数理解析研究所講究録
983 巻 1997 年 150-155

A Remark on Finiteness and Duality of D-Modules

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30 September 1996

The purpose of this paper is to prove a theorem on finite dimensionality of the cohomology groups of analytic differential complexes on compact real manifolds. This generalizes the classical finiteness theorem for elliptic differential complexes.

In this paper, a manifold is always assumed to be paracompact. For sheaves and functors, we follow the notations of [KS]. For a sheaf $\mathcal{F}$ on a topological manifold X, $\Gamma(\mathcal{F})$ denotes the set of global sections of $\mathcal{F}$, and $\Gamma_c(\mathcal{F})$ of global sections with compact support. $\mathrm{R}\Gamma$ and $\mathrm{R}\Gamma_c$ denote their right derived functors.

1. Main Result

Let X be a complex manifold, $n = \dim_{\mathbf{C}} X$. Let $\mathcal{O}$ denote the sheaf of holomorphic functions on X, Ω^n the sheaf of holomorphic n-forms on X, and $\mathcal{D}$ the sheaf of rings of differential operators on X (of finite order).

Let T^*X denote the cotangent bundle of X.

For a coherent right $\mathcal{D}$-module $\mathcal{M}$ on X, $\mathrm{Ch}(\mathcal{M})$ denotes its characteristic variety; $\mathrm{Ch}(\mathcal{M})$ is a $\mathbf{C}^\times$-invariant closed analytic subset of T^*X and $\dim(\mathrm{Ch}(\mathcal{M})) \geq n$ (see [SKK]). Let $\mathrm{Mod}(\mathcal{D}^\circ)$ be the abelian category of right $\mathcal{D}$-modules, $\mathrm{D^b}(\mathcal{D}^\circ)$ its derived category with bounded cohomology. Let $\mathrm{D^b_{g,coh}}(\mathcal{D}^\circ)$ be the full triangulated subcategory of $\mathrm{D^b}(\mathcal{D}^\circ)$ consisting of bounded complexes with good coherent cohomology groups. [We say that a $\mathcal{D}$-module $\mathcal{M}$ is good coherent if, for any relatively compact open subset U of X, there exists a finite filtration G of $\mathcal{M}|_U$ by $\mathcal{D}$-modules such that

[Sai, 1.15].] For an object $\mathcal{M}^\bullet$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{g,coh}}(\mathcal{D}^\circ)$, $\mathrm{Ch}(\mathcal{M}^\bullet)$ denotes the union of $\mathrm{Ch}(H^k\mathcal{M}^\bullet)$, $k \in \mathbf{Z}$.

Let $\mathrm{D}^{\mathrm{b}}(X)$ denote the derived category with bounded cohomology of the abelian category of $\mathbf{C}_X$-modules, $\mathrm{D}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ consisting of $\mathbf{R}$-constructible objects [KS, Sect.8.4]. For an object F of $\mathrm{D}^{\mathrm{b}}(X)$, $\mathrm{SS}(F)$ denotes the micro-support of F [KS, Sect.5.1]. If F is $\mathbf{R}$-constructible, $\mathrm{SS}(F)$ is then an $\mathbf{R}_+$-invariant closed subanalytic subset of T^*X. (But we do not need this fact in this paper.)

For an object $(\mathcal{M}^\bullet, F)$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{g,coh}}(\mathcal{D}^\circ) \times \mathrm{D}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(X)$, $\mathcal{M}^\bullet \otimes F$ denotes the tensor product over $\mathbf{C}$ and is an object of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}^\circ)$.

Theorem 1. *Let $(\mathcal{M}^\bullet, F)$ be an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{g,coh}}(\mathcal{D}^\circ) \times \mathrm{D}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(X)$. Assume that, for any irreducible component V of $\mathrm{Ch}(\mathcal{M}^\bullet)$, $V \cap \mathrm{SS}(F)$ is contained in the zero section T^*_XX if $\dim V \neq n$. Suppose $\mathrm{Supp}(\mathcal{M}^\bullet) \cap \mathrm{Supp}(F)$ is compact. Then every cohomology group of $\mathrm{R}\Gamma(\mathcal{M}^\bullet \otimes F \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{O})$ and $\mathrm{RHom}_{\mathcal{D}}(X; \mathcal{M}^\bullet \otimes F, \Omega^n)$ is finite dimensional and*

$$(1.0) \quad \mathrm{RHom}_{\mathcal{D}}(X; \mathcal{M}^\bullet \otimes F, \Omega^n)[n] \longrightarrow \mathrm{Hom}_{\mathbf{C}}(\mathrm{R}\Gamma(\mathcal{M}^\bullet \otimes F \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{O}), \mathbf{C})$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{C})$. Hence, for any $k \in \mathbf{Z}$,

$$\mathrm{Tor}^{\mathcal{D}}_k(\mathcal{M}^\bullet \otimes F, \mathcal{O}) \qquad \text{and} \qquad \mathrm{Ext}^{k+n}_{\mathcal{D}}(X; \mathcal{M}^\bullet \otimes F, \Omega^n)$$

are vector spaces of finite dimension and dual to each other.

Remark. We say that $(\mathcal{M}^\bullet, F)$ is an elliptic pair if $\mathrm{Ch}(\mathcal{M}^\bullet) \cap \mathrm{SS}(F) \subset T^*_XX$ [SS]. In that case, Theorem 1 is proved in [SS]. On the other hand, if $\mathcal{M}^\bullet$ is holonomic, $(\mathcal{M}^\bullet, F)$ satisfies the hypothesis of Theorem 1 for any object F of $\mathrm{D}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(X)$.

Let M be a real analytic manifold of dimension n, X a complex neighborhood of M. Let T^*_MX denote the conormal bundle of M. $\mathcal{A}_M$ denotes the sheaf of real analytic functions on M, and $\mathcal{B}_M$ of hyperfunctions; $\mathcal{A}_M$ and $\mathcal{B}_M$ are $\mathcal{D}|_M$-modules. Let

$$\mathcal{B}^{(n)}_M = \mathcal{B}_M \otimes_{\mathcal{A}} (\Omega^n \otimes \mathrm{or}_{M/X}),$$

where $\mathrm{or}_{M/X}$ is the relative orientation sheaf of M in X; $\mathcal{B}^{(n)}_M$ is a right $\mathcal{D}|_M$-module.

As an immediate corollary of Theorem 1, we have the following finiteness and duality theorem of analytic differential complexes on compact real manifolds.

Corollary 2. *Let M be a compact real analytic manifold of dimension n. Let $\mathcal{M}^\bullet$ be an object of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{g,coh}}(\mathcal{D}^\circ)$. *Assume that, for any irreducible component V of* $\mathrm{Ch}(\mathcal{M}^\bullet)$, *$V \cap T^*_M X$ is contained in the zero section if* $\dim V \neq n$. *Then, for any $k \in \mathbf{Z}$,* $\mathrm{Tor}^{\mathcal{D}}_k(\mathcal{M}^\bullet, \mathcal{A}_M)$ *and* $\mathrm{Ext}^k_{\mathcal{D}}(M;\, \mathcal{M}^\bullet, \mathcal{B}^{(n)}_M)$ *are vector spaces of finite dimension and dual to each other.*

Let E^k, $0 \le k \le k_0$, be holomorphic vector bundles over X and let

$$\mathcal{O}(E^0) \xrightarrow{L_0} \mathcal{O}(E^1) \xrightarrow{L_1} \cdots \longrightarrow \mathcal{O}(E^{k_0}) \tag{1.1}$$

be a differential complex of vector bundles, where L_k is a differential operator mapping $\Gamma(\mathcal{O}(E^k))$ to $\Gamma(\mathcal{O}(E^{k+1}))$. [For a holomorphic vector bundle E, $\mathcal{O}(E)$ denotes the sheaf of holomorphic sections of E.]

Let $\mathcal{M}^k = \mathcal{O}(E^k) \otimes_{\mathcal{O}} \mathcal{D}$ and

$$\mathcal{M}^\bullet = \left[0 \longrightarrow \mathcal{M}^0 \xrightarrow{L_0} \mathcal{M}^1 \xrightarrow{L_1} \cdots \longrightarrow \mathcal{M}^{k_0} \longrightarrow 0\right], \tag{1.2}$$

where L_k acts on $\mathcal{M}^k$ by left multiplication; $\mathcal{M}^\bullet$ is then an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{g,coh}}(\mathcal{D}^\circ)$. Then $\mathrm{R}\Gamma(\mathcal{M}^\bullet \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{A}_M)$ is represented by a differential complex

$$0 \longrightarrow \Gamma(M,\, E^0) \xrightarrow{L_0} \Gamma(M,\, E^1) \xrightarrow{L_1} \cdots \longrightarrow \Gamma(M,\, E^{k_0}) \longrightarrow 0 \tag{1.3}$$

and $\mathrm{Tor}^{\mathcal{D}}_{-k}(\mathcal{M}^\bullet, \mathcal{A}_M)$ is its k-th cohomology group, where $\Gamma(M,\, E^k)$ denotes the space of analytic sections of E^k on M. For a vector bundle E, let us set $\mathcal{B}^{(n)}(E) = \mathcal{O}(E) \otimes_{\mathcal{O}} \mathcal{B}^{(n)}_M$. $\mathrm{RHom}_{\mathcal{D}}(M;\, \mathcal{M}^\bullet, \mathcal{B}^{(n)}_M)$ is represented by

$$0 \longleftarrow \Gamma(M,\, \mathcal{B}^{(n)}(E^*_0)) \xleftarrow{L_0} \Gamma(M,\, \mathcal{B}^{(n)}(E^*_1)) \xleftarrow{L_1} \cdots \longleftarrow \Gamma(M,\, \mathcal{B}^{(n)}(E^*_{k_0})) \longleftarrow 0,$$

where E^*_k is the dual bundle of E^k and L_k acts on $\mathcal{B}^{(n)}(E^*_k)$ by right multiplication; $\mathrm{Ext}^{-k}_{\mathcal{D}}(M;\, \mathcal{M}^\bullet, \mathcal{B}^{(n)}_M)$ is its k-th homology group. The pairing of $\mathrm{Tor}^{\mathcal{D}}_{-k}(\mathcal{M}^\bullet, \mathcal{A}_M)$ and $\mathrm{Ext}^{-k}_{\mathcal{D}}(M;\, \mathcal{M}^\bullet, \mathcal{B}^{(n)}_M)$ is induced from

$$\Gamma(M,\, E^k) \times \Gamma(M,\, \mathcal{B}^{(n)}(E^*_k)) \to \mathbf{C}, \qquad (u,\, v) \mapsto \int_M \langle u,\, v\rangle,$$

$\langle u,\, v\rangle$ being the pairing of E^k and E^*_k.

Remark. If (1.3) is an elliptic complex of vector bundles on M, for $\mathcal{M}^\bullet$ given by (1.2), $\mathrm{Ch}(\mathcal{M}^\bullet) \cap T^*_M X$ is contained in the zero section. The converse is not true in general.

2. Proof of Theorem 1

We can assume that $H^k\mathcal{M}^\bullet = 0$ for any $k \neq 0$; in what follows, $\mathcal{M}$ denotes a coherent right $\mathcal{D}$-module on X.

Let $\mathcal{M}^* = \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$; then $\mathcal{M}^*$ is a holonomic left $\mathcal{D}$-module, and we have an injective $\mathcal{D}$ homomorphism $\mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}^*, \mathcal{D}) \to \mathcal{M}$. Let $\mathcal{M}^{**} = \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}^*, \mathcal{D})$, and $\mathcal{N} = \mathcal{M}/\mathcal{M}^{**}$; then $\mathcal{M}^{**}$ is a holonomic $\mathcal{D}$-module, and the sequence

$$(2.0) \qquad 0 \longrightarrow \mathcal{M}^{**} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$$

is exact. Since $\mathcal{E}xt^{n-1}_{\mathcal{D}}(\mathcal{M}^{**}, \mathcal{D}) = 0$ and $\mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \to \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}^{**}, \mathcal{D})$ is an isomorphism, we see that $\mathcal{E}xt^n_{\mathcal{D}}(\mathcal{N}, \mathcal{D}) = 0$. Hence, by [K2, 2.11], $\mathrm{Ch}(\mathcal{N})$ has no irreducible components of codimension n. Since $\mathrm{Ch}(\mathcal{N}) \subset \mathrm{Ch}(\mathcal{M})$, by the hypothesis of the theorem, $\mathrm{Ch}(\mathcal{N}) \cap \mathrm{SS}(F)$ is contained in the zero section; therefore $(\mathcal{N}, F)$ is elliptic in the sense of [SS]. Moreover, by the definition of $\mathcal{N}$, if $\mathcal{M}$ is a good coherent $\mathcal{D}$-module, $\mathcal{N}$ is also good coherent.

Since Theorem 1 is proved for elliptic pairs in [SS, Part 1], by exact sequence (2.0), we may assume from the beginning $\mathcal{M}$ to be holonomic. If $\mathcal{M}$ is holonomic, by Kashiwara's theorem [K1], $\mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{O}$ is $\mathbf{C}$-constructible. Hence $(\mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{O}) \otimes F$ is an $\mathbf{R}$-constructible sheaf on X. Its support being compact by assumption, by [KS, Prop.8.4.8], $H^k\mathrm{R}\Gamma((\mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{O}) \otimes F)$ is finite dimensional for all $k \in \mathbf{Z}$. In the same way, the $\mathbf{C}$-constructibility of $\mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega^n)$ yields the finite dimensionality of $H^k\mathrm{R}\Gamma(\mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n))$. This completes the proof of the finiteness part.

We now prove (1.0) to be an isomorphism for a holonomic $\mathcal{D}$-module $\mathcal{M}$, assuming $\mathrm{Supp}(\mathcal{M}) \cap \mathrm{Supp}(F)$ is compact. Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^\circ)$ denote the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}^\circ)$ consisting of bounded complexes with holonomic cohomology groups. Letting $\mathrm{DR}(\mathcal{M}) = \mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{O}[-n]$ for an object $\mathcal{M}$ of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}^\circ)$, we have first :

Lemma 2.1. *Let $\mathcal{M}$ be an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^\circ)$, F of $\mathrm{D}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(X)$. Then there is an isomorphism*

$$(2.1) \qquad \mathrm{DR}(\mathcal{M}) \otimes F \cong \mathrm{D}'\, \mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n),$$

where $\mathrm{D}' = \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\bullet, \mathbf{C}_X)$.

The proof will be given later.

Since $\mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n)$ is $\mathbf{R}$-constructible, from (2.1), we have

$$\mathrm{D}'(\mathrm{DR}(\mathcal{M}) \otimes F) \cong \mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n).$$

By the Verdier duality, we get

$$\begin{aligned}\mathrm{R}\Gamma\,(\mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n))\,[n] &\cong \mathrm{RHom}_{\mathbf{C}}(\mathrm{R}\Gamma_c(\mathrm{DR}(\mathcal{M}) \otimes F), \mathbf{C})[-n] \\ &= \mathrm{RHom}_{\mathbf{C}}(\mathrm{R}\Gamma_c(\mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{D}} \mathcal{O} \otimes F), \mathbf{C}).\end{aligned}$$

This completes the proof of Theorem 1. QED

Proof of Lemma 2.1. If $F = \mathbf{C}_X$, this duality formula is contained in [KK] and [M2], and we have

$$\mathrm{DR}(\mathcal{M}) \cong \mathrm{D}'\,\mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega^n). \tag{2.2}$$

Let $C = \mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega^n)$; then, by Kashiwara's theorem [K1], C is $\mathbf{C}$-constructible. Hence

$$\mathrm{D}'(\mathrm{D}'\,C \otimes F) \cong \mathrm{R}\mathcal{H}om_{\mathbf{C}}(F, C)$$

(see [KS, 3.4.6]), and, since $\mathrm{D}'\,C \otimes F$ is $\mathbf{R}$-constructible,

$$\mathrm{D}'\,C \otimes F \cong \mathrm{D}'\,\mathrm{R}\mathcal{H}om_{\mathbf{C}}(F, C).$$

By (2.2), we get

$$\mathrm{DR}(\mathcal{M}) \otimes F \cong \mathrm{D}'\,\mathrm{R}\mathcal{H}om_{\mathbf{C}}(F, C).$$

QED

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