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# ON THE STRUCTURE OF CERTAIN CLASS OF VOLTERRA INTEGRAL OPERATORS AND ESTIMATES OF APPROXIMATION NUMBERS

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## Introduction.

We study the integral operators of the form

$$Kf(x) = v(x)\int_0^x k(x,y)u(y)f(y)\,dy, \quad x>0, \tag{1}$$

where the weight real functions $v(t)$ and $u(t)$ are locally integrable and the kernel $k(x,y)\geq 0$ satisfies the following condition: there exists a constant $D\geq 1$ such that

$$D^{-1}\Big(k(x,y)+k(y,z)\Big)\leq k(x,z)\leq D\Big(k(x,y)+k(y,z)\Big), \quad x>y>z\geq 0, \tag{2}$$

where $D$ does not depend on $x,y,z$.

A few standard examples of the kernel $k(x,y)\geq 0$ satisfying (2) are

(1) $k(x,y)=(x-y)^\alpha,\ \alpha\geq 0$,

(2) $k(x,y)=\log^\beta(1+x-y),\quad k(x,y)=\log^\beta\Big(\dfrac{x}{y}\Big);\ \beta\geq 0$,

(3) $k(x,y)=\Big(\displaystyle\int_y^x h(s)\,ds\Big)^\alpha,\ \alpha\geq 0,\ h(s)\geq 0$,

as well as their various combinations. However, for instance, the kernel of the first kind with negative value of $\alpha$ does not satisfy (2).

The operators (1) with kernels satisfying (2) were intensively studied during the last decade and many authors made contributions in this topic, e.g. see the author's survey [$\mathrm{St}_1$] with history and literature given there, and where the $L^p-L^q$ mapping properties of (1) were investigated.

Here, in Section 1, we give further extension of some characterization results of [$\mathrm{St}_1$] on the Banach function spaces for the following problems:

(B) Boundedness,

(C) Compactness and measure of non-compactness.

In Section 2 we give the more detailing structural results for the operators (1), when they are compact in the Lebesgue spaces, namely

(S) Two-sided estimates of the Schatten-von Neumann ideal norms,

(N) Asymptotic behaviour and two-sided estimates of $p$-norms of approximation numbers.

## 1. Problems (B) and (C) in Banach function spaces.

Assume $X$ and $Y$ be two Banach spaces of measurable functions defined on $\mathbf{R}^+=(0,\infty)$. First we consider the problem of the boundedness $K:X\to Y$ for the integral operator (1). This case was recently investigated by E. Berezhnoi [Ber], who, in particular, characterized the weak type estimates

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for operator (1) with the kernel $k(x,y) \geq 0$ increasing with respect to the first variable and also the strong estimates, when $k(x,y) = 1$ and the spaces $X$ and $Y$ satisfy $\ell$-condition (see Definition 3 below). E. Berezhnoi [Ber] has also obtained some necessary and/or sufficient conditions for the boundedness of operators (1) with the restrictions on $k(x,y) \geq 0$, stronger then (2).

**Definition 1** [BS]. A real normed linear space $X = \left\{f : \ \|f\|_X < \infty\right\}$ of Lebesgue-measurable functions on $\mathbf{R}^+$ is called a *Banach function space* (BFS), if in addition to the usual norm axioms $\|f\|_X$ satisfies the following properties:

1) $\|f\|_X$ is defined for every Lebesgue-measurable function $f$ on $\mathbf{R}^+$, and $f \in X$ if, and only if, $\|f\|_X < \infty$; and $\|f\|_X = 0$ if, and only if, $f = 0$ almost everywhere (a.e);
2) $\|f\|_X = \||f|\|_X$ for all $f \in X$;
3) if $0 \leq f \leq g$ a.e., then $\|f\|_X \leq \|g\|_X$;
4) if $0 \leq f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$;
5) if $mesE < \infty$, then $\|\chi_E\|_X < \infty$;
6) if $mesE < \infty$, then $\int_E f(x)dx \leq C_E\|f\|_X$ for all $f \in X$.

Given BFS $X$, its *associate space* $X'$ is defined by

$$X' = \left\{g : \ \int_0^\infty |fg| < \infty \text{ for all } f \in X\right\}$$

and endowed with the associate norm

$$\|g\|_{X'} = \sup\left\{\int_0^\infty |fg| : \|f\|_X \leq 1\right\}.$$

$X'$ is also the Banach function space satisfying axioms (1-6) and, moreover, $X'$ is the norm fundamental subspace of the dual space $X^*$, that is the inequality

$$\|f\|_X = \sup\left\{\int_0^\infty |fg| : \|g\|_{X'} \leq 1\right\}$$

holds for all $f \in X$ [BS].

The spaces $X, X'$ are the complete normed linear spaces and $X'' = X$ [BS].

$X$ has *absolutely continuous norm* (AC norm), if for all $f \in X, \|f\chi_{E_n}\|_X \to 0$ for every sequence of sets $\{E_n\} \subset \mathbf{R}^+$ such, that $\chi_{E_n} \to 0$ a.e. We assume throughout the paper that *$X'$ and $Y$ have the AC-norms.*

Let $\ell$ be a it Banach sequence space (BSS), what means that axioms (1-6) are fulfilled with respect to the count measure and let $\{e_n\}$ denote the standard basis in $\ell$.

**Definition 2.** Given BFS $X$ and BSS $\ell$, $X$ is said to be *$\ell$ − concave*, if for any sequence of disjoint intervals $\{J_k\}$ such that $\bigcup J_k = \mathbf{R}^+$, and for all $f \in X$

$$\left\|\sum_k e_k \|\chi_{J_k} f\|_X\right\|_\ell \leq d_1\|f\|_X,$$

where a finite positive constant $d_1$ independent on $f \in X$ and $\{J_k\}$. Analogously, BFS $Y$ is said to be *$\ell$ − convex*, if for any sequence of disjoint intervals $\{I_k\}$ such that $\bigcup I_k = \mathbf{R}^+$, and for all $g \in X$

$$\|g\|_Y \leq d_2\left\|\sum_k e_k \|\chi_{I_k} g\|_Y\right\|_\ell$$

with a finite positive constant $d_2$ independent on $g \in Y$ and $\{I_k\}$.

**Definition 3** [Ber]. We say, that Banach function spaces $X, Y$ satisfy $\ell$ – *condition*, if there exist a Banach sequence space $\ell$ such that $X$ is $\ell$-concave and $Y$ is $\ell$-convex simultaneously.

Throughout the paper the uncertainties of the form $0 \cdot \infty$, $0/0$, $\infty/\infty$ are taken equal to zero, the inequality $A \ll B$ means $A \leq cB$, where $c$ depends only on $D$ and, possibly, on $d_1$ and $d_2$ from Definition 2; however the relationship $A \approx B$ is interpreted as $A \ll B \ll A$ or $A = cB$. $\chi_E$ denotes the characteristic function of a set $E \subset \mathbf{R}^+$.

**1.1. Boundedness.** Put for all $t \geq 0$

$$A_0 = \sup_{t>0} A_0(t) = \sup_{t>0} \|\chi_{[t,\infty]} v\|_Y \|\chi_{[0,t]}(\cdot) k(t,\cdot) u(\cdot)\|_{X'}, \tag{3}$$

$$A_1 = \sup_{t>0} A_1(t) = \sup_{t>0} \|\chi_{[t,\infty]}(\cdot) k(\cdot,t) v(\cdot)\|_Y \|\chi_{[0,t]} u\|_{X'} \tag{4}$$

and let $A = A_0 + A_1$. Note, that $A_0 = A_1$, if $k(x,y) = 1$.

**THEOREM 1.1.** *Let $X$ and $Y$ be* BFS *satisfying the $\ell$-condition and let $K$ be the integral operator of the form* (1) *with the kernel $k(x,y) \geq 0$ satisfying* (2). *Then $K : X \to Y$ is bounded, if and only if, $A$ is finite. Moreover,*

$$\|K\|_{X \to Y} \approx A. \tag{5}$$

**Example 1.** If $X = L^p$, $Y = L^q$, $1 \leq p, q \leq \infty$ are the Lebesgue spaces with the usual norms, then the $\ell$-condition holds if, and only if, $p \leq q$ and quantities (3) and (4) transform into

$$A_0 = \sup_{t>0} A_0(t) = \sup_{t>0} \Big( \int_t^\infty k^q(x,t) |v(x)|^q \, dx \Big)^{1/q} \Big( \int_0^t |u(y)|^{p'} \, dy \Big)^{1/p'},$$

$$A_1 = \sup_{t>0} A_1(t) = \sup_{t>0} \Big( \int_t^\infty |v(x)|^q \, dx \Big)^{1/q} \Big( \int_0^t k^{p'}(t,y) |u(y)|^{p'} \, dy \Big)^{1/p'}$$

in this case, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. For the case $1 < q < p$ in $L^p - L^q$ setting the following criterion is true [St$_1$]: $\|K\|_{L^p \to L^q} \approx B$, where $B = B_0 + B_1$ defined by

$$B_0 = \Big\{ \int_0^\infty \Big( \int_t^\infty k^q(x,t) |v(x)|^q \, dx \Big)^{r/q} \Big( \int_0^t |u(y)|^{p'} \, dy \Big)^{r/q'} |u(t)|^{p'} \, dt \Big\}^{1/r},$$

$$B_1 = \Big\{ \int_0^\infty \Big( \int_t^\infty |v(x)|^q \, dx \Big)^{r/p} \Big( \int_0^t k^{p'}(t,y) |u(y)|^{p'} \, dy \Big)^{r/p'} |v(t)|^q \, dt \Big\}^{1/r},$$

where $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{p}$. If $k(x,y) = 1$, then $B_0 = \left(\frac{p'}{q}\right)^{1/r} B_1$ and the above criterion is valid for the range $0 < q < p, p \geq 1$ [S] with suitable modification, when $p = 1$ [SS].

**Example 2.** For $0 < r < \infty$, $0 < s \leq \infty$ and a locally integrable function $\varphi(x)$ on $\mathbf{R}^+$, the Lorentz space $L_\varphi^{rs} \equiv L_\varphi^{rs}(\mathbf{R}^+)$ consists of all measurable functions $f$ such that $\|f\|_{rs,\varphi} < \infty$, where

$$\|f\|_{rs,\varphi} = \Big( \int_0^\infty \Big( t^{1/r} f^{**}(t) \Big)^s \frac{dt}{t} \Big)^{1/s} \qquad \textit{for } \ 0 < s < \infty,$$

$$\|f\|_{rs,\varphi} = \sup_{t>0} t^{1/r} f^{**}(t) \quad for \ \ s = \infty,$$

and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds,$$

$$f^*(t) = \inf\left\{x > 0: \ \ \lambda_f(x) = \int_{\{y\in\mathbf{R}^+:|f(y)|>x\}} \varphi(z)\,dz \le t\right\}.$$

If $r = s$, then

$$\|f\|_{rr,\varphi} = \left(\int_0^\infty |f(x)|^r \varphi(x)\,dx\right)^{1/r}.$$

If $X = L_\varphi^{rs}, Y = L_\psi^{pq}$ then the $\ell$-condition holds if, and only if, $\max(r,s) \le \min(p,q)$ and in this case the norm of $K : X \to Y$ is sandwiched by $A = A_0 + A_1$, where

$$A_0 = \sup_{t>0} A_0(t) = \left\|\chi_{[0,t]}(\cdot)k(t,\cdot)(u(\cdot)/\varphi(\cdot))\right\|_{r's',\varphi}\left\|\chi_{[t,\infty]}v\right\|_{pq,\psi},$$

$$A_1 = \sup_{t>0} A_1(t) = \left\|\chi_{[0,t]}(u/\varphi)\right\|_{r's',\varphi}\left\|\chi_{[t,\infty]}(\cdot)k(\cdot,t)v(\cdot)\right\|_{pq,\psi}.$$

**Remark 1.1.** (i) If the $\ell$-condition fails, then the lower bound in (5) is nevertheless true. However, there exist an operator, when (5) is valid for the spaces with no $\ell$-condition. Indeed, if we take $k(x,y) = 1, v(x) = 1$, then in Lorentz space setting above the criterion (5) holds for $1 < r = s, q \ge r, 0 < p < \infty$ ([Sa], Theorem 2).

(ii) If $1 < r = s, 0 < q < r < \infty, 0 < p < \infty$, then the criterion for the boundedness of this operator is the following ([St$_1$], Theorem 2.2). Put $Uf(x) = \int_0^x f(y)u(y)dy$. Then

$$\sup_{f\ne 0}\frac{\|Uf\|_{pq,\psi}}{\|f\|_{rr,\varphi}} \approx \left(\int_0^\infty \left(\|\chi_{[t,\infty]}\|_{pq,\psi}\right)^\gamma d\left(\|\chi_{[0,t]}(u/\varphi)\|_{r'r',\varphi}\right)^\gamma\right)^{1/\gamma},$$

where $\frac{1}{\gamma} = \frac{1}{q} - \frac{1}{r}$.

**1.2. Compactness and measure of non-compactness.**

**THEOREM 1.2.** *Let the assumptions of Theorem 1 be fulfilled and the spaces $X'$ and $Y$ have the AC-norms. Then the operator $K : X \to Y$ is compact if, and only if, $A$ is finite and*

$$\lim_{t\to a_i} A_i(t) = \lim_{t\to b_i} A_i(t) = 0, \quad i = 0, 1, \tag{6}$$

*where*

$$a_i = \inf\{t > 0 : A_i(t) > 0\}, \quad b_i = \sup\{t > 0 : A_i(t) > 0\} \quad i = 0, 1.$$

**Remark 1.1.** (i) In fact, it follows from the proof of Theorem 1.2, that $a_0 = a_1, b_0 = b_1$.

(ii) By many authors the condition (6) used to be formulated for the end-points, however it is easy to point out a formal counterexamle, when $A$ is finite and (6) is valid with $a_0 = a_1 = 0, b_0 = b_1 = \infty$, but $K$ is non-compact. The matter is, that the condition (6) has to formulated for the end-points of the real interval of non-zero action of $K$.

(iii) For the case $1 < q < p$ in $L^p - L^q$ setting the operator $K : L^p \to L^q$ is compact if, and only if, $B < \infty$ (see Example 1). It follows from the Ando theorem [A].

In the non-compact case we estimate the measure of non-compactness of $K$ or, equivalently, the distance between $K$ and the set of finite rank operators defined by

$$\alpha(K) = \inf \|K - P\|,$$

where the infimum is taken over all bounded linear maps $P : X \to Y$ of finite rank. To this end we need additional portion of notations. For $0 < a < z < b < \infty$ we put

$$J_L^0(a) = \sup_{0<t<a} \|\chi_{[t,a]} v\|_Y \|\chi_{[0,t]}(\cdot)k(t,\cdot)u(\cdot)\|_{X'},$$

$$J_L^1(a) = \sup_{0<t<a} \|\chi_{[t,a]}(\cdot)k(\cdot,t)v(\cdot)\|_Y \|\chi_{[0,t]} u\|_{X'},$$

$$J_L(z) = \max\left(J_L^0(z), J_L^1(a)\right), J_L = \lim_{z\to a_0} J_L(z),$$

$$J_R^0(b) = \sup_{b<t<\infty} \|\chi_{[t,a]} v\|_Y \|\chi_{[0,t]}(\cdot)k(t,\cdot)u(\cdot)\|_{X'},$$

$$J_R^1(b) = \sup_{b<t<\infty} \|\chi_{[t,a]}(\cdot)k(\cdot,t)v(\cdot)\|_Y \|\chi_{[0,t]} u\|_{X'},$$

$$J_R(z) = \max\left(J_R^0(z), J_R^1(a)\right), J_R = \lim_{z\to a_0} J_R(z),$$

$$J = \max\left(J_L, J_R\right).$$

**THEOREM 1.3.** *Let the assumptions of Theorem 2 be valid and $K : X \to Y$ be bounded. Then*

$$D^{-1} J \le \alpha(K) \le (d_1, d_2, D) J.$$

## 2. Problems (S) and (N) in Lebesgue spaces

**2.1. Schatten-von Neumann ideal norms.** Let $H$ be a separable Hilbert space. Then the set of all linear bounded operators $T : H \to H$ forms the normed algebra B, where $\sigma_\infty$-the ideal of all compact operators. The theory of simmetrically normed (s.n.) ideals $\sigma_\Phi \subset \sigma_\infty$ was developed by using the s.n. functions $\Phi$ defined on the space of sequences with a finite number of non-zero terms ([GK], Chapter 3). If $T \in \sigma_\infty$, then $T^* \in \sigma_\infty$ and $\left(T^*T\right)^{1/2} \in \sigma_\infty$. To construct $\sigma_\Phi$ the sequences of singular numbers $s_j(T) = \lambda_j\left[\left(T^*T\right)^{1/2}\right]$ were used, with the eigenvalues $\lambda_j \ge 0$ taken according to their multiplicity and decrease. Formula $\|T\|_{\sigma_\Phi} = \Phi\big(s_j(T)\big)$ defines the norm (quasinorm) in the s.n. ideal $\sigma_\Phi$. The most well-known are the s.n. ideals $\sigma_p$ related to the space of sequences $l_p$, $0 < p \le \infty$. The norm (quasinorm) $\|T\|_{\sigma_p} = \left(\sum_j s_j^p(T)\right)^{1/p}$ is usually called by the Schatten-von Neumann norm (quasinorm). Thus, $\|T\|_{\sigma_2} = \|T\|$ and $\|T\|_{\sigma_2}$ is the Hilbert-Schmidt norm expressed for an integral operator $Tf(x) = \int T(x,y)f(y)\,dy$ by the formulae $\|T\|_{\sigma_2} = \left(\int\int \left|T(x,y)\right|^2 dx\,dy\right)^{1/2}$. It is known [BS], that in general the norm $\|T\|_{\sigma_p}$ of an integral operator substansially depends on the smoothness of its kernel, when $p < 2$, however for some particular operators of complex harmonic analysis the effective two-sided estimates of the Schatten-von Neumann norms are well known, e.g.

see [Pa], [P].The aim of the section is to present a brief account of some results from [ES$_2$] and [St$_2$] about the Schatten-von Neumann ideal norms for the integral operators (1) with the condition (2) for their kernels.

Let $H = L^2(0,\infty)$ and

$$A_0^2 = \sup_{t>0} \int_t^\infty k^2(x,t)|v(x)|^2\,dx \int_0^t |u(y)|^2\,dy,$$

$$A_1^2 = \sup_{t>0} \int_t^\infty |v(x)|^2\,dx \int_0^t k^2(t,y)|u(y)|^2\,dy.$$

Theorem 1.1 and the Hilbert-Schmidt formula bring

$$\|K\|_{\sigma_\infty} \approx A_0 + A_1,$$

$$\|K\|_{\sigma_2} = \Big(\int_0^\infty |v(x)|^2\,dx \int_0^x k^2(t,y)|u(y)|^2\,dy\Big)^{1/2} = \Big(\int_0^\infty |u(y)|^2\,dy \int_y^\infty k^2(x,y)|v(x)|^2\,dx\Big)^{1/2}.$$

Using these formulas and applying the real method of interpolation we obtain the following

**THEOREM 2.1.** *Let $K$ be an operator of the form* (1) *with the kernel satisfying* (2) *and $K \in \sigma_\infty$. Then*

$$\|K\|_{\sigma_p} \approx \Bigg(\int_0^\infty \Bigg[\Big(\int_0^x k^2(x,y)|u(y)|^2\,dy\Big)^{p/2}\Big(\int_x^\infty |v(y)|^2\,dy\Big)^{\frac{p}{2}-1}|v(x)|^2 +$$

$$\Big(\int_x^\infty k^2(y,x)|v(y)|^2\,dy\Big)^{p/2}\Big(\int_0^x |u(y)|^2\,dy\Big)^{\frac{p}{2}-1}|u(x)|^2\Bigg]\,dx\Bigg)^{1/p}, \quad 2 \le p < \infty. \tag{7}$$

**Remark 2.1.** The upper bound of (7) is proved in [ES$_2$] and the lower one in [St$_2$]. In case $k(x,y) \equiv 1$ the formula (7) can be simplified and extended as follows. If

$$Hf(x) = v(x)\int_0^x f(y)u(y)\,dy, \tag{8}$$

then

$$\|H\|_{\sigma_p} \approx \Bigg(\int_0^\infty \Big(\int_0^x |u(y)|^2\,dy\Big)^{p/2}\Big(\int_x^\infty |v(y)|^2\,dy\Big)^{\frac{p}{2}-1}|v(x)|^2\,dx\Bigg)^{1/p}, \quad 1 < p < \infty. \tag{9}$$

**Remark 2.1.** In alternate form the equivalence (9) for the case $u(y) = 1$ has been established in [N] and later this result has been widely extended in [NS] for the operator

$$I_\nu f(x) = \frac{v(x)}{x^\nu}\int_0^x (x-y)^{\nu-1}f(y)dy, \quad \nu > 1/2.$$

**2.2. Approximation numbers.** Recall that if $T : X \to Y$, then the $n$-th approximation number of $T$ is defined by

$$a_n = \inf\{\|T-P\|, \quad rank P < n\}, \quad n = 1,2,...$$

The problem of asympthotic behaviour of the approximation numbers is well known and was treated in the monograghs [EE], [K] and others and for the operator (8) in the papers [EEH$_1$], [EEH$_2$],

[$ES_1$], [$LS_1$], [$LS_2$]. Here we present the new result for the operator (8) easy comparable with formula (9).

**THEOREM 2.2** *Let $1 < p, s < \infty$ and the integral operator $H : L^p(0,\infty) \to L^p(0,\infty)$ given by (8) be compact and $\{a_n\}$ is the sequence of the approximation numbers of $H$. Then*

$$\Big(\sum_{n=1} a_n^s\Big)^{1/s} \approx \left(\int_0^\infty \Big(\int_0^x |u(y)|^{p'}\,dy\Big)^{s/p'} \Big(\int_0^x |v(y)|^p\,dy\Big)^{\frac{s}{p}-1} |v(x)|\,dx\right)^{1/s},$$

$$\overline{\lim_{n\to\infty}}\, na_n \le \gamma_p \int_0^\infty |uv| \le 2 \underline{\lim_{n\to\infty}}\, na_n, \quad p \neq 2,$$

$$\lim_{n\to\infty} na_n = \frac{1}{\pi}\int_0^\infty |uv|, \quad p = 2.$$

The last two formulas are proved in [$EEH_2$] and the proof of the first is based on the results of [$EEH_2$].

**Remark 2.2.** All the assertions of the paper have natural analogs for a finite interval instead of $(0,\infty)$ and for the dual operator $K^*$ as well (see [$St_1$] for details).

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