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Introduction to the classification of purely infinite simple C*-algebras - Kirchberg 氏の仕事の紹介-

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1 Introduction

In this note we try to present the framework of the following classification thery of purely infinite simple C*-algebras by Kirchberg [17].

Theorem B. If A and B are purely infinite simple, separable, unital, nuclear C*-algebras (*pi-sun* algebras) with $A=A^{st}$ and $B=B^{st}$, then for every KK-equivalence $z\in KK(A,B)$ there exists a unital *-isomorphism h from A onto B which induces the equivalent KK-element to z, where A^{st} denotes the Cuntz standard form, that is, the K_0 element $[1_A]_0$ is a zero element in $K_0(A)$.

Since any purely infinite simple C*-algebra is stable isomorphic to some purely infinite simple unital C*-algebra in Cuntz standard form, we get

Corollary. Let A and B be *pi-sun* algebras.

(1) A and B are KK-equivalent if and only if they are stable isomorphic.

(2) If there exists a KK-equivalence x in $KK(A,B)$ with

$$\gamma_0(x)([1_A]_0)=[1_B]_0,$$

then A and B are isomorphic, where γ_0 is a natural map from $KK(A,B)$ to $Hom(K_0(A),K_0(B))$ which is induced by Kasparov product.

Therefore,

$$(K_0(A),[1_A]_0,K_1(A))$$

is a complete system of invariant in the classification of *pi-sun* algebras satisfying the Universal Coefficient Theorem for their KK-Theory.

From this classification theorem we obtain that

1) For every separable simple unital nuclear C*-algebra A

$$A \otimes O_2 \cong O_2.$$

2) A separable simple unital nuclear C*-algebra A is purely infinite if and only if $A \cong A \otimes O_\infty$.

Now we shall look through Kirchberg's approach to Theorem B

Kirchberg proved Theorem B using his deepest result of the following characterization of exact C*-algebras [17]:

Theorem A.

(1) A separable C*-algebra A is isomorphic to a C*-subalgebra of O_2 if and only if A is exact.

(2) A separable unital C*-algebra A is isomorphic to the range of a conditinal expectation from O_2 onto a C*-subalgebra of O_2 if and only if A is nuclear.

Using Theorem A (1) Kirchberg defined a semigroup $EK(A,B)$ constructed by *-monomorphisms from A into $M(C_0(\mathbf{R}_+) \otimes B)/C_0(\mathbf{R}_+) \otimes B$, and show that the Grohtendieck group of $EK(A,B)$ is isomorphic to $KK(A,B)$. Next, he defines a natural semigroup morphism from $EK(A,B)$ to a semigroup $EK_\omega(A,B)$ which is constructed by *-monomorphisms from A into B_ω, where ω is a ultrafilter on $\mathbf{N}$ and B_ω is the limit algebra $\ell_\infty(B)/C_\omega(B)$ which is also purely infinite simple. Then, applying the approximate intertwining argument in the sence of Elliott and the property of Kasparov product, Theorem B is obtained.

This note is a survey of Kirchberg's draft "The classification of purely infinite C*-algebras using Kasparov's Theory ", and it is based on Dr. Rajarama Bhat's lecture and lecture note by Peter Frris [13] in the program year (Sept., 1994 - Aug., 1995) in Operator Algebras and Applications at the Fields Institute. The author also has been writing the more detail explanation about it [22].

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2 Basic results on purely infinite simple C*-algebras

Definition 2.1 *A projection p is called infinite if p is Murray-von Neumann equivalent to a proper subprojection of p*

For projection p, q, let us write $p \preceq q$ if p is equivalent to a subprojection of p.

Definition 2.2 *A simple C*-algebra A is purely infinite(= pi) if every hereditary C*-subalgebra contains an infinite projection.*

In the rest part of this note a "*pi* algebra" is simple in particular.

Example 2.3 *The Cuntz algebra* $O_n(n = 2, 3, \cdots, \infty)$ *is typical examples of simple and purely infinite [7].*

Proposition 2.4 (Cuntz[9],Kirchberg[17]) *Let A be a simple C*-algebra(*$A \neq \mathbf{C}, 0$*). Then the following conditions are equivalent:*

(i) A is pi.

(ii) For positive elements $a, b \in A$ *with* $\|a\| = \|b\| = 1$ *and* $\epsilon > 0$ *there exists* $c \in A$ *with* $\|c\| = 1$ *such that* $\|b - c^*ac\| < \epsilon$.

When A has a unit, we get the following characterization.

Proposition 2.5 (Cuntz [9]) *Let A be a simple unital C*-algebra. Suppose that A is not the scalars. Then the following conditions are equivalent:*

(i) A is a pi algebra.

(ii) For any non-zero a in A there are elements x and y such that

$$1 = xay.$$

(iii) For any non-zero positive element a in A and $\epsilon > 0$ *there is an element x such that*

$$1 = xax^* \quad and \quad \|x\| \leq \|a\|^{-\frac{1}{2}} + \epsilon.$$

Corollary 2.6 *(i) every hereditary C^*-subalgebra of a pi algebra is pi.*

(ii) $A \otimes \mathbf{K}$ is pi if A is a pi algebra, where $\mathbf{K}$ is a C^*-algebra generated by all compact operators on some Hilbert space.

(iii) A is a pi algebra if(and only if) A is simple and contains in multiplier algebra $M(A)$ a central sequence(for elements in A) of unital copies of E_2(or of O_2), where E_2 is the C^*-subalgebra generated by s_1, s_2 of $O_3 = C^*(s_1, s_2, s_3)$.

(iv)(Rørdam [28], Lin [18]) Let A be a σ-unital C^*-algebra.

If $M(A)/A$ is simple, then $A \cong \mathbf{K}$ or A is pi.

Proof. (iii): Let a and b be positive elements in A with $\|a\| = \|b\| = 1$ and $\epsilon > 0$. Since A is simple there are $x_1, \cdots, x_n$ in A such that $\|b - \sum x_i^* a x_i\| < \varepsilon$. From assumption there are isometries $s_1, \cdots, s_n$ such that $s_i^* s_j = \delta_{i,j}$ for all i, j and $\|s_i a - a s_i\| < \frac{1}{\sum_{i=1}^n \|x_i\|^2}\varepsilon$ for i. Put $c = \sum s_i x_i$ and $p = \sum s_j s_j^*$, then

$$\begin{aligned}
\|c^* papc - b\| &\leq \|c^* papc - c^*(\textstyle\sum s_j a s_j^*)c\| \\
&\leq \|c\|^2 \|\textstyle\sum s_j s_j^* a s_k s_k^* - \sum s_j s_j^* s_k a s_k^*\| \\
&\leq \|c\|^2 \|\textstyle\sum_j s_j s_j^*\| \|\sum_k (a s_k - s_k a) s_k\| \\
&\leq (\textstyle\sum \|x_i\|^2) \frac{1}{\sum_{i=1}^n \|x_i\|^2}\varepsilon = \varepsilon.
\end{aligned}$$

Hence, if we take ε as a small, A is purely infinite(see proof of $(ii) \to (i)$ in Proposition 2.4).

For a unital C*-algebra A we denote by $U(A)$ the group of unitaries of A, and by $U_0(A)$ the connected compornent of the unit.

Proposition 2.7 (Cuntz[9]) *Let A be a unital pi algebra. Then:*

(i) For every $g \in K_0(A)$ there is a non-zero projection p such that $[p] = g$.

(ii) $K_0(A)^+ = K_0(A)$.

(iii) If p and q are non-zero projections then $[p] = [q]$ iff $p \sim q$.

(iv) The canonical map $U(A)/U_0(A) \to K_1(A)$ is an isomorphism.

3 The generalized Weyl-von Neumann Theorem

In this section we present the Weyl-von Neumann type theorem along lines of Voiculescu and Kasparov(see [1],[14],[33]), which plays an important role in [17].

Definition 3.1 *For a unital C^*-algebra A which has a unital copy of O_2 and a fixed pair of generators s_1, s_2 of O_2, we define the Cuntz addition $\oplus = \oplus_{s_1,s_2}$ in A as follows:*

$$a \oplus b = s_1 a s_1^* + s_2 b s_2^*, \quad a, b \in A.$$

Remark 3.2 *Upto unitary equivalence, Cuntz addition is independent of which particular copy of O_2 we take: If s_1, s_2 and t_1, t_2 are generators of two copies of O_2 in A, then*

$$a \oplus_{s_1,s_2} b = u(a \oplus_{t_1,t_2}) u^* \quad \textit{for all} \quad a, b \in A,$$

where $u = s_1 t_1^* + s_2 t_2^*$ is a unitary.

Theorem 3.3 (Kircberg [17]) *Let A be a σ-unital C^*-algebra. Then the following properties of A are equivalent:*

(i) $A \otimes \mathbf{K} \cong \mathbf{K}$ or there exists a unital pi algebra B such that $A \otimes \mathbf{K} \cong B \otimes \mathbf{K}$.

(ii) For every unital separable C^*-algebra C of $M(A \otimes \mathbf{K})$ and every weakly nuclear unital completely positive map $V : C \to M(A \otimes \mathbf{K})$ with $V(C \cap (A \otimes \mathbf{K})) = \{0\}$ there exists a sequence of isometries $s_n \in M(A \otimes \mathbf{K})$ with

$$\alpha) s_n^* d s_n - V(d) \in A \otimes \mathbf{K}, \forall d \in C$$
$$\beta) \lim_{n\to\infty} \|s_n^* d s_n - V(d)\| = 0, \forall d \in C.$$

(iii) For every unital separable C^*-algebra C of $M(A \otimes \mathbf{K})$ and every weakly nuclear unital representation $h : C \to M(A \otimes \mathbf{K})$ with $h(C \cap (A \otimes \mathbf{K})) = \{0\}$ there exists a sequence of unitaries $u_n \in M(A \otimes \mathbf{K})$ with

$$\alpha) u_n^* d u_n - d \oplus h(d) \in A \otimes \mathbf{K}, \forall d \in C$$
$$\beta) \lim_{n\to\infty} \|u_n^* d u_n - (d \oplus h(d))\| = 0, \forall d \in C.$$

The following proposition is a kye result to prove Theorem 3.3 $(i) \to (ii)$.

Proposition 3.4 (Kirchberg [17]) *Let A be a pi algebra and let B be a separable C^*-algebra of A. Let Ω be a compact Hausdorff space. Then for every nuclear map $\phi : B \to C(\Omega, A)$ there exists a sequence (d_n) of contractions in $C(\Omega, A)$ and a sequence (h_n) in A^+ with $\|h_n\| = 1$ such that $\lim_{n\to\infty} \|h_n d_n\| = 0$ and*

$$\phi(b) = \lim_{n\to\infty} d_n^* b d_n, \forall b \in B.$$

We shall prove the assertion in the case that Ω is a single point, that is, Corollary 3.5. The proof of general case is almost the same.

Corollary 3.5 *Let A be a σ-unital pi algebra and let B be a separable C^*-algebra of $M(A)$. For every nuclear map $\phi : B \to A$ there exists a sequence (d_n) of contructions in A such that*

$$\phi(b) = \lim_{n\to\infty} d_n^* b d_n, \forall b \in B.$$

Proof.

Let ϕ be a nuclear map from B to A. Fix elements $b_1, b_2, \cdots, b_m$ and $\varepsilon > 0$. By the nuclearity of ϕ there are completely positive contractive maps $\phi' : B \to M_n$ and $\phi'' : M_n \to A$ such that

$$\|\phi(b_k) - \phi'' \circ \phi'(b_k)\| < \varepsilon, \quad 1 \leq k \leq m.$$

Since $B \subset A \subset A \otimes \mathbf{K}(= C)$, ϕ' extends to C by Arveson's extension theorem (let us denote the extention by ϕ' again) and we can assume that $(\phi')^{**}(1_{C^{**}}) = 1$. Since A is infinite dimensional, there is a positive element $h = h_1 \otimes e_{1,1}$, $h_1 \in A^+$, $\|h_1\| = 1$ such that $\|\phi'(h)\| < \varepsilon$. Let ρ be a pure state on C.

Claim 1: There are contractions $f_1, \cdots, f_n$ in C such that $f_i^* f_j = 0$ for $i \neq j$, $\rho(f_i^* f_i) = 1$ for all i, and $\|\phi'(b_k) - F^{(k)}\| < \varepsilon$, $k = 1, \cdots, m+1$, where $F_{i,j}^{(k)} = \rho(f_i^* b_k f_j)$ and $b_{m+1} = h$.

Proof of Claim 1.

Let $\pi : C \to B(H_\rho)$ be the GNS-representation induced by ρ with a cyclic vector η. Since C is simple and π is faithful, we have a map $\widehat{\phi}' : \pi(C) \to M_n$ defined by $\widehat{\phi}'(\pi(c)) = \phi'(c)$. Note that $\widehat{\phi}'$ extends to $\pi(\tilde{C})$.

Using Glimm's lemma (Lemma 3.6), we can find orthogonal vectors $x_1, \cdots, x_n$ in H_ρ such that

$$\|\phi'(b_k) - F^{(k)}\| < \varepsilon$$

for $k = 1, \cdots, m+1$, where $F_{i,j}^{(k)} = < \pi(b_k)x_j, x_i >$.

By Kadison's transitivity theorem, there are elements $f_1, \cdots, f_n$ in C such that $\pi(f_i)\eta = x_i$ and it follows that

$$\begin{aligned} < \pi(b_k)x_j, x_i > &= < \pi(b_k)\pi(f_j)\eta, \pi(f_i)\eta > \\ &= < \pi(f_i^* b_k f_j)\eta, \eta > \\ &= \rho(f_i^* b_k f_j) \end{aligned}$$

for all i, j.

End of the proof of Claim 1.

Since ϕ'' is a contraction, we get

$$\|\phi'' \circ \phi'(b_k) - \phi''(F^{(k)})\| < \varepsilon, \quad k = 1, \cdots, m+1.$$

Note that the multiplier algebra $M(C)$ of C contains a unital copy of O_n.

Claim 2: There exist $e_1, \cdots, e_n$ in C such that $\phi''(F^{(k)}) = \sum_{i,j} F_{i,j}^{(k)} e_i^* e_j$ for $k = 1, \cdots, m+1$.

Proof of Claim 2.

Let $e_{i,j}$ be a canonical matrix units for M_n. Then, $[e_{i,j}]$ is a positive matrix in M_{n^2}. Since ϕ'' is completely positive, $[\phi''(e_{i,j})]$ $(= G)$ is positive. Now define $e_1, \cdots, e_n$ in C by

$$[e_1, \cdots, e_n] = [s_1, \cdots, s_n] G^{\frac{1}{2}},$$

where $s_1, \cdots, s_n$ are generators of O_n in $M(C)$. Then, $G = [e_1, \cdots, e_n]^*[e_1, \cdots, e_n]$ and hence for each $[\alpha_{i,j}] \in M_n$ we have

$$\phi''([\alpha_{i,j}]) = \sum \alpha_{i,j} \phi''(e_{i,j}) = \sum_{i,j} \alpha_{i,j} e_i^* e_j.$$

End of the proof of Claim 2.

Therefore,

$$\|\phi'' \circ \phi'(b_k) - \sum_{i,j} F_{i,j}^{(k)} e_i^* e_j\| < \varepsilon, \quad k = 1, \cdots, m+1.$$

Choosing an approximate unit x for C such that $\|xe_i - e_i\|$ is sufficiently small for all $i = 1, \cdots, n$, we can get

$$\|\sum_{i,j} F_{i,j}^{(k)} e_i^* e_j - \sum_{i,j} F_{i,j}^{(k)} (e_i^* x)(x e_j)\| < \varepsilon, \quad k = 1, \cdots, m+1.$$

Claim 3: There is a contraction $y \in C$ such that

$$\|\sum_{i,j} F_{i,j}^{(k)} (e_i^* x)(x e_j) - \sum_{i,j} e_i^* y^* (f_i^* b_k f_j) y e_i\| < \varepsilon$$

for all $k = 1, \cdots, m+1$.

Proof of Claim 3. From Lemma 3.7 and claim 1, there exists $y \in C$ such that $\|F_{i,j}^{(k)} x^2 - y^*(f_i^* b_k f_j) y\|$ are small enough for

$$\|\sum_{i,j} F_{i,j}^{(k)} (e_i^* x)(x e_j) - \sum_{i,j} e_i^* y^* (f_i^* b_k f_j) y e_i\| < \varepsilon$$

for $k = 1, \cdots, m+1$.

End of the proof of Claim 3.

So, with $d = \sum_i f_i y e_i$ we have

$$\|\sum_{i,j} F_{i,j}^{(k)}(e_i^* x)(x e_j) - d^* b_k d\| < \varepsilon, \quad k = 1, \cdots, m+1.$$

From the above arguments, we obtain that

$$\|\phi(b_k) - d^* b_k d\| < \varepsilon \quad k = 1, \cdots, m.$$

Moreover, since $\|\phi'' \circ \phi'(h)\| < \varepsilon$, we get $\|d^* h d\| < 4\varepsilon$. Hence, $\|hd\|$ can be made as small as we like.

If we replace d by $(1 \otimes e_{1,1})d(1 \otimes e_{1,1})$, then we get the assetion.

We put some observations about completely positive maps which were used in the proof of Corollary 3.5.

Lemma 3.6 (Glimm [1]) *Let $A \subset B(H)$ be a unital C^*-algebra. Suppose that $\phi : A \to M_n$ is a completely positive map and annihilates $A \cap \mathbf{K}$. Then there is a net (v_λ) of operators : $\mathbf{C}^n \to H$ such that*

$$\|\phi(a) - v_\lambda^* a v_\lambda\| \to 0$$

for all $a \in A$. If ϕ is unital, then the v_λ's can be chosen to be isometries.

Lemma 3.7 (Kirchberg[17]) *Let A be a pi algebra, let ρ be a pure state on A and let $x \in A$ be a contraction. Then for $\delta > 0$ and compact subset K of A there exists a contractions $y \in A$ such that*

$$\|\rho(a)x^* x - y^* a y\| < \delta$$

for all $a \in K$.

4 Proof of Theorem A (1)

In this section we present the sketch of the proof of Theorem A (1), which is used to construct a semigroup $EK_\omega(B, A)$.

The following two results are key points in Theorem A.

Lemma 4.1 (Glimm [24]) *Let A be a non-type I, separable, unital C^*-algebra. Then, there are a C^*-subalgebra B of A and a closed left ideal L of A such that*

$$\begin{cases} B + L \cap L^* = N(L) \\ L + L^* + N(L) = A \\ N(L)/L \cap L^* \cong M_{2^\infty}, \end{cases}$$

where $N(L) = \{a \in L : La + La^* \subset L\}$.

Proof. From [20,Theorem 6.7.3] there is a C*-subalgebra B of A and a closed projection q in A^{**}, commuting with B, such that $qAq = qB$ and qB is isomorphic to M_2^∞.

Put $L = A^{**}(1-q) \cap A$, then L is a closed left ideal as required.

Theorem 4.2 (Kirchberg [16]) *Let A be a separable unital C*-algebra. Then A is exact if and only if there is a untal C*-subalgebra C of M_{2^∞} and a closed two-sided AF-ideal J of C such that A is *-isomorphic to C/J.*

We say that a subalgebra B of a C*-algebra A is essential if it has no left annihilators in A, i.e. if $aB = \{0\}$ implies $a = 0$ for a in A.

An ideal I of A is essential if and only if it has non-trivial intersection with every non-trivial ideal of A

Remark 4.3 *When A is a Cuntz algebra O_2, then $L \cap L^*$ in Lemma 4.1 is an essential hereditary algebra. In fact, if it is not essential, there exists a non-zero hereditary subalgebra K of O_2 such that $LK = \{0\}$. Let r be a open projection corresponding to K, then $r = rq = qr$, where q is a closed projection corresponding to L. Hence, K can be embedded into M_2^∞. But, since K is purely infinite simple C*-algebra, this is a contradiction.*

Proposition 4.4 *Let A be a separable, unital exact C*-algebra. Then there exist a C*-subalgebra E of O_2 and a closed two-sided ideal D of E such that*

(i) D is an essential hereditary subalgebra of O_2.

(ii) $E/D \cong A$.

Proof. From Glimm's theorem there is a closed left ideal L_1 of O_2 such that $L_1 + L_1^* + N(L_1) = O_2$ and $N(L_1)/L_1 \cap L_1^* \cong M_{2^\infty}$. Let q be a closed projection corresponding to L_1. Note that q is a identity of M_2^∞ and commute with elements in $N(L_1)$(cf. $N(L_1) = \{(1-q)O_2^{**}(1-q) + qO_2^{**}q\} \cap O_2$). From Theorem 4.2, there is a unital C*-subalgebra C of M_2^∞ and a closed two-sided AF-ideal J of C such that A is *-isomorphic to C/J. Then, $D = \{d \in N(L_1) : qd, qd^* \in J\}$ and $E = \{d \in N(L_1) : qd, qd^* \in C\}$ satisfy conditions (i) and (ii) in the statement from the previous remark.

Lemma 4.5 *For any essential hereditary proper subalgebra D of O_2 we have*

$$D \cong O_2 \otimes \mathbf{K}.$$

Proof. Since D is an hereditary subalgebra of O_2, it is a *pi* algebra by Corollary 2.6(i), thus it is either unital or stable by [38]. If D has a unit p, then $(1-p)D = \{0\}$ and $1-p=0$ because D is essential. So, $D \cong O_2$ and this is a contradiction. Therefore, D must be stable. By Brown's theorem [3] D is stable isomorphic to O_2, hence $D \cong O_2 \otimes \mathbf{K}$.

Recall a few basic definitions from extension theory.

For every short exact sequence $0 \to B \to E \to A \to 0$, we consider the Busby diagram:

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E & \to & A & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & M(B) & \xrightarrow{\pi} & Q(B) & \to & 0, \end{array}$$

where $Q(B)$ denotes the corona algebra $M(B)/B$. For $\phi, \psi \in Hom(A, Q(B))$, we write $\phi \approx \psi$ if there exists a unitary $u \in M(B)$ such that $\psi(a) = \pi(u)\phi(a)\pi(u^*), a \in A$. An element $\phi \in Hom(A, Q(B))$ is called trivial if there exists an element $\hat{\phi}$ such that $\pi \circ \hat{\phi} = \phi$, i.e., if ϕ is liftable. Furthermore we write $\phi \sim \psi$ if there exist trivial elements $\tau_1, \tau_2 \in Hom(A, Q(B))$ such that $\phi \oplus \tau_1 \approx \psi \oplus \tau_2$.

We recall that $Ext(A, B)$ is the semigroup $Hom(A, Q(B))/\sim$, where the zero element is the class of trivial elements. By $Ext^{-1}(A, B)$ is denoted the group of invertible elements in $Ext(A, B)$, i.e. the classes of c.p. liftable elements of $Hom(A, Q(B))$ [1].

By what we have seen so far, there exists for every separable unital exact C*-algebra A, an exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & D & \to & E & \xrightarrow{\alpha} & A & \to & 0 \\ & & \downarrow\cong & & \downarrow\beta & & \downarrow\tau & & \\ 0 & \to & O_2 \otimes \mathbf{K} & \to & M(O_2 \otimes \mathbf{K}) & \xrightarrow{\pi} & Q(O_2 \otimes \mathbf{K}) & \to & 0 \end{array}$$

where E is a subalgebra of O_2 and $D \cong O_2 \otimes \mathbf{K}$ is essential in O_2. Then, since D is nuclear and E is exact, we know that this extension is a semisplit extension from Effros-Haagerup Lifting Theorem [11, Theorem 3.4] (see also [35, Remark 9.5]), i.e., there is a unital completelely positive map : $A \to E$ which is right inverse for $E \to A$. So, If τ is a Busby invariant of this extension, then $[\tau]$ is invertible in $Ext(A, O_2 \otimes \mathbf{K})$. Then , from the following fact we know that $[\tau] = 0$, hence there exist liftable elements τ_1 and τ_2 in $Hom(A, Q(O_2 \otimes \mathbf{K}))$ such that $\tau \oplus \tau_1 \cong \tau_2$.

Theorem 4.6 *$Ext(A, O_2 \otimes K) \cong KK^1(A, O_2 \otimes \mathbf{K})$ is trivial for every C^*-algebra A.*

Proof. Since id_{O_2} and $id_{O_2} \oplus id_{O_2}$ are homotopic on O_2 [9], $KK(O_2, O_2) = \{0\}$. Using the property of Kasparov product we get $KK^1(A, O_2 \otimes \mathbf{K}) = \{0\}$.

Proof of Theorem A (1)

Let A be a separable exact C*-algebra. We may assume that A is unital. From the above argument, we have only to show that τ is liftable. Denote by C the image of E in $M(O_2 \otimes \mathbf{K})$ by β and define $\phi : C \to M(O_2 \otimes \mathbf{K})$ by

$$\phi = \hat{\tau}_1 \circ \tau^{-1} \circ \pi.$$

Then, ϕ is a unital weakly nuclear map such that $\phi(C \cap (O_2 \otimes \mathbf{K})) = \{0\}$.

Now we employ the generalized Weyl-von Neumann theorem (Theorem 3.3). This gives a unitary $u \in M(O_2 \otimes \mathbf{K})$ such that

$$\pi(\phi(c)) \oplus \pi(c) = \pi(u^*)\pi(c)\pi(u), c \in C.$$

Let $a \in A$. For every $e \in E$ such that $\alpha(e) = a$ we have $\pi(c) = \tau(a)$, where $c = \beta(e)$. Thus $\phi(c) = \hat{\tau}_1(a)$, hence for all a

$$\pi(u^*)\tau(a)\pi(u) = \tau_1(a) \oplus \tau(a) = \tau_2(a).$$

Therefore, $[\tau] = [\tau_2]$ in $Ext(A, O_2 \otimes \mathbf{K})$, and τ is liftable.

5 Limit algebras

Definition 5.1 *A filter on* $\mathbf{N}$ *is a set* ω *of subsets of* $\mathbf{N}$ *satisfying the following conditions:*

(i) $\emptyset \notin \omega$.

(ii) $L_1 \cap L_2 \in \omega$ *for all* $L_1, L_2 \in \omega$.

(iii) $L \in \omega$ *whenever* $L' \subset L$ *for some* $L' \in \omega$.

We say that ω *is an ultrafilter if in addition it satisfies*

(iv) For every $L \subset N$, *either* $L \in \omega$ *or* $L^c \in \omega$.

or equivalently

(iv)' ω *is not properly contained in any other filter.*

Note that for any ultrafilter ω, the intersection $\cap_{L \in \omega} L$ is either empty or contains exactly one element.

In order to avoid pathological behaviour, we usually restrict our attension to ultrafilter ω for which $\cap_{L \in \omega} L = \emptyset$. Such ultrafilters are called *free.* Note that $\omega \in \beta\mathbf{N}$ is free if and only if $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$.

Definition 5.2 *Let ω be an (ultra)filter and let A be a C^*-algebra. A sequence (a_n) in A said to converge to an element $a \in A$ along ω (written $a_n \xrightarrow{\omega} 0$ or $\lim_\omega a_n = 0$) if for $\varepsilon > 0$ there is an $L \in \omega$ such that $\|a - a_n\| < \varepsilon$ for all $n \in L$.*

Let A be a unital C*-algebra and ω be a ultrafilter on $\mathbf{N}$. Let $\ell_\infty(A) = \{(a_n) : a_n \in A, \sup \|a_n\| < +\infty\}$ and $c_\omega(A) = \{(a_n) \in \ell_\infty(A) : a_n \xrightarrow{\omega} 0\}$. Then $c_\omega(A)$ is a closed tow-sided ideal in $\ell_\infty(A)$. Set $A_\omega = \ell_\infty(A)/c_\omega(A)$, and let π_ω be the quotient mapping $\ell_\infty(A) \to A_\omega$. We call A_ω the limit algebra of A. Note that the quotient map π_ω satisfies

$$\|\pi_\omega(a)\| = \lim_\omega \|a_n\|$$

, where the crucial point is that the limit always exists.

Remark 5.3 *If ω is free, then*

$$c_0(A) = \{(a_n) | \lim a_n = 0\} \subset c_\omega(A).$$

Hence A_ω is a quotient of $A_\infty (= \ell_\infty(A)/c_0(A))$.

Proposition 5.4 *Let A be a unital C^*-algebra. Then, A_ω is a pi algebra if and only if A is a pi algebra.*

Proof. Suppose that A_ω is a *pi* algebra. Let x be a non-zero element in A, and consider a canonical image of x in A_ω. Since A_ω is *pi*, there are sequences $(y_n), (z_n)$ of A such that

$$(y_1 x z_1, y_2 x z_2, \cdots) - (1, 1, \cdots) \in C_\omega.$$

So, there is a non-zero set $L \in \omega$ such that $\|y_n x z_n - 1\| < 1$ for $n \in L$. Hence, there are $y, z \in A$ such that $yxz = 1$. This implies that A is a *pi* algebra from Proposition 2.5(ii).

Suppose that A is a *pi* algebra. Take a non-zero positive element $a \in A_\omega$ with $\|a\| = 1$, write $a = [(a_n)]$. Since $\|a\| = 1$ and ω is an ultrafilter, there is a non-zero set $L \in \omega$ such that $\|a_n\| > \frac{1}{2}$ for $n \in L$. Define $\widetilde{a_n}$ by

$$\widetilde{a_n} = \begin{cases} \frac{a_n}{\|a_n\|} & n \in L \\ y \quad (\|y\| = 1) & n \notin L \end{cases}$$

Note that for any $\varepsilon > 0$ there is a non-zero subset L_ε of L such that $|\|a_n\| - 1| < \varepsilon$. Set $\tilde{a} = [(\widetilde{a_n})]$. Then, $\tilde{a} = a$ in A_ω.

Since $\|\widetilde{a_n}\| = 1$, there is a $x_n \in A$ such that $\|x_n\| \leq 1 + \frac{1}{n}$ and $x_n^* \widetilde{a_n} x_n = 1$ for any $n \in \mathbf{N}$. Set $x = [(x_n)] \in A_\omega$. Then, we know that $x^*ax = x^*\tilde{a}x = 1$, and this implies that A_ω is a *pi* algebra from Proposition 2.5(iii).

The following is a limit version of Corollary 3.5.

Proposition 5.5 *Let ω be a free ultrafilter in* $\mathbf{N}$. *If A is a unital pi algebra, B a separable C^*-subalgebra of A_ω containing 1_A, and $V : B \to A_\omega$ is a nuclear unital completely positive map. Then there exists a nonunitary isometry $s \in A_\omega$ with $V(b) = s^*bs$ for $b \in B$.*

Proof. Take a compact subset Δ_B of B such that B is the closed linear span of Δ_B. Fix $0 < \varepsilon < 1$ and let Y be the set of $d = (d_k) \in A_\omega$ such that $\|h_k d_k\| \leq \varepsilon$ for some $(h_k) \in A_\omega$, $h_k \in A^+$, $\|h_k\| = 1$. By Proposition 3.5 we have

$$\inf_{d\in Y} \sup_{b\in\Delta_B} \|\phi(b) - d^*bd\| = 0.$$

Claim: There is an element $s \in Y$ such that $\phi(b) = s^*bs$ for all $b \in B$.

Since ϕ is unital, s is an isometry. Moreover, since $\|hs\| \leq \varepsilon < 1$ for some $h \in A_\omega^+$, $\|h\| = 1$, s is not a unitary.

Proof of Claim. We may prove the following result:

Let A be a C*-algebra and let ω be an ultrafilter in $\mathbf{N}$. Let Z be a set of contractions $A \to A$ and let Z_ω denote the set of all sequences of elements of Z. Let Ω be a compact metric space and let f_1, f_2 be continuous maps : $\Omega \to A_\omega$. Then the infimum

$$\mu = \inf_{g\in Z_\omega} \sup_{x\in\Omega} \|f_1(x) - g(f_2(x))\|$$

is attained by some $g \in Z_\omega$.

Proof.

Choose a sequence $(g^{(n)})$ in Z_ω such that

$$\sup_{x\in\Omega} \|f_1(x) - g^{(n)}(f_2(x))\| < \mu + \frac{1}{n}$$

for all n. Fix the lifting of $A_\omega \to \ell_\infty(A)$, $y \mapsto (y_k)$. By the compactness of Ω, there are sets $L_n \in \omega$ such that

$$\sup_{x\in\Omega} \|f_1(x)_k - g^{(n)}(f_2(x))_k\| < \mu + \frac{1}{n}, \quad k \in L_n.$$

We may assume that $L_1 \supset L_2 \supset \cdots$. Now let g be the diagonal element given by

$$g_k = \begin{cases} g_k^{(1)}, & \text{if} \quad k \in \mathbf{N}\setminus L_1 \\ g_k^{(n)} & \text{if} \quad k \in L_n\setminus L_{n+1} \end{cases}$$

Then for any $n \geq 2$ $\sup_{x\in\Omega} \|f_1(x)_k - g(f_2(x))_k\| < \mu + \frac{1}{n}$ for all $k \in L_n$. But then $\sup_{x\in\Omega} \|f_1(x) - g(f_2(x))\| = \mu$ as desired.

We can construct more general limit algebra as follows:

Let X be a locally compact Hausdorff space, let $C_b(X, A)$ be a C*-algebra of C*-algebra A-valued bounded continous functions, and let $C_0(X, A)$ be a C*-algebra of C*-algebra A-valued continuous functions with vanishing at infinity. For every point ω in the corona space $\beta X\backslash X$ of X we consider a two-sided ideal in $C_b(X, A)$ consisting of all functions $f \in C_b(X, A)$ such that $\omega(\{x \in X \rightarrow \|f(x)\|\}) = 0$, where βX is a Stone-Čech compactification of X.

Lemma 5.6 (Rørdam[28]) *Let A be a simple unital C*-algebra. Then, A is pi if and only if for any positive elements a, b in A with norm one there are c_1, c_2 in A such that $c_1ac_1^* + c_2ac_2^* = b$.*

Proof. Suppose that for any positive elements a, b in A with norm one there are c_1, c_2 in A such that $c_1ac_1^* + c_2ac_2^* = b$.

Define a continuous function $f_\varepsilon : \mathbf{R}^+ \rightarrow [0, 1]$ by

$$f_\varepsilon(t) = \begin{cases} 0 & t \in [0, \varepsilon] \\ 1 & t \in [2\varepsilon, \infty) \\ \text{linear} & t \in [\varepsilon, 2\varepsilon] \end{cases}$$

Let a be a positive element in A with norm one. Since $\overline{f_{\frac{1}{3}}(a)Af_{\frac{1}{3}}(a)}$ is simple and infinite dimensional, there are positive elements x', y' in $\overline{f_{\frac{1}{3}}(a)Af_{\frac{1}{3}}(a)}$ with norm one such that $x'y' = 0$.

Set $z = 1 - f_{\frac{1}{6}}(y')$, $y'' = f_{\frac{1}{3}}(y')$. Since A is simple, there is a unitary u in A such that

$$\overline{x'Ax'} \cap \overline{uy''Ay''u^*} \neq \emptyset.$$

Choose x in $\overline{x'Ax'} \cap \overline{uy''Ay''u^*} \neq \emptyset$. Put $y = u^*xu$. Then, x, y in $\overline{aAa}$, $xz = x$, and $yz = 0$.

By assumption there are c_1, c_2 in A such that $c_1xc_1^* + c_2xc_2^* = 1$. Put $t = c_1z + c_2u(1 - z)$. Then, $t(x + y)t^* = 1$. Since $x + y$ in $\overline{aAa}$, there is an infinite projection in $\overline{aAa}$ which is equivalent to 1. Hence, A is purely infinite.

Proposition 5.7 *Let A be a unital C*-algebra and ω in $\beta X\backslash X$. Then, $Q_\omega(A) = C_b(X, A)/J_\omega$ is a pi algebra if A is a pi algebra.*

Proof. Take positive elements $a, b \in Q_\omega(A)$ with $\|a\| = \|b\| = 1$. Then, we may assume that

$\|\tilde{a}(x)\| = \|\tilde{b}(x)\| = 1$ for all $x \in X$, where $\tilde{a}$ and $\tilde{b}$ are preimage in $C_b(X, A)_+$ of a and b, respectively.

Claim: For any compact set $\Omega \subseteq X$ and $\varepsilon > 0$ there is a continuous function l in $C(\Omega, A)$ with $\|l\| \leq 1$ such that $\|\tilde{b}|\Omega - l^*\tilde{a}|\Omega l\| < \varepsilon$..

Proof. Take a state γ of $C^*\{1, \tilde{a}|\Omega\}$ $(= C)$ such that $\gamma(\tilde{a}|\Omega) = 1$. Define a map V from C into $C(\Omega, A)$ by $V(f) = \gamma(f)\tilde{b}|\Omega$ for $f \in C$. Then, V is completely positive contractive. Hence, there exists a contraction l in $C(\omega, A)$ such that $\|\tilde{b}|\Omega - l^*\tilde{a}|\Omega l\| < \varepsilon$ from Proposition 3.4.

(End of the proof of claim)

Since ω is free ultrafilter, there is a sequence of open sets X_i of X such that $X_i^c \in \omega$, $\Omega_i = \overline{X_i} \subseteq X_{i+1}$, and $X = \cup_i X_i$. Then, take continuous functions $g_i : X \to [0,1]$ such that $g_i|\Omega_i = 1$ and $g_i|\Omega_{i+1}^c = 0$. From the claim there are contractions $l_i \in C(\Omega_i, A)$ such that $\|l_i(x)^*\tilde{a}(x)l_i - \tilde{b}(x)\| < \frac{1}{2^i}$ for any $x \in \Omega_i$.

Set

$$c_1 = g_1^{\frac{1}{2}} l_1 + (g_3 - g_2)^{\frac{1}{2}} l_3 + (g_4 - g_3)^{\frac{1}{2}} l_5 + \cdots$$
$$c_2 = (g_2 - g_1)^{\frac{1}{2}} l_2 + (g_3 - g_2)^{\frac{1}{2}} l_4 + \cdots.$$

Then,

$\|\pi_\omega(c_1^*\tilde{a}c_1 + c_2^*\tilde{a}c_2 - \tilde{b})\| = 0$, where π_ω is a canonical quotient map from $C_b(X, A)$ to $Q_\omega(A)$. Hence, $Q_\omega(A)$ is a *pi* algebra from the previous lemma.

6 Elliott's intertwining principle

In this section we shall give a brief introduction to this principle [12]. Fix two sequences

$$(1) A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} A_4 \longrightarrow \cdots$$
$$(2) B_1 \xrightarrow{\psi_{1,2}} B_2 \xrightarrow{\psi_{2,3}} B_3 \xrightarrow{\psi_{3,4}} B_4 \longrightarrow \cdots$$

of separable C*-algebras and *-homomorphisms. And fix dense sequences $F_i \subset A_i$ and $G_i \subset B_i$, respectively, for $i \in \mathbf{N}$. Let $A = \lim A_i$ and $B = \lim B_i$ denote the corresponding inductive limit C*-algebras and $\phi_i : A_i \to A$ and $\psi_i : B_i \to B$ be the canonical homomorphisms.

Lemma 6.1 (Elliott[12]) *Let $\{\delta_n\}$ be a sequence in $[0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$. Let $\alpha_i : A_i \to B_i$, $i \in \mathbf{N}$ be *-homomorphisms such that*

$$\|\psi_{i,i+1} \circ \alpha_i(x) - \alpha_{i+1} \circ \phi_{i,i+1}(x)\| < \delta_i,$$

whenever $x \in S_i$, where S_i is the finite set of A_i consisting of the images in A_i of the first i terms of the sequences $F_1, \cdots, F_{i-1}$.

Then the sequence $\{\psi_l \circ \alpha_l \circ \phi_{i,l}(x)\}$, $l \geq i$, *converges in* B *for each* $x \in A_i$ *and all* $i \in \mathbf{N}$. *Furthermore, there is a *-homomorphism* $\alpha : A \to B$ *such that*

$$\alpha(\phi_i(x)) = \lim \psi_l \circ \alpha_l \circ \phi_{i,l}(x),$$

$x \in A_i$, $i = 1, 2, \cdots$.

Proof. Let $F_i = \{a_{i,1}, a_{i,2}, \cdots\}$. From the assumption, $\{\psi_k \circ \alpha_k \circ \phi_{i,k}(a_{i,l})\}_{k=l}^{\infty}$ $(k \geq l \geq i)$ is a cauchy sequence for $a_{i,l} \in A_i$. So, we can define $\alpha'_i : F_i \to B$ by $\alpha'_i(x) = \lim_{k\to\infty} \psi_k \circ \alpha_k \circ \phi_{i,k}(x)$. Since $\alpha'_{i+1} \circ \phi_{i,i+1} = \alpha'_i$ and the density of F_i in A_i, we obtain a *-homomorphism $\alpha : A \to B$, which is required one.

Definition 6.2 *By an Elliott's approximate intertwinig between the sequences (1) and (2), we mean *-homomorphisms* $\alpha_i : A_i \to B_i$ *and* $\beta_i : B_i \to A_{i+1}$ *such that*

$$\begin{aligned}
&||\alpha_{i+1} \circ \beta_i(y) - \psi_{i,i+1}(y)|| < 2^{-i} \\
&\textit{for} \quad y \in T_i \\
&||\beta_i \circ \alpha_i(x) - \phi_{i,i+1}(x)|| < 2^{-i} \\
&\textit{for} \quad x \in S_i, i \in \mathbf{N},
\end{aligned}$$

where S_i *is the finite subset of* A_i *consisting of the images in* A_i *of the first* i *terms of the sequences* $F_1, \cdots, F_{i-1}$, *and* $G_1, \cdots, G_{i-1}$, *along all possible paths in the diagram. Similarily,* T_i *is the finite subset of* B_i *consisting of the images in* B_i *of the first* i *terms of the sequences* $F_1, \cdots, F_i$ *and* $G_1, \cdots, G_{i-1}$, *along all possible paths in the diagram.*

Theorem 6.3 (Elliott[12]) *An approximate intertwinig between (1) and (2) induces a *-isomorphism between* $A = \lim A_i$ *and* $B = \lim B_i$.

Proof. From the definition of an approximate intertwining it follows that

$$\begin{aligned}
||\psi_{i,k} \circ \alpha_i(x) - \alpha_k \circ \phi_{i,k}(x)|| &\leq 2^{-i+2}, \\
&x \in S_i \\
||\phi_{i+1,k} \circ \beta_i(y) - \beta_{k-1} \circ \psi_{i,k-1}(y)|| &\leq 2^{-i+1}, \\
&y \in T_i.
\end{aligned}$$

Therefore, there are *-homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ from Lemma 6.1. By using the original estimates from the definitions it is easily seen that α and β are inverses of each other.

Definition 6.4 (Rørdam[29]) *Two *-homomorphisms $\phi, \psi : A \to B$ between C^*-algebras A and B are said to be approximately unitary equivalent if for every finite subset F of A and every $\varepsilon > 0$ there is a unitary $v \in B$ (or $\tilde{B}$ if B has no unit) so that*

$$\|v\phi(a)v^* - \psi(a)\| < \varepsilon$$

for all $a \in F$.

7 Kirchberg's semigroup(discrete)

In this section we introduce Kirchberg's semigroup ($EK_\omega(B,A)$) and present basic observations. This notion (idea) is an important tool to get Theorem B. We shall prove that for every simple separable unital nuclear C*-algebra B $B \otimes D_2 \cong D_2$, where $D_2 = O_2 \otimes O_2 \otimes O_2 \cdots$.

We use the following definitions and observations to define and study a semigroup $EK_\omega(B,A)$.

Let D be a C*-algebra such that its multiplier algebra $M(D)$ contains a unital copy of O_2, let B be a C*-algebra and $h_1, h_2 : B \to D$ *-homomorphisms.

We say that h_1 *n-dominates* h_2 if there exist $d_1, d_2, \cdots, d_n \in M(D)$ such that $d_1^*d_1 + \cdots + d_n^*d_n = 1$ and $h_2(\cdot) = d_1^*h_1(\cdot)d_1 + \cdots + d_n^*h_1(\cdot)d_n$.

h_1 *dominates* h_2 if there is an isometry $s \in M(D)$ with $h_2(\cdot) = s^*h_1(\cdot)s$. We will write $h_2 \prec h_1$.

Remark 7.1 *In this case, ss^* is in the relative commutant $h_1(B)' \cap M(D)$ of $h_1(B)$ in $M(D)$.*

Proof. Set $q = ss^*$. Then, since $s^*h_1(b^*b)s - s^*h_1(b^*)ss^*h_1(b)s = 0$, $qh_1(b^*)h_1(b)q = qh_1(b^*)qh_1(b)q$. So, $qh_1(b^*)(1-q)h_1(b)q = 0$. Hence, $(1-q)h_1(b)q = 0$, and $h_1(b)q = qh_1(b)q$ for any $b \in B$. Therefore, $h_1(b)q = qh_1(b)$ for any $b \in B$.

The following are simple observations:

Lemma 7.2 *(i) If h_1 dominates h_2 and $h_2(B)' \cap M(D)$ contains a unital copy of O_2, then Cuntz addition (see Definition 5.1) $h_1 \oplus h_2$ and h_1 are unitary equivalent in $M(D)$.*

(ii) If h_1 n-dominates h_2 and $h_1(B)' \cap M(D)$ contains a unital copy of E_2, then h_1 dominates h_2.

*(iii) If $h_0(B)' \cap M(D)$ contains a unital copy of O_2, then a set of unitary equivalence classes of the *-homomorphisms $h : B \to M(D)$*

which are dominated by h_0 forms a semigroup $S(h_0, B, D)$ under Cuntz addition.

We use $[h]$ as an unitary equivalence class of $h : B \to M(D)$.

Proof. (i): From the assumption there is an isometry $d \in M(D)$ such that $h_2 = d^*h_1d$. Take a generator $\{s_1, s_2\}$ of O_2 which is contained in $h_2(B)' \cap M(D)$. Then, $h_1 \oplus_{s_1,s_2} h_2 = s_1h_1s^* + s_2h_2s_2^*$. Set $u = (1-dd^*)s_1^* + ds_1d^*s_1^* + ds_2s_2^*$, then u is a unitary in $M(D)$ and $u(h_1 \oplus_{s_1,s_2} h_2)u^* = h_1$.

(ii): From the assumption there are n elements $\{d_i\}_{i=1}^n$ in $M(D)$ such that $h_2 = \sum_{i=1}^n d_i^*h_1d_i$ and $\sum_{i=1}^n d_i^*d_i = 1$. Since $h_1(B)' \cap M(D)$ contains a unital copy of E_2, it contains O_∞. Take n elements $\{s_i\}_{i=1}^n$ of generators of O_∞.

Set $d = \sum_{i=1}^n s_id_i$. Then, d is an isometry in $M(D)$, and

$$\begin{aligned} h_2 &= \textstyle\sum_{i=1}^n d_i^*h_1d_i \\ &= \textstyle\sum_{i,j} d_i^*s_i^*h_1s_jd_j \\ &= d^*h_1d. \end{aligned}$$

(iii): Note that the definition of $h \oplus_{O_2} k$ is a independent from the choice of a unital copy of O_2 in $M(D)$. Take $[h_1], [h_2] \in S(h_0, B, D)$, and write $h_i = d_i^*h_0d_i$ $(i = 1, 2)$, where d_1, d_2 are isometries in $M(D)$.

Then,

$$\begin{aligned} h_1 \oplus_{\sigma_1,\sigma_2} h_2 &= \textstyle\sum_{i=1}^2 \sigma_id_i^*h_0d_i\sigma_i^* \\ &= \textstyle\sum_{i,j,k} \sigma_id_i^*\sigma_i^*\sigma_jh_0\sigma_j^*\sigma_kd_k\sigma_k^* \\ &= (\textstyle\sum_i \sigma_id_i^*\sigma_i^*)(\sum_j \sigma_jh_0\sigma_j^*)(\sum_k \sigma_kd_k\sigma_k^*) \\ &= d^*h_0d, \end{aligned}$$

where σ_1, σ_2 are generators of O_2 in $h_0(B)' \cap M(D)$ and $d = \sigma_1d_1\sigma_1^* + \sigma_2d_2\sigma_2^*$.

Hence, the unitary equivalence class of $h_1 \oplus h_2$ is contained in $S(h_0, B, D)$.

Proposition 7.3 *Let $h_0 : B \to M(D)$ be a *-homomorphism so that $h_0(B)' \cap M(D)$ contains a unital copy of O_2. Then*

(1) the set $G(h_0, B, D) = \{[h \oplus h_0] : h \prec h_0\}$ forms a subgroup of $S(h_0, B, D)$. Moreover,

(2) $G(h_0, B, D)$ is isomorphic to the Grothendieck group of $S(h_0, B, D)$ $(= Groth(S(h_0, B, D)))$.

Proof. (1): From Lemma 7.2(i), we know that

$$[h_0] + [h_0] = [h_0 \oplus h_0] = [h_0],$$

so $[h_0]$ is a zero element in $G(h_0, B, D)$. Take nonzero element $[h \oplus h_0]$. Since $h \prec h_0$, there is an isometry d in $M(D)$ such that $h = d^* h_0 d$. Set $k = (1 - dd^*)h_0 + dh_0 d^*$. Then,

$$k \oplus_{s_1, s_2} h = s_1(1 - dd^*)h_0 s_1 + s_1 dh_0 d^* s_1^* + s_2 d^* h_0 ds_2^*,$$

where s_1, s_2 are generators of a unital copy of O_2 in $M(D)$.

Set $u = \sigma_1 as_2^* + \sigma_1(1 - dd^*)s_1^* + \sigma_2 d^* s_1^*$, where σ_1, σ_2 are generators of O_2 in $h_0(B)' \cap M(D)$. Then, u is a unitary in $M(D)$, and

$$\begin{aligned} u(k \oplus_{s_1,s_2} h)u^* &= \sigma_1 dd^* h_0 dd^* \sigma_1^* + \sigma_1(1 - dd^*)h_0 \sigma_1^* + \sigma_2 h_0 d^* d\sigma_2^* \\ &= \sigma_1 h_0 \sigma_1^* + \sigma_2 h_0 \sigma_2^* \\ &= h_0. \end{aligned}$$

Hence,

$$\begin{aligned} [k \oplus h_0] + [h \oplus h_0] &= [k \oplus h \oplus h_0 \oplus h_0] \\ &= [h_0 \oplus h_0 \oplus h_0] = [h_0] \end{aligned}.$$

Therefore, $G(h_0, B, D)$ is a group.

(2) Exercise.

Now we will define a semigroup $EK_w(B, A)$.

Definition 7.4 *Let A be a unital pi in Cunts standard form ($A = A^{st}$), that is, A contains a unital copy of O_2. and let B be a separable unital C^*-algebra. Let ω be a free ultrafilter of $\mathbf{N}$, that is, $\omega \in \beta\mathbf{N}\backslash\mathbf{N}$.*

Then $EK_w(B, A)$ is a set of unitary equivalence classes $[h]$ of nuclear unital $*$-monomorphisms $h : B \longrightarrow A_\omega$.

Under the Cuntz addition, that is, $[h] + [k] = [h \oplus_{O_2} k]$, $EK_\omega(B, A)$ becomes an abelian semigroup. Note that if h, k are nuclear, then it is easily seen that $h \oplus_{O_2} k$ is also nuclear.

We show $EK_\omega(B, A)$ is a **group** if B is exact.

Lemma 7.5 *Let A be a unital pi algebra with $A = A^{st}$ and let B be a unital separable exact C^*-algebra. Then for any $[h_1], [h_2] \in EK_\omega(B, A)$,*

$$h_2 \prec h_1.$$

Proof. Let $C = h_1(B)$. Define $h : C \longrightarrow A_\omega$ by $h = h_2 \circ h_1^{-1}$. Then, from Proposition 5.5 there is a proper isometry $s \in A_\omega$ such that $h(c) = s^* cs$ for $c \in C$. Put $c = h_1(b)$ ($b \in B$). Then, $h_2(b) = s^* h_1(b)s$ for any $b \in B$. This implies the conclusion.

Proposition 7.6 *Under the same assumption as in the previous lemma*

(i) $EK_\omega(B, A)$ *is a group.*

(ii) $[h] = 0$ *in* $EK_\omega(B, A)$ *if and only if* $h(B)' \cap A_\omega$ *contais a unital copy of* O_2.

Proof. (i): Let h_0 be an inclusion map

$$h_0 : B \hookrightarrow O_2 \subseteq O_2 \otimes O_2 \subseteq O_2 \subseteq A \subseteq A_\omega$$

which is guaranteed by Theorem A. From the previous lemma we know that for any element $[h]$ in $EK_\omega(B, A)$ $h \prec h_0$. Note that $h_0(B)' \cap A_\omega$ contains a unital copy of O_2.

Hence, $EK_\omega(B, A)$ is a group from Lemma 7.2(i) and Proposition 7.3.

(2): This comes from Lemm 7.2(iii).

Remark 7.7 *Let* D_2 *be a decoy of* O_2*, that is,* $O_2 \otimes O_2 \otimes \cdots$*. Then,*

$$EK_\omega(B, D_2) = 0$$

for any separable unital exact C-algebra B.*

Proof. Note that D_2 is a *pi* algebra from the next lemma and Corollary 2.6. The statement comes from Corollary 7.9 and the freeness of ω.

Lemma 7.8 *C*-algebra* D_2 *contains a central sequence of unital copies of* O_2*. Hence,* D_2 *is a pi algebra.*

Proof. Let Δ_D be a compact subset of D such that the linear span of elements in Δ_D is dense in D_2. We have only to show that for any ε there is a generator $\{s_1, s_2\}$ of O_2 such that $\|s_i x - x s_i\| < \varepsilon$ $(i = 1, 2)$ for any $x \in \Delta_D$. For each $x \in D_2$ and for any $\varepsilon > 0$ there is an element $y \in D_2$ such that

$y \in \underbrace{O_2 \otimes \cdots O_2}_{n(x)} \otimes 1 \otimes \cdots$. Since Δ_D is compact, there are n elements $x_1, \cdots x_n \in \Delta_D$ such that $\Delta_D \subseteq \cup_{i-1}^n U(x_i, \varepsilon)$. Set $k = \max\{n(x_i)\}$. Then, if we take a generator $\{s_1, s_2\}$ of O_2 from $\underbrace{1 \otimes \cdots 1}_{k} \otimes O_2 \otimes \cdots$, $\|x s_i - s_i x\| < 4\varepsilon$ $(i = 1, 2)$ for all $x \in \Delta_D$.

Corollary 7.9 *Let A be a unital C*-algebra and let* $h : D_2 \to A$ *be a unital *-monomorphism. Then,* $h(D_2)' \cap A_\omega$ *contains a unital copy of* O_2.

The following is a reformation of Theorem 6.3.

We call a C*-algebra A *pi-sun* algebra if A is purely infinite simple, separable, unital, nuclear algebra.

Lemma 7.10 *If A and B are pi-sun algebra with $A = A^{st}, B = B^{st}$, and let $h : A \to B$, $k : B \to A$ unital *-homomorphisms such that $[kh] = [id_A]$ in $EK_\omega(A, A)$ and $[hk] = [id_B]$ in $EK_\omega(B, B)$ then there exists an isomorphism $\phi : A \to B$ which is approximately unitary equivalent to h.*

Proof. Since $[kh] = [id_A]$, there is a unitary $u \in A_\omega$ such that $kh(a) = uau^*$ for $a \in A$. Since u can be lifted to a sequence $\{u_n\}$ of unitaries in $\ell_\infty(A)$, we may know that kh and id_A are approximately unitary equivalent. Similarily, we know that hk and id_B are approximately unitary equivalent. Let X_A and X_B be dense sequenses in A and B, respectively.

Then we can find sequences of unitaries $\{u_n\}$ in A and $\{v_n\}$ in B such that $A_i = A$, $\phi_{i,i+1} = u_i(\cdot)u_i^*$, $B_i = B$, $\psi_{i,i+1} = (\cdot)v_i^*$, $\alpha_i = h$, $\beta_i = k$, $F_i = X_A$, and $G_i = X_B$ for all $i \in \mathbf{N}$ induces approximate intertwining in Definition 6.2. So, there is an isomorphism $\phi : A \to B$ by Theorem 6.3.

It is easily seen that ϕ is approximately unitary equivalent to h from the construction.

Proposition 7.11 *If A is a pi-sun algebra with $A = A^{st}$ and $[id_A] = 0$ in $EK_\omega(A, A)$, then $A \cong D_2$.*

Proof. Let $h : A \hookrightarrow D_2 \subseteq A$ and $k : D_2 \hookrightarrow A$ be *-monomorphisms which is guranteed by Theorem A (note that D_2 is nuclear). Then $kh : A \to D_2 \to A \subseteq A_\omega$ and $kh(A)' \cap A_\omega$ contains a unital copy of O_2 from the previous corollary. Hence, $[kh] = 0$ in $EK_\omega(A, A)$. From the assumption $[kh] = [id_A]$.

On the contrary, since $[hk] \in EK_\omega(D_2, D_2)$ ($= 0$), we know that $[hk] = 0 = [id_{D_2}]$.

Hence, from Lemma 7.10 $A \cong D_2$.

Corollary 7.12 *(i) If B is simple separable unital nuclear and contains a central sequence of unital copies of O_2, then $B \cong D_2$.*

(ii) For every simple separable unital nuclear B, we have $B \otimes D_2 \cong D_2$.

Proof. (i): Let Δ_B be a compact set of B such that linear span of elements in Δ_B is dense B. From the assumption there is a central sequence $\{s_1^j, s_2^j\}_{j=1}^\infty$ of unital copies of O_2 such that $\|[s_k^j, x]\| < \frac{1}{j}$ for any $j \in \mathbf{N}$ and $x \in \Delta_B$.

Note that B is *pi* (Corollary 2.6(iii)) Now set $T_1 = (s_1^1, s_1^2, \cdots)$ and $T_2 = (s_2^1, s_2^2, \cdots)$, then $C^*(T_1, T_2) \cong O_2 \subseteq B_\omega$. Moreover, $id_B(B)' \cap B_\omega$ contains $C^*(T_1, T_2)$, hence $[id_B] = 0$ in $EK_\omega(B, B)$ from Lemma 7.2(iii). So, from Proposition 7.11 $B \cong D_2$.

(ii): Since D_2 has a central sequence of unital copies of O_2, $B \otimes D_2$ has also the same property. Hence, from (i) $B \otimes D_2 \cong D_2$.

8 Kirchberg's semigroup(continous)

In this section we introduce a continuous version of $EK_\omega(B, A)$.

We assume that A is a unital *pi* algebra with $A = A^{st}$ and B is a separable unital exact C*-algebra.

Definition 8.1 *Let* $\omega \in \beta(\mathbf{R}_+)\backslash\mathbf{R}_+$ *is in the corona of* $\mathbf{R}_+$*, there is a canonical epimorphism* $\pi_\omega : C_b(\mathbf{R}_+, A)/C_0(\mathbf{R}_+, A) \to A_\omega^{\mathbf{R}_+}$*, where* $A_\omega^{\mathbf{R}_+}$ *is pi (Proposition 5.7)*

$$C_b(\mathbf{R}_+, A)/C_0(\mathbf{R}_+, A)/J_\omega/C_0(\mathbf{R}_+, A)(\cong C_b(\mathbf{R}_+, A)/J_\omega).$$

*We call a unital *-monomorphism*

$$h : B \to C_b(\mathbf{R}_+, A)/C_0(\mathbf{R}_+, A) = Q(\mathbf{R}_+, A)$$

completely faithful if $\pi_\omega \circ h$ *is a *-monomorphism for every* $\omega \in \beta\mathbf{R}_+\backslash\mathbf{R}_+$.

h is constant if $h(B) \subseteq A \subseteq Q(\mathbf{R}_+, A)$.

h is D_2-factorizable if $h(B)$ is contained in a unital copy of D_2 in $Q(\mathbf{R}_+, A)$.

h is scaling invariant if for every topological isomorphism σ of $\mathbf{R}_+$ *onto* $\mathbf{R}_+$ *we have that h and $\hat{\sigma} \circ h$ are approximately unitary equivalent, where $\hat{\sigma}$ is the *automorphism of* $Q(\mathbf{R}_+, A)$ *by σ.*

From the above definitions we introduce the following three abelian semigroups under the Cuntz addition:

$CEK(B, A) = \{[h] : h$ is constant unital nuclear *-monomorphism$\}$.

$EK(B, A) = \{[h] : h$ is completely faithful unital nuclear *-monomorphisms$\}$.

$SEK(B, A) = \{[h] : h$ is unital nuclear *-monomorphism$\}$.

Remark 8.2 *One has* $CEK(B, A) \subseteq EK(B, A)$ *and*

$$SEK(B, A) + CEK(B, A) \subseteq EK(B, A) + SEK(B, A) \subseteq EK(B, A).$$

Proposition 8.3 *(i) Any two completely faithful unital nuclear *-monomorphisms from B into* $Q(\mathbf{R}_+, A)$ *2-dominates each other.*

*(ii) Any two D_2-factorizable *-monomorphisms are unitarily equivalent, and their class is contained in $CEK(B, A)$. In particular, if*

$$h_0 : B \hookrightarrow O_2 \subseteq O_2 \otimes O_2 \subseteq O_2 \subseteq A \subseteq Q(\mathbf{R}_+, A),$$

then $[h_0] = 2[h_0]$ *in* $SEK(B,A)$.

(iii) $SEK(B,A)+[h_0]$ *is a subgroup of* $EK(B,A)$ *which is isomorphic to* $Groth(SEK(B,A)) = Groth(EK(B,A))$.

(iv) If $0 = X_1 < X_2 < \cdots$ *is a sequence in* $\mathbf{R}_+$ *with* $\lim_{n\to\infty} X_n = \infty$, ω *is a free ultrafilter on* $\mathbf{N}$ *and* $\overline{\omega} \in \beta\mathbf{R}_+ \backslash \mathbf{R}_+$ *defined by* (X_n) *and* ω, *then there is a canonical isomorphism from* $A_{\overline{\omega}}^{\mathbf{R}_+}$ *onto* A_ω *and* $\pi_{\overline{\omega}} : Q(\mathbf{R}_+, A) \to A_{\overline{\omega}}^{\mathbf{R}_+} \cong A_\omega$ *defines a semigroup morphism from* $SEK(B,A)$ *into* $EK_\omega(B,A)$.

Proof. Hint: For the proof of (i) we use the following claim.

Claim. Let A be a unital *pi* algebra, X a locally compact σ-compact Hausdorff space, $1_A \in B \subseteq Q(X,A) = C_b(X,A)/C_0(X,A)$ a separable unital completely faithfully embedded C*-subalgebra of $Q(X,A)$ and $V : B \to Q(X,A)$ a nuclear unital completely positive map.

Then there exists $a_1, a_2 \in Q(X,A)$ with $V(b) = a_1^* b a_1 + a_2^* b a_2$ for every $b \in B$.

Here $B \subseteq Q(X,A)$ is called completely faithfully embedded if any free ultrafilter ω in $\beta X \backslash X$ $\pi_\omega | B$ is faithful, where π_ω is a canonical quotient map from $Q(X,A)$ to $Q(X,A)/(J_\omega + C_0(X,A))$ ($\cong Q_\omega(A)$ in Proposition 5.7).

Corollary 8.4 *Let* ω *be a free ultrafilter in* $\mathbf{N}$. *Then, the natural map from* $Groth(SEK(B,A))$ *into* $EK_\omega(B,A)$ *is injective.*

Proposition 8.5 *If* A *and* B *are pi-sun algebra with* $A = A^{st}$ *and* $B = B^{st}$ *and* $h : A \to B$, $k : B \to A$ *unital *-homomorphisms such that* $[kh] + [h_0^A] = [id_A] + [h_0^A]$ *in* $Groth(SEK(A,A))$ *and* $[hk] + [h_0^B] = [id_B] + [h_0^B]$ *in* $Groth(SEK(B,B))$, *then there exits an isomorphism* $\phi : A \to B$ *which is approximately unitary equivalent to* h.

Proof. This comes from Lemma 7.10 and Corollary 8.4.

9 Proof of Theorem B

In this section we present the out line of Theorem B. If a reader is interested in the detail proof of main theorem (Theorem 11.1), he or she may try to read (or complete) arguments in [17, Appendix].

The follwoing is a main theorem in a article of Kirchberg.

Theorem 9.1 *Let* B *be a unital separable and exact* C^**-algebra such that* B *contains a unital copy of* O_2 *and* A *a pi algebra with* $A = A^{st}$, *then there is a group isomorphism* ϕ *from* $SEK(B,A) + [h_0] = Groth(SEK(B,A))$ *onto* $KK_{nuc}(B,A)$.

Here $KK_{nuc}(B, A)$ is a group which was studied by Skandalis [31]: (Note that if B is nuclear, then $KK_{nuc}(B, A)$ is a usual KK-group $KK(B, A)$)

Let's see the outline of Theorem 9.1.

Let $Q(X, D) = C_b(X, D)/C_0(X, D)$ for a C*-algebra D and a locally compact Hausdorff space X. We define a new abelian semigroup $ES(B, A)$ as a set of unitary equivalence classes of nuclear*-monomorphisms $h : B \otimes \mathbf{K} \to Q(\mathbf{R}_+, A \otimes \mathbf{K})$. Then, we define a map τ from $SEK(B, A)$ to $ES(B, A)$ by

$\tau([h]) = [h \otimes id_{\mathbf{K}}]$.

Proposition 9.2 *Let A be a pi algebra with $A = A^{st}$ and let B be a separable unital exact C^*-algebra that contains a unital copy of O_2. Then, τ is a semigroup isomorphism.*

Next, we define one more semigroup $SExt_{nuc}(B, A)$ as a set of unitary equivalence classes of nuclear *-monomorphisms $h : B \otimes \mathbf{K} \to M(C_0(\mathbf{R}, A \otimes \mathbf{K})/C_0(\mathbf{R}, A \otimes K)$. Define a map ψ from $ES(B, A)$ to $SExt_{nuc}(B, A)$ by

$\psi([h]) = [(h_0, h)]$, where h_0 is an absorbing element from $B \otimes \mathbf{K}$ into $Q(\mathbf{R}_-, A \otimes \mathbf{K})$ and (h_0, h) is a nuclear *-monomorphism from $B \otimes \mathbf{K}$ into $Q(\mathbf{R}_-, A \otimes \mathbf{K}) \oplus Q(\mathbf{R}_+, A \otimes \mathbf{K}) \cong Q(\mathbf{R}, A \otimes \mathbf{K}) \subset M(C_0(\mathbf{R}, A \otimes \mathbf{K})/C_0(\mathbf{R}, A \otimes \mathbf{K})$.

Proposition 9.3 *Under the same assumption in the previous proposition ψ maps $ES(B, A)$ onto the absorbing classes of $SExt_{nuc}(B, A)$, that is, $SExt_{nuc}(B, A) + [h_0]$ ($\cong Groth(SExt_{nuc}(B, A))$. Moreover, this induces a group isomorphism from $Groth(ES(B, A))$ from $Groth(SExt_{nuc}(B, A))$.*

From the construction we know that if B is nuclear, then $Groth(SExt_{nuc}(B, A))$ is a usual extension group $Ext^{-1}(B \otimes \mathbf{K}, C_0(\mathbf{R}, A \otimes \mathbf{K}))$. On the contrary , from the KK-Theory there is a correspondence between $Ext^{-1}(B \otimes \mathbf{K}, C_0(\mathbf{R}, A \otimes \mathbf{K}))$ and $KK(B, A)$. Hence, if B is nuclear, then there is a group isomorphism from $SEK(B, A) + [h_0]$ to $KK(B, A)$. If B is a general exact C*-algebra, see [31].

Remark 9.4 *Since $KK_{nuc}(B, A)$ is the homotopy invariant, we have*

$\phi([\hat{\sigma} \circ h]) = \phi([h])$ for any topological isomorphism σ from $\mathbf{R}_+$ to $\mathbf{R}_+$. Hence,

$$[\hat{\sigma} \circ (h \oplus h_0)] = [\hat{\sigma} \circ h] + [h_0] = [h] + [h_0] = [h \oplus h_0].$$

Corollary 9.5 *(1) $O_2 \cong D_2$.*

(2) $B \otimes O_2 \cong O_2$ for any simple separable unital nuclear C^*-algebra B.

Proof. (1): Since $KK(O_2, O_2) = 0$, $Groth(SEK(O_2, O_2)) = 0$ from Theorem 9.1. Hence, $O_2 \cong D_2$ from Corollary 8.4(ii) and Proposition 7.11.

(2): This comes from (1) and Corollary 7.12(ii).

Lemma 9.6 *A nuclear *-monomorphism $h : B \to Q(\mathbf{R}_+, A)$ is scaling invariant (modulo unitary equivalence) if and only if h is unitary equivalent to a constant *-monomorphism $k : B \to A \subseteq Q(\mathbf{R}_+, A)$.*

Corollary 9.7 *If B is separable unital nuclear C^*-algebra with a unital copy of O_2 , and A is a pi algebra with $A = A^{st}$, then every element $z \in KK_{nuc}(B, A)$ is of the form $\phi([h \oplus h_0])$, where h is a unital nuclear *-monomorphism from B into A.*

Theorem B

Let A and B be *pi-sun* algebras with $A = A^{st}$ and $B = B^{st}$. If $z \in KK(A, B)$ is a KK-equivalence, then there exists an isomorphism ψ from A onto B such that $\phi([\psi] + [h_0]) = z$ in $KK(A, B)$.

Proof. From assumption there is an inverse y of z in $KK(B, A)$ such that $zy = Id_B$ and $yz = Id_A$. From the previous result, there are nuclear *-monomorphisms $h : A \to B$ and $k : B \to A$ such that $\phi([h \oplus h_0^A]) = z$ and $\phi([k \oplus h_0^B]) = y$.

Using Kasparov product we get

$$[hk] + [h_0^B] = [id_B] + [h_0^B]$$

$$[kh] + [h_0^A] = [id_A] + [h_0^A].$$

Hence, from Proposition 8.5 there exists an isomorphism ψ from A to B which is approximately unitary equivalent to h. Therefore, they induce the same class in $EK_\omega(B, A)$, where ω is a free ultrafilter on $\mathbf{N}$, that is, $\omega \in \beta\mathbf{N}\backslash\mathbf{N}$. Therefore, we get $[\psi] = [h]$ in $SEK(B, A)$ from Corollary 8.4

Corollary 9.8 *Let A and B be pi-sun algebras.*

(1) A and B are KK-equivalent if and only if they are stable isomorphic.

(2) If there exists KK-equivalence x in $KK(A, B)$ with

$$\gamma_0(x)([1_A]_0) = [1_B]_0,$$

then A and B are isomorphic, where γ_0 is a nutural map from $KK(A, B)$ to $Hom(K_0(A), K_0(B))$ which is induced by Kasparov product.

Proof. (1): Take projections p in A and q in B such that $[p]_0 = 0$ in $K_0(A)$ and $[q]_0 = 0$ in $K_0(B)$. Then, both pAp and qBq are in Cuntz standard form.

Since pAp and qBq are KK-equivalent, there are isomorphic from Theorem B. So, A and B are stable isomorphic from [2].

(2): As in the same argument we have an isomorphism τ from pAp to qBq for some projections $p \in A$ and $q \in B$ such that $\tau_0([1_A]_0) = [1_B]_0$. Take projections $p_1 \in pAp$ and $q_1 \in qBq$ so that p_1 is equiavalent to 1_A, and q_1 is equivalent to 1_B.

Since $\tau_0([p_1]_0) = [q_1]_0$, there is a partial isometry u in qBq such that $u^*u = \tau(p_1)$ and $uu^* = q_1$. So, p_1Ap_1 is isomorphic to q_1Bq_1, hence A is isomorphic to B.

Corollary 9.9 *Let A be a simple separable unital nuclear C^*-algebra. Then, A is pi if and only if $A \cong A \otimes O_\infty$.*

Proof. Let $\phi : A \to A \otimes O_\infty$ be a *-homomorphism defined by $\phi(a) = a \otimes 1$ Then, this induces KK-equivalence in $KK(A, A \otimes O_\infty)$. Note that $\phi_0([1_A]_0) = [1_{A\otimes O_\infty}]_0$. Therefore, A and $A \otimes O_\infty$ are isomorphic from Corollary 9.8(2).

Corollary 9.10 *If A and B are pi-sun algebras satisfying the UCT, and if $\sigma : K_*(A) \to K_*(B)$ is an isomorphism with $\sigma_0([1_A]_0) = [1_B]_0$, then there exists an isomorphism τ from A onto B with $K_*(\tau) = \sigma_*$.*

Proof. This comes from the same argument as in Corollary 9.9.

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