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INSTABILITY OF THE MOTION OF AN IDEAL FLUID CONTAINED IN 2-SPHERE AS A GEODESIC EQUATION

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ABSTRACT. The Euler's equation for an incompressible fluid filled in a Riemannian manifold D is regarded as a geodesic equation on the group of volume-preserving diffeomorphisms of D provided with a one-sided invariant metric. A negative sectional curvature implies instability of the geodesic with respect to the corresponding flow and perturbation. The exponential growth of the perturbation is estimated from the values of the sectional curvatures.

The expression of the components of Riemannian curvature tensor of the group of area-preserving diffeomorphisms on 2-sphere is given in explicit formulas through $3-j$ coefficients.

The mean sectional curvature associated with the flow described by the flow function Y_2^0 is estimated for example and the result suggests that the initial perturbation ε grows to $3\times 10^2\varepsilon$ after the period during which the particles on $u = cos\theta = \frac{1}{2}$ rotate around $z-$ axis.

1. INTRODUCTION

Let $T(t)$ be the map which maps the configuration of some fluid particles at the time t_0 to the configuration at the time t. Then the motion of the fluid is regarded as this time dependent map $T(t)$. Continuity of the fluid requires this map $T(t)$ to be diffeomorphism. We will regard the diffeomorphism $T(t)$ as a point element in the group of the diffeomorphisms. From now on, $T(t)$ will be denoted by x_t.

The Euler's equation for an ideal incompressible fluid filled in a Riemannian manifold D is regarded as a geodesic equation on the group of volume-preserving diffeomorphisms of D provided with a one-sided invariant metric [2]. Let the group be denoted by $\mathfrak{D}_{vol}(D)$. The *geodesic* $x_t \in \mathfrak{D}_{vol}(D)$, $t \in \mathbf{R}$, satisfies the following *geodesic equation*:

$$\nabla_{\dot{x}_t}\dot{x}_t = 0, \tag{1.1}$$

$$\text{where} \qquad \dot{x}_t = \frac{d}{dt}x_t \in T_{x_t}\mathfrak{D}_{vol}(D), \qquad x_t \in \mathfrak{D}_{vol}(D).$$

The metric g, the Riemannian connection ∇, the Riemannian curvature tensor R and the sectional curvature κ will be defined later in Section 2. A *Jacobi field* Y is a vector field along the geodesic x_t which satisfies the following *Jacobi equation*:

$$\nabla^2_{\dot{x}_t} Y + R(Y, \dot{x}_t)\dot{x}_t = 0. \tag{1.2}$$

A Jacobi field is an infinitesimal variation of the geodesic x_t and is uniquely determined by the values of Y and $\nabla_{\dot{x}_a} Y$ at one point x_a (or an initial point x_0) on the geodesic. We obtain from (1.2) the following equation:

$$\frac{d^2}{dt^2}\|Y\|^2 = 2(\|\nabla_{\dot{x}_t} Y\|^2 - g(R(Y, \dot{x}_t)\dot{x}_t, Y)). \tag{1.3}$$

The length $\|Y\|$ of the Jacobi field Y grows exponentially with respect to t if the value of the sectional curvature $\kappa(Y, \dot{x}_t)$ is negative. For a geodesic x_t parameterized to satisfy the property $\|\dot{x}_t\| = 1$ for all t, the minimum exponent of the exponential growth with respect to t for the fixed value of Y at one point x_a is given by $\sqrt{-\kappa(Y, \dot{x}_a)}$ and this is the case when $\nabla_{\dot{x}_a} Y = 0$. If we regard the values of Y and $\nabla_{\dot{x}_t} Y$ at x_0 as initial perturbations, the negative sectional curvature implies the instability of the geodesic. This instability suggests that it is impossible to predict the passive scalar advected by the motion of the ideal incompressible fluid over a certain period, and the period is estimated from the value of the sectional curvature κ.

The flow over 2-sphere(S^2) may be regarded as a simplified model of the motion of atmosphere over the earth.

2. Basic Notations

Let S^2 be defined in $\mathbf{R}^3$ by the equation $x^2 + y^2 + z^2 = r^2$ and its *Riemannian metric* $(\ , \ )$ be defined as the restriction of the standard Euclidean metric of $\mathbf{R}^3$. Let (θ, φ) and (u, φ) be coordinates defined by:

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta = ru. \end{aligned} \tag{2.1}$$

and the Riemannian metric in S^2 $(\ , \ )$ has components

$$\begin{aligned} (\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) &= \frac{r^2}{1-u^2} \\ (\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}) &= r^2(1-u^2) \\ (\frac{\partial}{\partial u}, \frac{\partial}{\partial \varphi}) &= 0. \end{aligned} \tag{2.2}$$

The *Riemannian volume (area) element* μ is given by:

$$\mu = \frac{S}{4\pi} d\varphi \wedge du, \tag{2.3}$$

where $S = 4\pi r^2$ is the area of the sphere.

Let the *group of volume-preserving diffeomorphisms* $\mathfrak{D}_{vol}(S^2)$ *of* S^2 acts on S^2 from right. The elements of $T\mathfrak{D}_{vol}(S^2)$ induce vector fields on S^2 with divergence-free. The set of vector fields on S^2 with divergence-free will be denoted by $\mathfrak{X}_{vol}(S^2)$, and the vector field induced by $A \in T\mathfrak{D}_{vol}(S^2)$ will be denoted by A^*. The *Riemannian metric* g *on* $\mathfrak{D}_{vol}(S^2)$ is induced from the metric (,) on S^2 as follows:

$$(2.4) \qquad g_a(A,B) = \int_{S^2} (A^*, B^*)\mu, \qquad a \in \mathfrak{D}_{vol}(S^2), A, B \in T_a\mathfrak{D}_{vol}(S^2).$$

This metric is *left*-invariant.

Remark. In [2] ,[3], $\mathfrak{D}_{vol}(D)$ is (implicitly) determined to act on D from *left*, so that the corresponding metric is *right*-invariant.

For $X \in \mathfrak{X}_{vol}(S^2)$, we have:

$$(2.5) \qquad 0 = divX = L_X\mu = d\iota_X\mu,$$

where L_x is the Lie derivative with respect to the vector field X and ι_X is the contraction with X defined as follows:

$$(2.6) \qquad \iota_X\mu(Y) = \mu(X,Y).$$

(2.5) is obtained from the identity $L_X = d \circ \iota_X + \iota_X \circ d$. Since de Rham cohomology group $H^1(S^2)$ is 0, there exists a unique function ψ_X on S^2 for each $X \in \mathfrak{X}_{vol}(S^2)$ satisfying:

$$(2.7) \qquad d\psi_X = \iota_X\mu.$$

$\psi_X$is said to be a *flow function* of the vector field X on S^2. Let X_ψ denote the vector field whose flow function is ψ. X_ψ is expressed in the coordinate (u,φ) by

$$(2.8) \qquad X_\psi = \frac{4\pi}{S}\left(\frac{\partial\psi}{\partial u}\frac{\partial}{\partial\varphi} - \frac{\partial\psi}{\partial\varphi}\frac{\partial}{\partial u}\right).$$

X_ψ is a Hamiltonian vector field of the function ψ with respect to the symplectic 2-form μ. The *bracket product* { , } of two functions f and g on S^2 is defined by:

$$(2.9) \qquad \{f,g\} = -\mu(X_f, X_g).$$

Note that our definition of the bracket differs in sign from the conventional definition of the Poisson bracket. Let A and B be the elements of the Lie algebra $\mathfrak{d}_{vol}(S^2)$ (i.e. a set of left invariant vector fields) of the group $\mathfrak{D}_{vol}(S^2)$ and e denote the identity element of the group $\mathfrak{D}_{vol}(S^2)$. In our definitions, the mapping $A \in \mathfrak{d}_{vol}(S^2) \to A_e{}^* \in$

$\mathfrak{X}_{vol}(S^2)$ and the mapping $X \in \mathfrak{X}_{vol}(S^2) \to \psi_X \in C^\infty(S^2)$ are both isomorphisms with respect to each corresponding bracket, namely:

$$[A_e{}^*, B_e{}^*] = [A, B]_e{}^*, \qquad A, B \in \mathfrak{d}_{vol}(S^2), \tag{2.10}$$

$$\{\psi_X, \psi_Y\} = \psi_{[X,Y]}, \qquad X, Y \in \mathfrak{X}_{vol}(S^2). \tag{2.11}$$

Consequently, we obtain:

$$\{\psi_{A_e{}^*}, \psi_{B_e{}^*}\} = \psi_{[A,B]_e{}^*}, \tag{2.12}$$

and from now on, we identify the function ψ_{A_e*} on S^2 with the element A of the Lie algebra $\mathfrak{d}_{vol}(S^2)$. The bracket product $\{\, , \}$ of two functions on S^2 is also identified with the Lie bracket of the Lie algebra $\mathfrak{d}_{vol}(S^2)$. We formally complexify the Lie algebra $\mathfrak{d}_{vol}(S^2)$ and the above identification enable to express the elements of the algebra $\mathfrak{d}_{vol}(S^2)$ by the linear combinations of the spherical harmonics Y_l^m. The orthonormal basis with respect to the metric g will be denoted by $\tilde{Y}_l^m$:

$$\tilde{Y}_l^m = \sqrt{\frac{(2l+1)}{4\pi l(l+1)}}\, Y_l^m, \tag{2.13}$$

$$g_e(\tilde{Y}_l^m, \tilde{Y}_{l'}^{-m'}) = (-1)^m \delta_{ll'} \delta_{mm'}. \tag{2.14}$$

Let C denote the *structure constants*, namely:

$$\{\tilde{Y}_{l_1}^{m_2}, \tilde{Y}_{l_2}^{m_2}\} = \sum_{lm} C \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \tilde{Y}_l^m. \tag{2.15}$$

We now consider the *Riemannian connection* ∇ associated with the Riemannian metric g. Let $\mathfrak{X}(M)$ denote the set of vector fields on M. $\nabla_X Y \in \mathfrak{X}(\mathfrak{D}_{vol}(S^2))$, the *covariant derivative* of Y in the direction of X, is a bilinear function of $X, Y \in \mathfrak{X}(\mathfrak{D}_{vol}(S^2))$. ∇ is uniquely defined to satisfy the following conditions:

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0, \tag{2.16}$$

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{2.17}$$

for any $X, Y, Z \in \mathfrak{X}(\mathfrak{D}_{vol}(S^2))$. For $X, Y, Z \in \mathfrak{d}_{vol}(S^2)$, we obtain the following formula from (2.16) and (2.17).

$$g(\nabla_X Y, Z) = \frac{1}{2}(g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X)). \tag{2.18}$$

The *Christoffel's symbols* Γ is defined by:

$$\nabla_{\tilde{Y}_{l_1}^{m_1}} \tilde{Y}_{l_2}^{m_2} = \sum_{lm} \Gamma \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \tilde{Y}_l^m. \tag{2.19}$$

From (2.18),(2.14) and (2.19), we obtain the following formula:

$$(2.20)\quad \Gamma\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix}$$
$$= \frac{1}{2}\left(C\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} + (-1)^{m_1} C\begin{pmatrix} l & l_1 & l_2 \\ -m & m_1 & -m_2 \end{pmatrix} + (-1)^{m_2} C\begin{pmatrix} l & l_2 & l_1 \\ -m & m_2 & -m_1 \end{pmatrix}\right).$$

The *Riemannian curvature transformation* R associates to each pair of vector fields X and Y the linear transformation:

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}. \tag{2.21}$$

The *Riemannian curvature tensor*, also denoted by R, is defined by:

$$R(X_1,X_2,X_3,X_4) = g(R(X_3,X_4)X_2,X_1), \tag{2.22}$$

where $X_1,\dots,X_4$ are vector fields. The Riemannian curvature tensor R satisfies the following properties:

$$\begin{gathered} R(X_1,X_2,X_3,X_4) = -R(X_2,X_1,X_3,X_4), \\ R(X_1,X_2,X_3,X_4) = -R(X_1,X_2,X_4,X_3), \\ R(X_1,X_2,X_3,X_4) + R(X_1,X_3,X_4,X_2) + R(X_1,X_4,X_2,X_3) = 0. \end{gathered} \tag{2.23}$$

If X_1 and X_2 are orthonormal, the value:

$$\kappa(X_1,X_2) = R(X_1,X_2,X_1,X_2) \tag{2.24}$$

is called the *sectional curvature* of the 2-dimensional plane containing the directions of X_1 and X_2. If X_1 and X_2 are not orthonormal, then the corresponding sectional curvature is given by:

$$\kappa(X_1,X_2) = \frac{R(X_1,X_2,X_1,X_2)}{g(X_1,X_1)g(X_2,X_2) - (g(X_1,X_2))^2}. \tag{2.25}$$

We employ the following abbreviations:

$$R\begin{pmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix} = R(\tilde{Y}_{l_1}^{m_1}, \tilde{Y}_{l_2}^{m_2}, \tilde{Y}_{l_3}^{m_3}, \tilde{Y}_{l_4}^{m_4}), \tag{2.26}$$

$$\kappa\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} = \kappa(\tilde{Y}_{l_1}^{m_1}, \tilde{Y}_{l_2}^{m_2}). \tag{2.27}$$

3. Discussions

The explicit formulas of the Riemannian curvature tensor R and the sectional curvature κ are obtained in [14]. They are expressed through $3-j$ *coefficients*. The $3-j$ coefficients

$$\begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix}$$

can be computed from the following formula [13].

$$\begin{aligned}
&\begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix} \\
&= (-1)^{2j-k+n}\sqrt{\frac{(j+k-l)!\,(k+l-j)!\,(l+j-k)!\,(l+p)!\,(l-p)!}{(j+k+l+1)!\,(j+m)!\,(j-m)!\,(k+n)!\,(k-n)!}} \\
&\quad\times\sum_t (-1)^t \frac{(l+j-n-t)!\,(k+n+t)!}{(l+p-t)!\,(k-j-p+t)!\,t!\,(l-k+j-t)!} \\
&= (-1)^{j-k-p}\sqrt{\frac{(j+k-l)!\,(j-k+l)!\,(-j+k+l)!}{(j+k+l+1)!}} \\
&\quad\times\sum_s \frac{(-1)^s\sqrt{(j+m)!\,(j-m)!\,(k+n)!\,(k-n)!\,(l+p)!\,(l-p)!}}{s!\,(k+n-s)!\,(j-m-s)!\,(l-k+m+s)!\,(l-i-n+s)!\,(j+k-l-s)!}
\end{aligned} \tag{3.1}$$

Here is the theorem for the explicit formulas of the Riemannian curvature tensor R and the sectional curvature κ.

Theorem 3.1 (Yoshida [14]). *If l_1+l_2+l is odd, the structure constants are expressed by:*

$$\begin{aligned}
&C\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \\
&= (-1)^m \frac{i}{S}\sqrt{\frac{4\pi l(l+1)(2l_1+1)(2l_2+1)(2l+1)(l_1+l_2-l)(l_1+l_2+l+1)}{(l_1+1)(l_2+1)}} \\
&\qquad\times\begin{pmatrix} l_1-\frac{1}{2} & l_2-\frac{1}{2} & l \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix},
\end{aligned} \tag{3.2}$$

otherwise

$$C\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = 0. \tag{3.3}$$

Corollary 3.1. *With the structure constants C, the Christoffel's symbols Γ and the Riemannian curvature tensor R are given by the formulas:*

(3.4)
$$\Gamma\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = \frac{1}{2\lambda}(\lambda - \lambda_1 + \lambda_2) C \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix},$$

(3.5)
$$\begin{aligned} R\begin{pmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix} = (-1)^{m_1} \sum_{lm} \Bigg(& \Gamma\begin{pmatrix} l_4 & l_2 & l \\ m_4 & m_2 & m \end{pmatrix} \Gamma\begin{pmatrix} l_3 & l & l_1 \\ m_3 & m & -m_1 \end{pmatrix} \\ & - \Gamma\begin{pmatrix} l_3 & l_2 & l \\ m_3 & m_2 & m \end{pmatrix} \Gamma\begin{pmatrix} l_4 & l & l_1 \\ m_4 & m & -m_1 \end{pmatrix} \\ & - C\begin{pmatrix} l_3 & l_4 & l \\ m_3 & m_4 & m \end{pmatrix} \Gamma\begin{pmatrix} l & l_2 & l_1 \\ m & m_2 & -m_1 \end{pmatrix} \Bigg). \end{aligned}$$

By virtue of this theorem, it is possible to compute the sectional curvature of any particular 2-dimensional plane in $T_e\mathfrak{D}_{vol}(S^2)$. We are going to study the stability of the azimuthal flow with the flow function $\hat{Y}_l^0$. We restrict the perturbations to the flows described by the flow function $R_a^*\hat{Y}_{l'}^0$, $a \in SU(2)$, where R_a^* is a pull back of the function associated with the right action of a on S^2 [14] . We define the *mean sectional curvature* $\kappa_m(l,l')$ *associated with* l,l' by:

(3.6)
$$\begin{aligned} \kappa_m(l,l') &= \frac{1}{\Omega}\int_{SU(2)} \kappa(\hat{Y}_l^0, R_a^*\hat{Y}_{l'}^0) da \\ &= \frac{1}{2l'+1}\sum_m \hat{\kappa}\begin{pmatrix} l & l' \\ 0 & m \end{pmatrix}, \end{aligned}$$

where da is the Haal measure of $SU(2)$ and Ω is the measure of the whole group defined to be $\int_{SU(2)} da$.

Fixing r to be 1, actual computations of the mean sectional curvatures $\kappa_m(l,l')$ are performed for some l and l', and the results are shown in Figure 1. Lukatskii [9] showed that $\lim_{l\to\infty}\hat{\kappa}\left(\begin{smallmatrix} 2 & l \\ 0 & \pm l \end{smallmatrix}\right) = -(15/8\pi) \approx -0.60$. Since $\hat{\kappa}\left(\begin{smallmatrix} 2 & 20 \\ 0 & 20 \end{smallmatrix}\right) \approx -0.54 \approx 0.90 \lim_{l\to\infty}\hat{\kappa}\left(\begin{smallmatrix} 2 & l \\ 0 & l \end{smallmatrix}\right)$, we may consider $l = 20$ is a sufficiently large value and reflects some behavior when l tends to be infinity.

Now we look at the flow $\hat{Y}_2^0$. The corresponding vector field $X_{\hat{Y}_2^0}$ is given by:

(3.7)
$$X_{\hat{Y}_2^0} = \frac{1}{S}\sqrt{30\pi}u\frac{\partial}{\partial\varphi}.$$

The particles near the poles $u = \pm 1$ rotates around z-axis faster and the inferior limit of the period T during which the particle rotates around z-axis is equal to $\sqrt{2\pi/15}S$. We estimate the mean curvature $\kappa_m(2)$ associated with $\hat{Y}_2^0$ to be

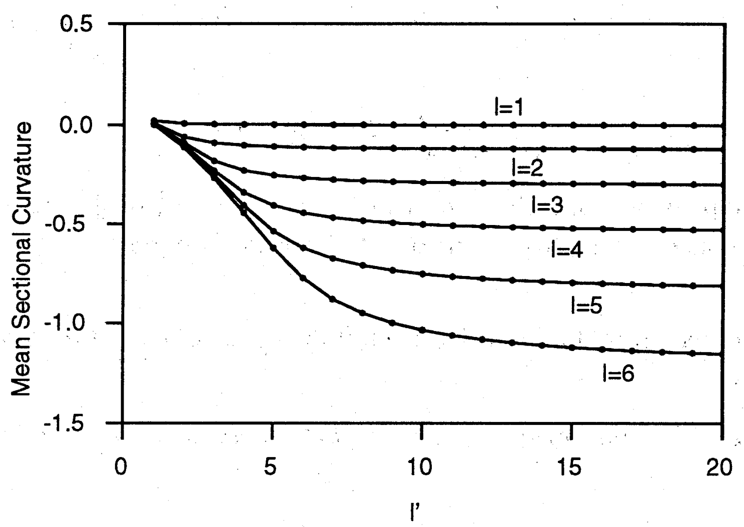

FIGURE 1. l' *vs* $\kappa_m(l,l')$

$\kappa_m(2,20) \approx -0.12(4\pi/S)^2$. Then the initial perturbation of the small order ε grows exponentially and becomes $e^{\sqrt{-\kappa_m(2)}T}\varepsilon \approx 17\varepsilon$ after the period of the rotation of the fastest particles. The particles on $u = \frac{1}{2}$ rotate around z-axis with the period $2T$ and the initial perturbation ε grows to $e^{2\sqrt{-\kappa_m(3)}T}\varepsilon \approx 3 \times 10^2\varepsilon$ after this period. This suggests that it is practically impossible to predict the passive scalar advected by the flow $\hat{Y}_2^0$ over the period $2T$.

4. STABILITY OF THE FLOW AS VECTOR FIELD

One should remark that *the instability of the motion of the fluid* discussed above (i.e. instability as the geodesic equation on the group $\mathfrak{D}_{vol}(S^2)$) is different from the /it instability of the flow in usual sense (i.e. the instability of the velocity field). The distance between two flow with flow functions ψ and ψ' is given by

$$\|\psi - \psi'\| = \sqrt{g(\psi - \psi', \psi - \psi')} = \sqrt{\int_{S^2} (X_\psi - X_{\psi'}, X_\psi - X_{\psi'})\mu}. \tag{4.1}$$

Then the Liapunov stability of the flow can be discussed with respect to this distance. It is possible that a flow is Liapunov stable where its corresponding motion of the fluid is unstable. For example, we have seen the instability of the motion of the fluid for the flow Y_0^2, and now we will see that the flow Y_2^0 is almost Liapunov stable.

Let Δ be the *Laplace-de Rham operator* (see [1]), which differs in sign from the Laplacian ∇^2. The function $\Delta\psi$ is said to be it vorticity of the flow function ψ.

Lemma 4.1. *Y_l^0 is a stationary flow. Let ϕ_0 be the initial perturbation and $Y_l^0 + \psi_t$ satisfy the Euler's equation. Then the following value conserves (is independent of time t).*

$$\frac{1}{l(l+1)} \int_{S^2} (\Delta\phi_t)^2 \mu - \|\phi_t\|^2 \tag{4.2}$$

Proof. This is proved from the fact that the energy $(\frac{1}{2}\|\psi_t\|^2)$ and the enstrophy $(\frac{1}{2}\int_{S^2}(\Delta\psi_t)^2)$ conserves for the solution of the Euler's equation ψ_t and that $\Delta Y_l^0 = l(l+1)Y_l^0$ (see [3]). □

It is easy to see from Lemma 4.1 that flow Y_2^0 is *almost Liapunov stable* in the following sense.

Theorem 4.1. *Fix some arbitrary $\varepsilon > 0$. For ϕ_t defined above, the following equation:*

$$\|\phi_t\|^2 \leq \frac{1}{\varepsilon}\Big(\int_{S^2} \frac{1}{6}(\Delta\phi_0)^2\mu - \|\phi_t\|^2\Big) \tag{4.3}$$

holds if

$$\int_{S^2} \frac{1}{6}(\Delta\phi_t)^2\mu \geq (1+\varepsilon)\|\phi_t\|^2. \tag{4.4}$$

Remark. Since any flow not containing Y_1^m (the rigid body motion mode) satisfies the equation:

$$\int_{S^2} \frac{1}{6}(\Delta\phi_t)^2\mu \geq \|\phi_t\|^2, \tag{4.5}$$

the condition in Theorem 4.1 may almost be satisfied, choosing ε to be sufficiently small.

5. Summary

(1) The structure constants C of the Lie algebra $\mathfrak{d}_{vol}(S^2)$ of the group of area-preserving diffeomorphism (motions of fluid) $\mathfrak{D}_{vol}(S^2)$ of S^2 are obtained in Theorem 3.1. They are expressed through $3-j$ coefficients. The components of the curvature tensor of the group $\mathfrak{D}_{vol}(S^2)$ are expressed with the structure constants. [14]

(2) The mean sectional curvature associated with $\hat{Y}_2^0$ is computed and it is estimated that the initial perturbation ε grows to $3 \times 10^2\varepsilon$ after the period during which the particles on $u = \frac{1}{2}$ rotate around z-axis,i.e.,the corresponding motion of the fluid is unstable in this sense.

(3) In spite of the instability of the motion of the fluid noted above for the flow $\hat{Y}_2^0$, the flow $\hat{Y}_2^0$ is almost Liapunov stable as a vector field.

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