrix}1&1&0&0&0&y\\0&0&1&1&0&1\\0&1&0&x&1&1\end{pmatrix}$$

$$\vec{z}_2(x,y) = \begin{pmatrix}1&1&0&0&0&1\\0&0&1&1&0&1\\0&y&0&x&1&1\end{pmatrix} \quad \vec{z}_3(x,y) = \begin{pmatrix}1&1&0&0&0&1\\0&0&1&1&0&x\\0&y&0&1&1&1\end{pmatrix}$$

$$\vec{z}_4(x,y) = \begin{pmatrix}1&1&0&0&0&1\\0&0&1&1&0&1\\0&y&0&1&1&x\end{pmatrix} \quad \vec{z}_5(x,y) = \begin{pmatrix}1&1&0&0&0&y\\0&0&1&1&0&1\\0&1&0&1&1&x\end{pmatrix}$$

$$\vec{z}_6(x,y) = \begin{pmatrix}1&1&0&0&0&1\\0&0&1&1&0&1\\0&1&0&x&1&y\end{pmatrix} \quad \vec{z}_7(x,y) = \begin{pmatrix}1&1&0&0&0&1\\0&0&1&1&0&x\\0&1&0&1&1&y\end{pmatrix}$$

$$\vec{z}_8(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & x \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad \vec{z}_9(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & x & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\vec{z}_{10}(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & x \end{pmatrix} \qquad \vec{z}_{11}(x,y) = \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\vec{z}_{12}(x,y) = \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & 1 \end{pmatrix} \qquad \vec{z}_{13}(x,y) = \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & x & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\vec{z}_{14}(x,y) = \begin{pmatrix} 1 & x & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & y \end{pmatrix}$$

and

$$\begin{aligned}
f_i(x,y) &= xy(1-x)(1-y)(1-x-y) && \text{for } i = 0,\dots,3 \\
f_i(x,y) &= xy(1-x)(x-y)(1-x+y) && \text{for } i = 4,5 \\
f_i(x,y) &= xy(y-x)(1-y)(1+x-y) && \text{for } i = 6,7 \\
f_i(x,y) &= xy(1-x)(xy-1)(1-xy+y) && \text{for } i = 8,9 \\
f_i(x,y) &= xy(1-x)(xy-1)(1-xy+y) && \text{for } i = 10 \\
f_i(x,y) &= xy(1-x)(1-y)(1+xy-y) && \text{for } i = 11,12 \\
f_i(x,y) &= xy(1-x)(1-y)(xy-x-y) && \text{for } i = 13 \\
f_i(x,y) &= xy(xy-1)(1-y)(1-xy+x) && \text{for } i = 14.
\end{aligned}$$

We set $N = \left\{ \vec{z_i} \,:\, 0 \le i \le 14 \right\}$. For the stratum s_0, we denote by N_{s_0} the set $\left\{ \vec{z} = \sigma \vec{z_i} \,:\, \sigma \in \mathfrak{S}_3, \vec{z_i} \in N \right\}$ and call $\vec{z} \in N_{s_0}$ a *normal form* of $z \in s_0$.

Proposition 3.2. *The normal forms of the matrices in any other stratum $s \in S$ can be obtained by the action of Weyl group $W \simeq \mathfrak{S}_6$ on N_{s_0}.*

Using the same method as above, we can obtain the normal forms of $z \in Z_\nu$ for $\nu = 1, 2, \dots, 8$.

§4 CHG functions and the classical CHG functions of 2 variables.

Definition 4.1. For $0 \le \nu \le 8$, the function Φ given by (0.1), i.e.,

$$\Phi(z;\alpha) = \int_\Delta \chi(\iota^{-1}(tz);\alpha) \cdot \omega \qquad \text{for} \quad z \in Z^\nu \tag{4.1}$$

is called the confluent hypergeometric function of type $\lambda^{(\nu)}$ (for short CHG function).

By the symmetry

$$\Phi(zw_\lambda;\alpha)=\Phi(z;\alpha^t w_\lambda) \qquad \text{for} \quad w_\lambda\in W_\lambda,$$

we can list up the functions Φ on Z^ν with normalized parameters (see Table III). On the other hand, there is a list of the classical CHG functions of two variables, which is known as Horn's list (see [Erd 1]). Integral representations of these functions have been investigated by M. Kita [Ki], M. Yoshida [Y], B. Dwork and F. Loeser [D-L].

We reinterpret some of the functions $F_2,\Psi_1,\Psi_2,F_3,\Xi_1,\Xi_2,H_2,\mathbf{H}_k(k=2,3,4,5,11)$ in list in terms of the CHG functions. The changes of variables

$$\begin{array}{ll}
(u,v)\longrightarrow(-1/u,-1/v) & \text{for } F_2,\ \Psi_1 \text{ and } \Psi_2,\\
(u,v)\longrightarrow(-u,-v) & \text{for } F_3 \text{ and } \Xi_1,\\
(u,v)\longrightarrow(-u,1/v) & \text{for } \Xi_2,\\
(u,v)\longrightarrow(-1/u,v) & \text{for } H_2 \text{ and } \mathbf{H}_k\ (k=2,3,4,5,11)
\end{array}$$

transform the integral representations of these functions into the following:

$$\begin{array}{ll}
F_2: & v^{\beta'-\gamma'}(v+y)^{-\beta'}u^{\beta-\gamma}(u+x)^{-\beta}(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv\\
\Psi_1: & v^{-\gamma'}\exp(-\frac{y}{v})u^{\beta-\gamma}(u+x)^{-\beta}(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv\\
\Psi_2: & v^{-\gamma'}\exp(-\frac{y}{v})u^{-\gamma}\exp(-\frac{x}{u})(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv\\
F_3: & (1+yv)^{-\beta'}v^{\alpha'-1}(1+u+v)^{\gamma-\alpha-\alpha'-1}u^{\alpha-1}(1+xu)^{-\beta}dudv\\
\Xi_1: & \exp(-yv)v^{\alpha'-1}(1+u+v)^{\gamma-\alpha-\alpha'-1}u^{\alpha-1}(1+xu)^{-\beta}dudv\\
\Xi_2: & \exp(-yv)v^{\gamma-\beta-2}\exp(-\frac{u+1}{v})u^{\beta-1}(1+xu)^{-\alpha}dudv\\
H_2: & v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}(1-yu)^{-\beta'}u^{\beta-\delta}(u+x)^{-\beta}dudv\\
\mathbf{H}_2: & v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})(1-yv)^{-\beta'}u^{\beta-\delta}(u+x)^{-\beta}dudv\\
\mathbf{H}_3: & v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})\exp(yv)u^{\beta-\delta}(u+x)^{-\beta}dudv\\
\mathbf{H}_5: & v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})\exp(yv)u^{-\delta}\exp(-\frac{x}{u})dudv
\end{array}$$

$$H_2: \quad u^{\beta-\delta}(u+x)^{-\beta}(1-yv)^{-\beta'}v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$
$$\mathbf{H}_{11}: \quad u^{-\delta}\exp(-\frac{x}{u})(1-yv)^{-\beta'}v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$
$$\mathbf{H}_4: \quad u^{-\delta}\exp(-\frac{x}{u})\exp(yv)v^{\gamma}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$

For these functions, the corresponding partitions λ and normal forms $\vec{x}_i = \vec{x}_i(x,y)$ are tabulated in the following:

TABLE II

Function (λ)	Normal form $\vec{x}_i = \vec{x}_i(x,y)$	$g_i(x,y)$
F_2 ($\lambda^{(0)}$)	$\vec{x}_1 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$xy(1-x-y)$ $(1-x)(1-y)$
Ψ_1 ($\lambda^{(1)}$)	$\vec{x}_2 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$xy(x-1)$
Ψ_2 ($\lambda^{(2)}$)	$\vec{x}_3 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	xy
F_3 ($\lambda^{(0)}$)	$\vec{x}_4 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 1 & 0 & 0 \end{pmatrix}$	$xy(xy-x-y)$ $(1-x)(1-y)$
Ξ_1 ($\lambda^{(1)}$)	$\vec{x}_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 1 & 0 & 0 \end{pmatrix}$	$xy(x-1)$
Ξ_2 ($\lambda^{(2)}$)	$\vec{x}_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 0 & 0 & 0 \end{pmatrix}$	$xy(x-1)$
H_2 ($\lambda^{(0)}$)	$\vec{x}_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(xy-y+1)$ $(1-x)(1-y)$
$\mathbf{H}_2$ ($\lambda^{(1)}$)	$\vec{x}_8 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(x-1)$

Function (λ)	Normal form $\vec{x}_i = \vec{x}_i(x,y)$	$g_i(x,y)$
$\mathbf{H}_3$ ($\lambda^{(2)}$)	$\vec{x}_9 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(x-1)$
$\mathbf{H}_5$ ($\lambda^{(3)}$)	$\vec{x}_{10} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	xy
H_2 ($\lambda^{(0)}$)	$\vec{x}_{11} = \begin{pmatrix} 0 & x & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	$xy(xy-y+1)$ $(1-x)(1-y)$
$\mathbf{H}_{11}$($\lambda^{(1)}$)	$\vec{x}_{12} = \begin{pmatrix} 0 & x & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	$xy(y-1)$
$\mathbf{H}_4$ ($\lambda^{(2)}$)	$\vec{x}_{13} = \begin{pmatrix} 0 & x & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	xy

For the normal forms $\vec{x}_i = \vec{x}_i(x,y)$, the variables $(x,y) \in \mathbb{C}^2$ are subject to the condition $g_i(x,y) \neq 0$.

Proposition 4.1. *Let $\lambda^{(\nu)}$ and the normal forms $\vec{x_i} = \vec{x}_i(x,y)$ be given in Table III. The CHG functions on $GL(3) \setminus Z^\nu/H_{\lambda^{(\nu)}}$ with the normalized parameters β_ν $(0 \leq \nu \leq 3)$ are related with the classical hypergeometric functions of two variables, for instence, as*

$$\begin{aligned}
\Phi_{\lambda^{(0)}}(\vec{x}_1;\beta_0) &= \int_{\Delta_1} v^{\alpha_0}(v+y)^{\alpha_1}u^{\alpha_2}(u+x)^{\alpha_3}(1+u+v)^{\alpha_5}dudv \\
&= C_1F_2(\alpha_4+1,-\alpha_3,-\alpha_1,-\alpha_2-\alpha_3,-\alpha_0-\alpha_1;x,y) \\
\Phi_{\lambda^{(1)}}(\vec{x}_2;\beta_1) &= \int_{\Delta_2} v^{\alpha_0}\exp(-\frac{y}{v})u^{\alpha_2}(u+x)^{\alpha_3}(1+u+v)^{\alpha_5}dudv \\
&= C_2\Psi_1(\alpha_4+1,-\alpha_3,-\alpha_3-\alpha_2,-\alpha_0;x,y) \\
\Phi_{\lambda^{(2)}}(\vec{x}_3;\beta_2) &= \int_{\Delta_3} v^{\alpha_0}\exp(-\frac{y}{v})u^{\alpha_2}\exp(-\frac{x}{u})(1+u+v)^{\alpha_5}dudv \\
&= C_3\Psi_2(\alpha_4+1,-\alpha_2,-\alpha_0;x,y).
\end{aligned}$$

The properties of those functions can be described in the following Table.

TABLE III

Type	Function	Stratum	Orbit
$(1,1,1,1,1,1)$	$G_1 = F_2$	$\{(4,0,1),(4,2,3)\}$	S
$(2,1,1,1,1)$	$G_2 = \Psi_1$	//	$O_{P_{\lambda(1)}}(s_1^2)$
$(2,2,1,1)$	$G_3 = \Psi_2$	//	$O_{P_{\lambda(2)}}(s_2^3)$
$(1,1,1,1,1,1)$	$G_4 = F_3$	$\{(0,1,2),(0,4,5)\}$	S
$(2,1,1,1,1)$	$G_5 = \Xi_1$	//	$O_{P_{\lambda(1)}}(s_1^1)$
$(2,2,1,1)$	$G_6 = \Xi_2$	//	$O_{P_{\lambda(2)}}(s_2^1)$
$(1,1,1,1,1,1)$	$G_7 = H_2$	$\{(2,0,3),(2,4,5)\}$	S
$(2,1,1,1,1)$	$G_8 = \mathbf{H}_2$	//	$O_{P_{\lambda(1)}}(s_1^3)$
$(2,2,1,1)$	$G_9 = \mathbf{H}_3$	//	$O_{P_{\lambda(2)}}(s_2^1)$
$(2,2,2)$	$G_{10} = \mathbf{H}_5$	//	$O_{P_{\lambda(3)}}(s_3^2)$
$(1,1,1,1,1,1)$	$G_{11} = H_2$	$\{(2,0,1),(2,3,4)\}$	S
$(2,1,1,1,1)$	$G_{12} = \mathbf{H}_{11}$	//	$O_{P_{\lambda(1)}}(s_1^2)$
$(2,2,1,1)$	$G_{13} = \mathbf{H}_4$	//	$O_{P_{\lambda(2)}}(s_2^2)$

G_i is a multi-valued holomorphic function in the domain:

$$X_i = \{(x,y) \in \mathbb{C}^2 \, : \, g_i(x,y) \neq 0\}, \tag{4.2}$$

where $g_i(x,y)$ $(1 \leq i \leq 13)$ are given in Table II.

Note that the functions $\{F_2,\ F_3,\ H_2\}$, $\{\Xi_2,\ \mathbf{H}_3\}$, $\{\Psi_1,\ \mathbf{H}_{11}\}$ belong to the same orbits, respectively.

§5 Transformation formulae.

We systematically deduce some transfomation fomulae for the systems of partial differental equations from the symmetries for the function Φ.

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)$$
$$=x^{-\beta}y^{-\beta'}F_2(\beta+\beta'-\gamma+1,\beta,\beta',\beta-\alpha+1,\beta'-\alpha'+1;\frac{1}{x},\frac{1}{y}).$$
$$H_2(\alpha,\beta,\beta',\gamma,\delta;x,y)$$
$$=y^{-\beta'}F_2(\alpha+\beta',\beta,\beta',\delta,\beta'-\gamma+1;x,-\frac{1}{y}),$$
$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)$$
$$=x^{-\alpha}H_2(\alpha-\gamma+1,\alpha,\alpha',\beta',\alpha-\beta+1;\frac{1}{x},-y),$$
$$\mathbf{H}_{11}(\alpha,\beta',\gamma,\delta;x,y)$$
$$=y^{-\beta'}\Psi_1(\alpha+\beta',\beta',\beta'-\gamma+1,\delta;-\frac{1}{y},x),$$
$$\mathbf{H}_3(\alpha,\beta,\delta;x,y)$$
$$=x^{-\beta}\Xi_2(\beta,\beta-\delta+1,-\alpha+\beta+1;\frac{1}{x},-y).$$

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