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A Weak Version of Poincaré-Bendixson Type Theorems for n Dimensional Spaces

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Abstract

The n-dimensional autonomous systems are discussed under some set of geometric conditions. It is shown that there exists a solution whose orbit stays on the boundary of a certain set for the above systems. This is a weaker version of a Poincaré-Bendixson type theorem. The method can also be applied to nonlinear difference equation systems.

Keywords: Autonomous system, Poincaré-Benedixson theorem, Periodic solution

1 Introduction

A well known theorem on the existence of periodic solutions due to Poincaré [?] and Bendixson [?] was established for the spaces of dimension two. Various applications for the systems on the plane thus include a set of conditions which are proved to be sufficient to satisfy the prerequisites for Poincaré-Bendixson theorem. The same has been true for the nonlinear second order differential equations.

Rauch [?] considered a third order autonomous equation which can be decomposed into a linear and a nonlinear part of special type. This result was extended by Williamson [?] into n-th order equations using the 'torus principle' (Pliss [?]). Cronin [?] took up an n-dimensional nonautonomous system whose nonautonomous part is a function of a fixed period, and showed the existence of a solution of the same fixed period under a set of assumptions. As she noted, these assumptions remind us of a popular variant of Poincaré-Bendixson theorem: if a given autonomous system on the plane has an unstable critical point, and if there is a simple closed curve C containing this unstable critical point but no other critical point in its interior or on

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C, and if every trajectory which intersects C crosses C inwardly, then the system has a periodic solution.

In this note, we consider n-dimensional autonomous systems which share the above geometric characteristic, and show that there exists a solution whose orbit stays on the boundary of a certain set. Some comments are made concerning the case of dimension less than four.

2 Assumptions and The Main Theorem

Let us consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where x is an n-vector in the n-dimensional Euclidean space R^n. We make the following assumptions.

Assumption 1. The system (1) has a unique solution when the initial vector $x(0)$ is given in a domain D which is homeomorphic to an open ball.

Assumption 2. Each solution starting in D is continuous (most probably differentiable in many applications) with respect to x and the initial point, and each orbit starting from a point on ∂D remains on the boundary or leaves the boundary inwardly, and is defined for all $t \geq 0$, while the negative semi-orbit starting from a point on ∂D never enters into the interior of D.

Assumption 3. There exists an open set S in D which is also homeomorphic to an open ball and each orbit starting from a point on ∂S(the boundary of S) either remains on the boundary or leaves the boundary outwardly.

Denote by $x(t, x_0)$ the solution with the initial vector x_0 in D, and fix a positive scalar T. And define a mapping F_T on D from x_0 to $x(T, x_0)$. This mapping F_T is a homeomorphism from D to $F_T(D)$, and satisfies

$$D \supset F_T(D) \supset F_T(S) \supset S.$$

We define $F_T^2(D) \equiv F_T(F_T(D))$, and similarly the symbol $F_T^i(D)$ for the i-th iterate of the transformation F_T. And we put $F_T^0(D) \equiv D$.

Now we have

Main Theorem. *Given the above assumptions* 1-3, *there exists a solution which remains on the boundary of a certain open set in* D.

proof. Let $\overline{X}$ denote the closure of a set X, and consider a sequence of closed sets $\overline{D} \supset \overline{F_T(D)} \supset \overline{F_T^2(D)} \supset \cdots \supset \overline{F_T^i(D)} \supset \cdots$, each of which contains the open set S. Put $L \equiv \bigcap_{i=1}^{\infty} \overline{F_T^i(D)}$. The set L is closed and contains S. Let L° be the interior of L.

We now prove that each orbit starting from a point ∂L° remains there. First, we pick up an arbitrary point x on ∂L°, and show that $F_T(x) \in \partial L^\circ$. If $F_T(x) \in L^\circ$, then there is a neighbourhood of x which is mapped into L° by F_T and contains an exterior point of $F_T^i(D)$ for some i (i may be 0) because of Assumption 2. (Fig.1). This contradicts the definition of L. On the other hand, if $F_T(x) \notin \overline{L^\circ}$, some points of L° are mapped into the outside of $\overline{L^\circ}$, contradicting the monotonicity of the sequence $\{F_T^i(D)\}_{i=1,2,\cdots}$.

Next we prove that even if we change the designated scalar T to another T', we end up with the same set L. Suppose that these sets are distinct, and without loss of generality suppose also that $T' < T$ and a point x on ∂L° is in the interior of L', L'°, where L' is the 'limit' set corresponding to T'. (Note that the word 'limit' set here is used in the set-theoretical meaning.) Then, within L'° there exists a neighbourhood of x which contains a point P on an orbit which starts from a boundary point Q of D such that $F_T^i(Q) = P$ for some i. Now by Assumption2, we can find a point Q' on this orbit which is either an exterior point of D or on the boundary of D such that $F_{T'}^j(Q') = P$ for some j. This implies L'° cannot be a 'limit' set of monotone shrinking sequence . A contradiction.

Thus we have shown that each orbit starting from a point on ∂L° remains there. This completes the proof. ∎

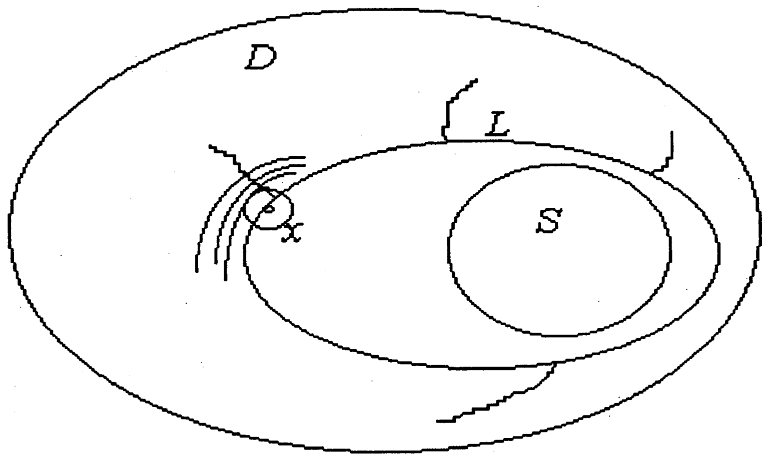

Fig. 1

3 Dimensions 2 and 3

Let us now turn to the case of dimension two, and consider

Assumption 4. There exists no critical point on the set $D - S$.

The following is a variant of Poincaré-Bendixson theorem.

Theorem. *When the dimension is two, given Assumptions* 1-4, *the system (1) has a periodic solution.*

We wished to prove the above theorem in a simple and elementary way without using the closed curve theorem due to Jordan. So far we are unable to do so.

If one can show in the three dimensional case that the interior of the limit set L° is homeomorphic to an open ball, thus ∂L° is homeomorphic to the sphere S^2, then we can apply Poincaré-Bendixson theorem (or our own result) to the trajectories on the two-dimensional sphere.

4 Remarks

(1) The trouble with our result is that the boundary set ∂L° need not be homeomorphic to a hypersphere when the dimension n is greater than two. Is the interior of the limit set connected? Assumption 4 should have played some role. (Cf. Yoneyama's example in Lefschetz [5].)

(2) The fact that a solution starting from a point of the boundary ∂L° remains there has nothing to do with periodicity. (In economics, however, this property, rather than the exact periodicity, can be used in business cycle theory.)

(3) When we impose more conditions on domains D and f(e.g., convexity or star-convexity etc. and the preservation of these properties by f), the sequence $\{F_T^i(D)\}_{i=1,2,\cdots}$ may converge to L° which is homeomorphic to an open ball. (Cf. Tarski [11].)

(4) In a dual way, with suitable modifications in Assumptions 2 and 3, we can conceive of the 'limit' set of expanding sequence $\{F_T^i(D)\}_{i=1,2,\cdots}$. If two 'limit' sets coincide, the boundary ∂L° is 'stable'.

(5) We avoid the use of limit points of a semi-orbit, and the notion of limit sets in the traditional sense. (For the conventional proof, see Coddington and Levinson [2] or Lefschetz [6].) More importantly, our method, if sound enough, is equally useful to prove a proposition similar to our Main Theorem for the system of nonlinear difference equations, possibly even in Banach spaces. The first motivation of the authors was to prove the existence of a quasi-periodic solution for discrete versions of Goodwin growth cycle model. (See Goodwin [4] and Semmler(ed.) [10].)

(6) In the conference, Professor Matano gave us useful comments, particularly on the inconsistency of Assumption 4 in the case of spaces of odd dimensions.

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