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M-Convex Function on Generalized Polymatroid

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Abstract

M-convex function is a generalization of valuated matroid of Dress–Wenzel, and a discrete analogue of convex function defined on a base polyhedron of a submodular system. We extend this concept to functions on generalized polymatroids, discuss the layer structure of M-convex functions on g-polymatroids, and give simultaneous exchange axioms.

Keywords: matroid, submodular system, convex function, generalized polymatroid.

1 Introduction

Generalizing the concept of valuated matroid due to Dress and Wenzel [4, 5], Murota [17, 18, 19] introduced the concept of M-convex function. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if it satisfies

(MB-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x)+f(y) \geq f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v),$$

where $\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$, $\mathrm{supp}^+(x-y) = \{w \in V \mid x(w) > y(w)\}$, $\mathrm{supp}^-(x-y) = \{w \in V \mid x(w) < y(w)\}$, and $\chi_w \in \mathbf{Z}^V$ is the characteristic vector of $w \in V$. An M-convex function f with $\mathrm{dom}\, f \subseteq \{0,1\}^V$ can be identified with a valuated matroid; to be specific, $-f$ is a valuated matroid in the sense of [4, 5]. The property (MB-EXC) implies that $\mathrm{dom}\, f$ is (the set of integral points of) a base polyhedron.

M-convex functions enjoy several nice properties: they can be extended to ordinary convex functions, and a Fenchel-type duality and a (discrete) separation theorem hold for them [14, 17, 18, 19]. These properties may be sufficient for us to regard M-convexity as convexity in

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discrete optimization. Applications of M-convex functions for polynomial matrices are described in [4, 5, 13].

The main aim of this paper is to extend the concept of M-convex functions to functions on generalized polymatroids. The concept of generalized polymatroid, or g-polymatroid for short, was introduced in 1981 by Frank [7] (see also Tardos [22] and Frank and Tardos [8]). G-polymatroid includes polymatroid, submodular polyhedron, supermodular polyhedron, and base polyhedron as its special cases. Although g-polymatroid is a generalization of those polyhedra mentioned above, it is also known to be equivalent to base polyhedron in the sense that any g-polymatroid can be obtained as a projection of a base polyhedron. Given a set $Q(\subseteq \mathbf{Z}^V)$, define $\widetilde{Q}(\subseteq \mathbf{Z}^{V\cup\{v_0\}})$ as $\widetilde{Q} = \{(x, -x(V)) \in \mathbf{Z}^{V\cup\{v_0\}} \mid x \in Q\}$, where v_0 is a new element not in V, and $x(V) = \sum\{x(w) \mid w \in V\}$.

Theorem 1.1 (Fujishige [9, 10]) *Q is (the set of integral points in) a g-polymatroid if and only if*

(G-PRJ) *$\widetilde{Q}$ is (the set of integral points in) a base polyhedron.*

In view of this theorem, it would be natural to define M-convexity for a function on a g-polymatroid as follows: a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is defined to be M-convex on a g-polymatroid if

(MG-PRJ) function $\widetilde{f} : \mathbf{Z}^{V\cup\{v_0\}} \to \mathbf{R} \cup \{+\infty\}$ satisfies (MB-EXC), where

$$\widetilde{f}(x, x_0) = \begin{cases} f(x) & (x_0 = -x(V)), \\ +\infty & (\text{otherwise}). \end{cases} \tag{1}$$

It is clear that $\mathrm{dom}\, f$ of an M-convex function f on a g-polymatroid is indeed a g-polymatroid.

Though M-convexity on a g-polymatroid is not entirely a new concept, we believe that it is worth investigating in its own right. One motivation for this paper is that we can talk of the layer structure of an M-convex function when it is defined on a g-polymatroid, where a layer of an M-convex function f on a g-polymatroid is defined as its restriction to $\{x \in \mathbf{Z}^V \mid x(V) = k\}$ for each $k \in \mathbf{Z}$. Then optimization on each layer naturally comes into a problem. Recently, many researchers analyze set systems and functions with respect to layer structures; for example, greedoid by Korte, Lovász, and Schrader [11], valuated bimatroid [13], valuation on independent sets [15], well-layered map and rewarding map by Dress and Terhalle [1, 2, 3], and so on. In particular, valuations on independent sets enjoy M-concavity on g-polymatroids, i.e., the negative of M-convex functions. We show that optimization of an M-convex function in a specified layer can be done efficiently in several different ways.

Another motivation is the richness of examples of M-convex functions on g-polymatroids, e.g., network flows, location problems, and polynomial matrices. It is well known that kinds of greedy algorithms work for those problems, but such phenomena cannot be explained by using the theory of g-polymatroid. The framework of M-convex functions on g-polymatroids explains why greedy algorithms work well for those problems. For example, the successive shortest path augmentation algorithm, which can be seen as a kind of greedy algorithm, works for the minimum-cost flow problem. Our result affords a new understanding to this fact through the M-convexity of the flow cost function.

In view of the exchange axiom (MB-EXC) for an M-convex function on a base polyhedron, it would be natural to ask how the M-convexity on a g-polymatroid can be characterized by an exchange property. We show in Theorem 3.2 that an M-convex function on a g-polymatroid is characterized by either of the following simultaneous exchange properties:

(MG-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$,

$$f(x)+f(y) \geq \min\left[f(x-\chi_u)+f(y+\chi_u), \min_{v\in \mathrm{supp}^-(x-y)}\{f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v)\}\right],$$

(MG-EXC$_{\mathbf{W}}$) $\forall x, y \in \mathrm{dom}\, f$ with $x(V) \geq y(V)$ and $x \neq y$,

$$f(x)+f(y) \geq \min_{u\in \mathrm{supp}^+(x-y)}\left[f(x-\chi_u)+f(y+\chi_u), \min_{v\in \mathrm{supp}^-(x-y)}\{f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v)\}\right].$$

2 Examples of M-Convex Functions on G-Polymatroids

Example 2.1 (Min-cost flow) Let $G=(V,A;V^+,V^-)$ be a directed graph with two specified vertex sets $V^+, V^- \subseteq V$ such that $V^+\cap V^-=\emptyset$. We denote an upper capacity function by $\overline{c}: A\to \mathbf{Z}\cup\{+\infty\}$, a lower capacity function by $\underline{c}: A\to\mathbf{Z}\cup\{-\infty\}$. A flow is a function $\varphi: A\to\mathbf{Z}$, and its boundary $\partial\varphi: V\to\mathbf{Z}$ is defined as

$$\partial\varphi(v)=\sum\{\varphi(a)\mid a \text{ leaves } v\}-\sum\{\varphi(a)\mid a \text{ enters } v\} \qquad (v\in V).$$

A flow φ is called feasible if it satisfies $\underline{c}(a)\leq\varphi(a)\leq\overline{c}(a)$ $(\forall a\in A)$ and $\partial\varphi(v)=0$ $(\forall v\in V-(V^+\cup V^-))$. Then, we see that $Q=\{(\partial\varphi)^-\mid\varphi:\text{feasible flow}\}(\subseteq\mathbf{Z}^{V^-})$ is a g-polymatroid, where $(\partial\varphi)^-$ is the restriction of $\partial\varphi$ to V^-.

Suppose we are given a family of convex functions $f_a:\mathbf{Z}\to\mathbf{R}$ indexed by $a\in A$. Here we call f_a convex if its piecewise linear extension $\overline{f}_a:\mathbf{R}\to\mathbf{R}$ is an ordinary convex function. We define a function $f_{\mathrm{mcf}}:\mathbf{Z}^{V^-}\to\mathbf{R}\cup\{\pm\infty\}$ as follows:

$$f_{\mathrm{mcf}}(x)=\begin{cases}\inf\{\Gamma(\varphi)\mid\varphi:\text{feasible flow}, (\partial\varphi)^-=x\} & (x\in Q),\\ +\infty & (x\notin Q),\end{cases}$$

where $\Gamma(\varphi)=\sum\{f_a(\varphi(a))\mid a\in A\}$. Then, the function f_{mcf} satisfies (MG-EXC) if f_{mcf} does not take the value $-\infty$, which can be proved in the similar way as in [17, 19].

Example 2.2 (k-tree-core) Suppose we are given a tree network $T = (V, E)$ with an edge length function $l : E \to \mathbf{R}_+$ and a vertex weight function $w : V \to \mathbf{R}_+$. For any $u, v \in V$, denote by $P(u, v)$ the unique path connecting u and v. We define the distance $d(u, v)$ between $u, v \in V$ as the sum of lengths of edges in $P(u, v)$. The distance-sum $\text{dis}(S)$ of a subtree S is given by $\text{dis}(S) = \sum\{w(u) \cdot \min\{d(u, v) \mid v \in S\} \mid u \in V\}$. A k-tree-core is a subtree with k leaves minimizing the distance-sum. It is clear that there exists a k-tree-core whose leaves are those of T. Hence, we may restrict ourselves to subtrees whose leaves are contained in $L = \{v \in V \mid v \text{ is a leaf of } T\}$. We represent such a subtree by the set of its leaves. Put $\mathcal{F} = \{X \subseteq L \mid |X| \geq 2\}$ and denote by $S(X)$ the subtree corresponding to $X \in \mathcal{F}$. Define a function $f_{\text{dis}} : \mathbf{Z}^L \to \mathbf{R} \cup \{+\infty\}$ by

$$f_{\text{dis}}(x) = \begin{cases} \text{dis}(S(X)) & (x = \chi_X \text{ for some } X \in \mathcal{F}), \\ +\infty & (\text{otherwise}), \end{cases}$$

where χ_X is the characteristic vector of $X \subseteq L$. We will prove the following theorem. See Peng et al.[20] and Shioura and Uno [21] for more about k-tree-core.

Theorem 2.1 *The function f_{dis} satisfies (MG-EXC).*

Before proving this theorem, we give a property of the distance-sum. For $u, v \in V$ with $(u, v) \in E$, set $W(u, v) = \sum\{w(t) \mid t \in V,\ v \in P(u, t)\}$. For any $u, v \in V$, put $\Delta(u, v) = \sum\{l(u_{i-1}, u_i)\ W(u_{i-1}, u_i) \mid i = 1, \cdots, r\}$, where $\{u_0 (= u), u_1, \cdots, u_r (= v)\}$ is the sequence of vertices on the path $P(u, v)$. It should be noted that $\Delta(u, v)$ is not equal to $\Delta(v, u)$.

Lemma 2.2 *Let $u, v \in V$ and S be a subtree such that $P(u, v) \cap S = \{u\}$. Then,*

$$\text{dis}(S \cup P(u, v)) - \text{dis}(S) = -\Delta(u, v).$$

Note that the value $\text{dis}(S \cup P(u, v)) - \text{dis}(S)$ does not depend on a subtree S.

Proof of Theorem 2.1 Let $X, Y \in \mathcal{F}$ and $u \in X - Y$. It suffices to show that (i) or (ii) holds, where

(i) $|X| \geq 3$ and $\text{dis}(S(X)) + \text{dis}(S(Y)) \geq \text{dis}(S(X - u)) + \text{dis}(S(Y + u))$,

(ii) $\text{dis}(S(X)) + \text{dis}(S(Y)) \geq \text{dis}(S(X - u + v)) + \text{dis}(S(Y + u - v))$ $(\exists v \in Y - X)$.

For each subtree S, we call $w \in S$ a branching vertex of S if there are at least three edges of S incident to w.

Case 1: $S(X)$ and $S(Y)$ contain a common vertex. Let c be the nearest vertex to u in the intersection of $S(X)$ and $S(Y)$. If $|X| \geq 3$, let b_X be the nearest branching vertex of $S(X)$ to u, and if $|X| = 2$ then let b_X be the unique element in $X - u$.

Case 1.1: $b_X \in P(c, u)$. It is easy to see that $|X| \geq 3$. Since $S(X - u) = S(X) - P(b_X, u)$ and $S(Y + u) = S(Y) \cup P(c, u)$, (i) is fulfilled by Lemma 2.2.

Case 1.2: $b_X \notin P(c, u)$. There necessarily exists a leaf v of $S(Y)$ with $c \in P(b_X, v)$. We also have $P(c, v) \cap S(X) = \{c\}$. If there exists a branching vertex of $S(Y)$ on $P(v, c)$, let b_Y be the nearest one to v, and otherwise set $b_Y = c$. Since $S(X - u + v) = (S(X) - P(c,u)) \cup P(c,v)$, and $S(Y + u - v) = (S(Y) - P(b_Y, v)) \cup P(c,u)$, we have the condition (ii) by Lemma 2.2.

CASE 2: $S(X)$ and $S(Y)$ contain no common vertex. Let c_X be the nearest vertex in $S(X)$ to $S(Y)$, and c_Y the nearest vertex in $S(Y)$ to $S(X)$. Note that $P(c_X, c_Y) \cap S(X) = \{c_X\}$ and $P(c_X, c_Y) \cap S(Y) = \{c_Y\}$. If there exists a branching vertex of $S(X)$ on $P(u, c_X)$, let b_X be the nearest one to u, and otherwise set $b_X = c_X$. Let v be any element of Y. If there exists a branching vertex on the path $P(v, c_Y)$ then let b_Y be the nearest one to v, and otherwise set $b_Y = c_Y$. The condition (ii) is obtained by Lemma 2.2 and the following equalities:

$$S(X - u + v) = (S(X) - P(b_X, u)) \cup (P(c_X, c_Y) \cup P(c_Y, v)),$$
$$S(Y + u - v) = (S(Y) - P(b_Y, v)) \cup (P(c_Y, c_X) \cup P(c_X, u)).$$

∎

Example 2.3 (Polynomial matrices [2, 4, 5, 13]) Let $A(t)$ be an $m \times n$ polynomial matrix, where each entry of $A(t)$ is a polynomial in t. Denote by R and C the row and column sets of $A(t)$, respectively. Define $\mathcal{J}$ to be the family of linearly independent column sets, and $f_{\mathrm{mat}} : \mathbf{Z}^C \to \mathbf{R} \cup \{+\infty\}$ by

$$f_{\mathrm{mat}}(x) = \begin{cases} -\max\{\deg_t \det A[I,J] \mid I \subseteq R,\ |I| = |J|\} & (x = \chi_J,\ J \in \mathcal{J}), \\ +\infty & (\text{otherwise}), \end{cases}$$

where $A[I,J]$ is the submatrix of $A(t)$ induced by the row set I and the column set J. Then, we can show that the function f_{mat} satisfies (MG-EXC) by using the Grassmann-Plücker identity.

3 Exchange Axioms for M-Convex Functions on G-Polymatroids

To derive exchange axioms for M-convex functions on g-polymatroids, we first recall a seemingly weaker exchange property than (MB-EXC) for M-convex functions on base polyhedra:

(MB-EXC$_\mathbf{W}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that $f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)$.

Theorem 3.1 ([16, 18]) *(MB-EXC) $\Longleftrightarrow$ (MB-EXC$_\mathrm{W}$).*

This equivalence is a quantitative generalization of the result of Tomizawa [23] for base polyhedra.

A straightforward translation of (MB-EXC) and (MB-EXC$_\mathrm{W}$) through the equation (1) leads to the following exchange axioms for M-convex functions on g-polymatroids:

(MG-EXC$_\mathbf{P}$) $\forall x, y \in \mathrm{dom}\, f$,

(i) $x(V) < y(V) \Longrightarrow f(x) + f(y) \geq \min_{v \in \mathrm{supp}^-(x-y)} \{f(x + \chi_v) + f(y - \chi_v)\}$,

(ii) $x(V) \leq y(V) \Longrightarrow \forall u \in \mathrm{supp}^+(x - y)$, $f(x) + f(y) \geq \min_{v \in \mathrm{supp}^-(x-y)} \{f(x - \chi_u + \chi_v) + f(y +$

$\chi_u - \chi_v)\}$,
(iii) $x(V) > y(V) \Longrightarrow \forall u \in \mathrm{supp}^+(x-y)$,

$$f(x)+f(y) \geq \min \left[f(x-\chi_u) + f(y+\chi_u), \min_{v \in \mathrm{supp}^-(x-y)} \{f(x-\chi_u+\chi_v) + f(y+\chi_u-\chi_v)\} \right],$$

(MG-EXC$_{\mathrm{pw}}$) $\forall x, y \in \mathrm{dom}\, f$,
(i) $x(V) > y(V) \Longrightarrow$

$$f(x)+f(y) \geq \min_{u \in \mathrm{supp}^+(x-y)} \left[f(x-\chi_u) + f(y+\chi_u), \min_{v \in \mathrm{supp}^-(x-y)} \{f(x-\chi_u+\chi_v) + f(y+\chi_u-\chi_v)\} \right],$$

(ii) $x(V) = y(V),\ x \neq y \Longrightarrow f(x)+f(y) \geq \min_{\substack{u \in \mathrm{supp}^+(x-y) \\ v \in \mathrm{supp}^-(x-y)}} \{f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v)\}$.

For example, (MG-EXC$_{\mathrm{p}}$) (i) is obtained from (MB-EXC) for $\widetilde{f}$ with $u = v_0$.

The objective of this section is to show that these axioms are equivalent to (MG-EXC) and (MG-EXC$_{\mathrm{W}}$), which look simpler and nicer.

Theorem 3.2 *(MG-PRJ) $\Longleftrightarrow$ (MG-EXC) $\Longleftrightarrow$ (MG-EXC$_{\mathrm{W}}$) $\Longleftrightarrow$ (MG-EXC$_{\mathrm{p}}$) $\Longleftrightarrow$ (MG-EXC$_{\mathrm{pw}}$).*

We can easily see from definitions and Theorem 3.1 that (MG-EXC$_{\mathrm{pw}}$) $\Longrightarrow$ (MG-EXC$_{\mathrm{p}}$) $\Longrightarrow$ (MG-EXC) $\Longrightarrow$ (MG-EXC$_{\mathrm{W}}$). Furthermore, it is obvious that (MG-EXC$_{\mathrm{W}}$) $\Longrightarrow$ (MG-EXC$_{\mathrm{pw}}$) (i). Thus, it suffices to show that (MG-EXC$_{\mathrm{W}}$) $\Longrightarrow$ (MG-EXC$_{\mathrm{pw}}$) (ii). For this purpose, we need some lemmas.

Lemma 3.3 *(MG-EXC$_{\mathrm{W}}$) $\Longrightarrow \forall x, y \in \mathrm{dom}\, f$ with $x(V) < y(V)$,*

$$f(x) + f(y) \geq \min_{v \in \mathrm{supp}^-(x-y)} \{f(x+\chi_v) + f(y-\chi_v)\}.$$

Proof. The proof is similar to and simpler than the one for Lemma 3.6 below and omitted. ∎

For any $x \in \mathbf{Z}^V$, we define $||x|| = \sum\{|x(w)| \mid w \in V\}$.

Lemma 3.4 *(MG-EXC$_{\mathrm{W}}$) $\Longrightarrow \forall x, y \in \mathrm{dom}\, f$ with $x(V) = y(V)$ and $||x-y|| = 4$,*

$$f(x) + f(y) \geq \min_{\substack{u \in \mathrm{supp}^+(x-y) \\ v \in \mathrm{supp}^-(x-y)}} \{f(x-\chi_u+\chi_v) + f(y+\chi_u-\chi_v)\}.$$

Proof. We can put $x = z + \chi_{w_1} + \chi_{w_2}$, $y = z + \chi_{w_3} + \chi_{w_4}$ with $w_i \in V$ $(i = 1,2,3,4)$ and $z \in \mathbf{Z}^V$ defined by $z(v) = \min\{x(v), y(v)\}$ for $v \in V$. In the following, we denote $\alpha_1 = f(z+\chi_{w_1})$, $\alpha_{23} = f(z+\chi_{w_2}+\chi_{w_3})$, $\alpha_{134} = f(z+\chi_{w_1}+\chi_{w_3}+\chi_{w_4})$, and so on. To the contrary suppose

$$\alpha_{12} + \alpha_{34} < \min\{\alpha_{13}+\alpha_{24}, \alpha_{14}+\alpha_{23}\}. \tag{2}$$

Then, we have $\alpha_{12} + \alpha_{34} = \min\{\alpha_1 + \alpha_{234}, \alpha_2 + \alpha_{134}\}$. In fact, LHS $\geq$ RHS is by (MG-EXC$_\mathrm{W}$) and (2), and the reverse inequality is by Lemma 3.3 and (2). Assume w.l.o.g. that $\alpha_{12} + \alpha_{34} = \alpha_1 + \alpha_{234}$. From (MG-EXC$_\mathrm{W}$), it holds that

$$2(\alpha_{12} + \alpha_{34}) = \alpha_{234} + \alpha_{12} + \alpha_{34} + \alpha_1 \geq \min\{\alpha_{123} + \alpha_{24}, \alpha_{124} + \alpha_{23}\} + \min\{\alpha_3 + \alpha_{14}, \alpha_4 + \alpha_{13}\}.$$

Again assume w.l.o.g. that $\min\{\alpha_{123} + \alpha_{24}, \alpha_{124} + \alpha_{23}\} = \alpha_{123} + \alpha_{24}$. In case that $\min\{\alpha_3 + \alpha_{14}, \alpha_4 + \alpha_{13}\} = \alpha_3 + \alpha_{14}$, we have a contradiction since

$$\alpha_{123} + \alpha_{24} + \alpha_3 + \alpha_{14} \geq \alpha_{13} + \alpha_{24} + \alpha_{23} + \alpha_{14} > 2(\alpha_{12} + \alpha_{34}),$$

where the first and second inequalities are by (MG-EXC$_\mathrm{W}$) and (2), respectively. If $\min\{\alpha_3 + \alpha_{14}, \alpha_4 + \alpha_{13}\} = \alpha_4 + \alpha_{13}$, then Lemma 3.3 and (2) yield another contradiction:

$$\alpha_{123} + \alpha_{24} + \alpha_4 + \alpha_{13} \geq \min\{\alpha_{12} + \alpha_{34}, \alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\} + \alpha_{13} + \alpha_{24} > 2(\alpha_{12} + \alpha_{34}).$$

∎

Lemma 3.5 *(MG-EXC$_\mathrm{W}$)* $\Longrightarrow \forall x, y \in \mathrm{dom}\, f$ *with* $x(V) = y(V)$ *and* $x \neq y$, $\exists u_1 \in \mathrm{supp}^+(x - y)$, $\exists v_1 \in \mathrm{supp}^-(x - y)$ *such that* $y + \chi_{u_1} - \chi_{v_1} \in \mathrm{dom}\, f$.

Proof. By applying (MG-EXC$_\mathrm{W}$) for x and y, either (a) or (b) holds, where

(a) $\exists u_1 \in \mathrm{supp}^+(x - y)$ such that $y + \chi_{u_1} \in \mathrm{dom}\, f$,
(b) $\exists u_1 \in \mathrm{supp}^+(x - y)$, $\exists v_1 \in \mathrm{supp}^-(x - y)$ such that $y + \chi_{u_1} - \chi_{v_1} \in \mathrm{dom}\, f$.

If (a) holds, then we can apply Lemma 3.3 for x and $y + \chi_{u_1}$, which yields that $y + \chi_{u_1} - \chi_{v_1} \in \mathrm{dom}\, f$ for some $v_1 \in \mathrm{supp}^-(x - (y + \chi_{u_1})) \subseteq \mathrm{supp}^-(x - y)$. ∎

In the following, we assume (MG-EXC$_\mathrm{W}$) and show a stronger statement than (MG-EXC$_\mathrm{PW}$) (ii). The proof is almost the same as the one for [18, Theorem 3.1].

Lemma 3.6 *(MG-EXC$_\mathrm{W}$)* $\Longrightarrow \forall x, y \in \mathrm{dom}\, f$ *with* $x(V) = y(V)$, $\forall u \in \mathrm{supp}^+(x - y)$,

$$f(x) + f(y) \geq \min_{v \in \mathrm{supp}^-(x-y)} \{f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\}.$$

Proof. Set

$$\begin{aligned}\mathcal{D} = \{(x, y) \mid x, y \in \mathrm{dom}\, f,\ x(V) = y(V),\ \exists u_* \in \mathrm{supp}^+(x - y),\\ \forall v \in \mathrm{supp}^-(x - y) :\ f(x) + f(y) < f(x - \chi_{u_*} + \chi_v) + f(y + \chi_{u_*} - \chi_v)\}.\end{aligned}$$

We assume $\mathcal{D} \neq \emptyset$ and derive a contradiction.

Let (x, y) be the element in $\mathcal{D}$ which minimizes the value $||x - y||$, and $u_* \in \mathrm{supp}^+(x - y)$ satisfy the condition for (x, y) to be in $\mathcal{D}$. Using $\varepsilon(> 0)$, we set $p \in \mathbf{R}^V$ as follows:

$$p(v) = \begin{cases} f(x) - f(x - \chi_{u_*} + \chi_v) & (v \in \mathrm{supp}^-(x-y),\ x - \chi_{u_*} + \chi_v \in \mathrm{dom}\, f), \\ f(y + \chi_{u_*} - \chi_v) - f(y) - \varepsilon & (v \in \mathrm{supp}^-(x-y),\ x - \chi_{u_*} + \chi_v \notin \mathrm{dom}\, f, \\ & \qquad y + \chi_{u_*} - \chi_v \in \mathrm{dom}\, f), \\ 0 & (\text{otherwise}). \end{cases}$$

Define $f_p(x) = f(x) + \sum\{p(w)x(w) \mid w \in V\}$ $(\forall x \in \mathbf{Z}^V)$.

Claim 1

$$f_p(x - \chi_{u_*} + \chi_v) = f_p(x) \qquad (v \in \mathrm{supp}^-(x-y),\ x - \chi_{u_*} + \chi_v \in \mathrm{dom}\, f), \tag{3}$$

$$f_p(y + \chi_{u_*} - \chi_v) > f_p(y) \qquad (v \in \mathrm{supp}^-(x-y)). \tag{4}$$

Suppose that $u_1 \in \mathrm{supp}^+(x-y)$, $v_1 \in \mathrm{supp}^-(x-y)$ satisfy

$$f_p(y + \chi_{u_1} - \chi_{v_1}) = \min_{\substack{u \in \mathrm{supp}^+(x-y) \\ v \in \mathrm{supp}^-(x-y)}} f_p(y + \chi_u - \chi_v). \tag{5}$$

Lemma 3.5 yields that $f_p(y + \chi_{u_1} - \chi_{v_1}) < +\infty$. Put $y' = y + \chi_{u_1} - \chi_{v_1}$.

Claim 2 $(x, y') \in \mathcal{D}$.

Proof. We have only to show that

$$f_p(x) + f_p(y') < f_p(x - \chi_{u_*} + \chi_v) + f_p(y' + \chi_{u_*} - \chi_v) \tag{6}$$

for each $v \in \mathrm{supp}^-(x - y')$. We can assume that $x - \chi_{u_*} + \chi_v \in \mathrm{dom}\, f$, which implies $f_p(x) = f_p(x - \chi_{u_*} + \chi_v)$ by (3) and the fact $v \in \mathrm{supp}^-(x - y') \subseteq \mathrm{supp}^-(x - y)$. Furthermore, it holds that

$$\begin{aligned} f_p(y' + \chi_{u_*} - \chi_v) &= f_p(y + \chi_{u_1} + \chi_{u_*} - \chi_{v_1} - \chi_v) + f_p(y) - f_p(y) \\ &\geq \min\{f_p(y + \chi_{u_1} - \chi_{v_1}) + f_p(y + \chi_{u_*} - \chi_v), \\ &\qquad f_p(y + \chi_{u_1} - \chi_v) + f_p(y + \chi_{u_*} - \chi_{v_1})\} - f_p(y) \qquad (\text{by Lemma 3.4}) \\ &\geq f_p(y') + \min\{f_p(y + \chi_{u^*} - \chi_v) - f_p(y),\ f_p(y + \chi_{u_*} - \chi_{v_1}) - f_p(y)\} \quad (\text{by (5)}) \\ &> f_p(y') \qquad (\text{by (4)}), \end{aligned}$$

which implies the inequality (6). ∎

Hence, we have $(x, y') \in \mathcal{D}$, and $||x - y'|| = ||x - y|| - 2$, which contradicts the selection of (x, y).

∎

4 Greedily Solvable Layer Structure

Suppose we are given a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. This section assumes that f satisfies (MG-EXC) unless otherwise stated explicitly. We discuss the layer structure of f, which is the restriction of f to $\{x \in \mathbf{Z}^V \mid x(V) = k\}$, and the following optimization problem in each layer $(k \in \mathbf{Z})$:

$$\text{minimize } f(x) \text{ subject to } x(V) = k.$$

Set $\lambda = \min\{x(V) \mid f(x) < +\infty\}$ and $\mu = \max\{x(V) \mid f(x) < +\infty\}$. For any integer k, define a function $f_k : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ as $f_k(x) = f(x)$ if $x(V) = k$, and $= +\infty$ otherwise. The following, a corollary of Theorem 3.2, shows that each layer has a nice structure.

Theorem 4.1 *f_k satisfies (MB-EXC) $(\lambda \leq \forall k \leq \mu)$.*

We can find a minimizer in each layer greedily by the following algorithm.
Exchanging Algorithm /* for minimization of an M-convex function on a base polyhedron */
STEP 0: Let x be any element in $\operatorname{dom} f$. Set $V^- = V$.
STEP 1: If $V^- = \emptyset$ then stop.
STEP 2: Choose any $u \in V^-$, and find $v \in V$ such that $f(x - \chi_u + \chi_v) = \min\{f(x - \chi_u + \chi_w) \mid w \in V\}$.
STEP 3: Set $x = x - \chi_u + \chi_v$, and if $v \in V^-$, set $V^- = V^- - \{v\}$. Go to STEP 1.

Note that with a slight modification, this algorithm also applies to global optimization for M-convex functions on g-polymatroids. The next lemma validates the exchanging algorithm.

Lemma 4.2 *Suppose $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies (MB-EXC). Given $x \in \operatorname{dom} f$ and $u \in V$, let $v \in V$ be such that $f(x - \chi_u + \chi_v) = \min\{f(x - \chi_u + \chi_w) \mid w \in V\}$.*
(i) If $v \neq u$, there exists $x^* \in \arg\min f$ with $x^*(v) > x(v)$.
(ii) If $v = u$, there exists $x^* \in \arg\min f$ with $x^*(v) \geq x(v)$.

Proof. We prove the first claim only. The second claim can be proved in a similar way. To the contrary suppose there is no $x^* \in \arg\min f$ with $x^*(v) > x(v)$, and let $x^* \in \arg\min f$ with the maximum value of $x^*(v)$. Then we have $v \in \operatorname{supp}^+((x - \chi_u + \chi_v) - x^*)$. By (MB-EXC), there exists $w \in \operatorname{supp}^-((x - \chi_u + \chi_v) - x^*)$ such that

$$f(x - \chi_u + \chi_v) + f(x^*) \geq f(x - \chi_u + \chi_w) + f(x^* + \chi_v - \chi_w).$$

The assumption for v and the fact $x^* \in \arg\min f$ imply $f(x^* + \chi_v - \chi_w) = f(x^*)$. However, it is a contradiction since $(x^* + \chi_v - \chi_w)(v) = x^*(v) + 1$. ∎

We propose different approaches for optimization in a layer, which use the following properties of the relationship between consecutive layers. For any integer k $(\lambda \leq k \leq \mu)$, define $\alpha_k^* = \min\{f(x) \mid x(V) = k\}$ and $M_k = \{x \in \mathbf{Z}^V \mid x(V) = k,\ f(x) = \alpha_k^*\}$.

Theorem 4.3 *(i) Let $x_k^* \in M_k$ ($\lambda \leq k \leq \mu - 1$), and $v \in V$ be such that $f(x_k^* + \chi_v) = \min\{f(x_k^* + \chi_w) \mid w \in V\}$. Then $x_k^* + \chi_v \in M_{k+1}$.*
(ii) Let $x_k^* \in M_k$ ($\lambda + 1 \leq k \leq \mu$) and $u \in V$ be such that $f(x_k^* - \chi_u) = \min\{f(x_k^* - \chi_w) \mid w \in V\}$. Then $x_k^* - \chi_u \in M_{k-1}$.

Proof. For (i) it suffices to show that $||y^* - x_k^*|| = 1$ holds for some $y^* \in M_{k+1}$. Let $y \in M_{k+1}$ with $||y - x_k^*|| > 1$. Note that $\text{supp}^+(y - x_k^*) \neq \emptyset$. For $u \in \text{supp}^+(y - x_k^*)$, the property (MG-EXC) yields either (a) or (b), where

(a) $f(y) + f(x_k^*) \geq f(y - \chi_u) + f(x_k^* + \chi_u)$,

(b) $f(y) + f(x_k^*) \geq f(y - \chi_u + \chi_v) + f(x_k^* + \chi_u - \chi_v) \quad (\exists v \in \text{supp}^-(y - x_k^*))$.

Since $x_k^* \in M_k$, $y \in M_{k+1}$, we have $y' = x_k^* + \chi_u \in M_{k+1}$ if (a) holds, and $y' = y - \chi_u + \chi_v \in M_{k+1}$ if (b) holds. In either case, we obtain $y' \in M_{k+1}$ with $||y' - x_k^*|| < ||y - x_k^*||$. By repeating this procedure, we can find a desired y^*. The proof of (ii) is similar. ∎

This property naturally yields the next algorithm:

Augmenting Algorithm

STEP 0: Find any $x_\lambda^* \in M_\lambda$. Set $k = \lambda$.
STEP 1: If $k = \mu$ then stop.
STEP 2: Find $v_k \in V$ such that $f(x_k^* + \chi_{v_k}) = \min\{f(x_k^* + \chi_w) \mid w \in V\}$.
STEP 3: Set $x_{k+1}^* = x_k^* + \chi_{v_k}$, $k = k + 1$. Go to STEP 1.

The exchanging algorithm can be used in STEP 0 of this algorithm. A reducing algorithm, which iteratively reduces k, can be constructed similarly. These algorithms work well if we can find an element $x_\lambda^* \in M_\lambda$ or $x_\mu^* \in M_\mu$ efficiently, in particular if $|\text{dom}\, f_\lambda| = 1$ or $|\text{dom}\, f_\mu| = 1$.

The next theorem shows the convexity of the sequence α_k^*.

Theorem 4.4 $\alpha_{k-1}^* + \alpha_{k+1}^* \geq 2\alpha_k^* \quad (\lambda + 1 \leq \forall k \leq \mu - 1)$.

Proof. By Theorem 4.3, there exist $x_{k-1}^* \in M_{k-1}$, $x_{k+1}^* \in M_{k+1}$ such that $x_{k-1}^* \leq x_{k+1}^*$. Apply (MG-EXC) to x_{k+1}^*, x_{k-1}^* and any $u \in \text{supp}^+(x_{k+1}^* - x_{k-1}^*)$ to obtain $f(x_{k+1}^*) + f(x_{k-1}^*) \geq f(x_{k+1}^* - \chi_u) + f(x_{k-1}^* + \chi_u) \geq 2\alpha_k^*$. Note that $\text{supp}^-(x_{k+1}^* - x_{k-1}^*) = \emptyset$. ∎

Therefore, we can use the augmenting algorithm for finding a global minimum, where we can stop the algorithm when k satisfies the condition $\alpha_{k+1}^* \geq \alpha_k^*$. As an immediate corollary of this theorem, we have $\{x \in \mathbf{Z}^V \mid x(V) = k,\ f(x) < +\infty\} \neq \emptyset$ $(\lambda \leq \forall k \leq \mu)$.

Finally, we mention that the local minimality characterizes a global minimum of an M-convex function on a g-polymatroid. This follows easily from the corresponding result [17, 18] for an M-convex function on a base polyhedron.

Theorem 4.5 *Suppose $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies (MG-EXC) and let $x \in \text{dom}\, f$. Then $f(x) \leq f(y)$ for any $y \in \mathbf{Z}^V$ if and only if*

$$f(x) \leq \min\left[\min_{u,v \in V} f(x - \chi_u + \chi_v), \min_{u \in V} f(x - \chi_u), \min_{v \in V} f(x + \chi_v)\right].$$

5 Concluding Remarks

Remark 5.1 Most properties of M-convex functions on base polyhedra [14, 17, 18, 19] extend to M-convex functions on g-polymatroids, according to its definition. For example,

- an M-convex function on a g-polymatroid is characterized by minimizers,
- M-convexity on g-polymatroids is preserved by addition of a linear function, translation, and negation of the argument,
- an M-convex function on a g-polymatroid can be extended to a convex function,
- convolution and network induction work,
- an intersection theorem, a Fenchel-type duality, and a discrete separation theorem hold.

Remark 5.2 As a corollary of Theorem 3.2, g-polymatroids are characterized by a simultaneous exchange property:

(G-EXC) $\forall x, y \in Q$, $\forall u \in \mathrm{supp}^+(x-y)$, either (i) or (ii) holds, where
(i) $x - \chi_u \in Q$ and $y + \chi_u \in Q$,
(ii) $x - \chi_u + \chi_v \in Q$ and $y + \chi_u - \chi_v \in Q$ $(\exists v \in \mathrm{supp}^-(x-y))$.

In fact, the axiom (MG-EXC) comes from this characterization. Alternatively, g-polymatroids are characterized by another exchange property:

(G-EXC$_0$) $\forall x, y \in Q$, $\forall u \in \mathrm{supp}^+(x-y)$, both (i) and (ii) hold, where
(i) either $x - \chi_u \in Q$, or $x - \chi_u + \chi_v \in Q$ $(\exists v \in \mathrm{supp}^-(x-y))$,
(ii) either $y + \chi_u \in Q$, or $y + \chi_u - \chi_w \in Q$ $(\exists w \in \mathrm{supp}^-(x-y))$,

which is a straightforward extension of the one for g-matroids due to Tardos [22]. This axiom, however, is not suitable for a quantitative generalization.

Remark 5.3 Suppose that we are given a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with (MB-EXC) and a specified subset $W \subseteq V$. Set $\lambda = \min\{x(W) \mid f(x) < +\infty\}$, $\mu = \max\{x(W) \mid f(x) < +\infty\}$, $\alpha_k^* = \min\{f(x) \mid x(W) = k\}$, and $M_k = \{x \in \mathbf{Z}^V \mid x(W) = k,\ f(x) = \alpha_k^*\}$. Then, $\alpha_{k-1}^* + \alpha_{k+1}^* \geq 2\alpha_k^*$ $\ (\lambda + 1 \leq \forall k \leq \mu - 1)$ as in Theorem 4.4, and Theorem 4.3 can be generalized as follows:

Theorem 5.1 *Let $x_k^* \in M_k$ $(\lambda \leq k \leq \mu - 1)$, and $u \in V - W$, $v \in W$ be such that $f(x_k^* - \chi_u + \chi_v) = \min\{f(x_k^* - \chi_s + \chi_t) \mid s \in V - W,\ t \in W\}$. Then $x_k^* - \chi_u + \chi_v \in M_{k+1}$.*

Theorems 4.3 and 4.4 are the translation by projection of these results when $|W| = 1$. Note that the similar properties of valuated bimatroid in [13] are also the special cases of the above results.

Remark 5.4 Several researchers considered discrete analogy of convex function, e.g., Lovász [12], and Favati and Tardella [6]; the latter investigated a class of discrete functions such that local minimality leads to global minimality.

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