数理解析研究所講究録
987 巻 1997 年 245-262

OPTIMIZATION APPROACHES TO VARIATIONAL INEQUALITY PROBLEMS[1]

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Abstract. The variational inequality problem (VIP) is to find a point $x\in S$ such that

$$\langle F(x), y-x\rangle\geq 0 \quad \text{for all } y\in S,$$

where S is a nonempty closed convex subset of $\Re^{n}$, F is a continuous mapping from $\Re^{n}$ into itself, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\Re^{n}$. An important special class of the VIP is the complementarity problem (CP), which is to find a point x such that

$$F(x)\geq 0,\ x\geq 0,\ \langle F(x), x\rangle=0.$$

The VIP and the CP have been widely used to formulate various equilibrium problems that arise in engineering, economics and operations research. Recently much effort has been made to reformulate the VIP and the CP as an equivalent optimization problem. Such reformulations turn out to be useful not only in designing a globally convergent algorithm for solving the VIP or the CP but also in analyzing the rate of convergence of an iterative method for solving those problems. This paper surveys recent developments in merit functions used to formulate equivalent optimization problems for the VIP and CP.

Key Words: Reformulation, variational inequality problem, complementarity problem, equivalent optimization problem, merit function.

1. INTRODUCTION

In the last several years, much effort has been made to reformulate the variational inequality problem (VIP) as an equivalent optimization problem with certain desirable properties, thereby designing new descent algorithms for solving the VIP and deriving error bounds which are useful in estimating how far any given point is to the solution set of the VIP. In this paper, we overview such attempts to the VIP and the complementarity problem (CP) which is an important subclass of the VIP.

The *variational inequality problem (VIP)* is a problem of finding a point $x\in S$ such that

$$\langle F(x), y-x\rangle\geq 0 \quad \text{for all } y\in S, \tag{1.1}$$

[1]This is an updated version of the author's survey paper [14]. The work of the author is supported in part by the Scientific Research Grant-in-Aid from the Ministry of Education, Science and Culture, Japan.

where S is a nonempty closed convex subset of $\Re^n$, F is a continuous mapping from $\Re^n$ into $\Re^n$, and $\langle \cdot , \cdot \rangle$ denotes the inner product in $\Re^n$. In the special case where $S = \{x \in \Re^n \,|\, x_i \geq 0, i = 1, \cdots, n\}$, the VIP (1.1) is equivalent to the following *complementarity problem (CP)*:

$$F(x) \geq 0, \;\; x \geq 0, \;\; \langle F(x), x \rangle = 0. \tag{1.2}$$

When F is an affine mapping such that $F(x) = Mx + q$ with an $n \times n$ matrix M and an n-vector q, the CP becomes the *linear complementarity problem (LCP)*:

$$Mx + q \geq 0, \;\; x \geq 0, \;\; \langle Mx + q, x \rangle = 0. \tag{1.3}$$

A comprehensive treatment of the VIP and CP may be found in [3, 18, 37, 42].

We say that the mapping $F : \Re^n \rightarrow \Re^n$ is *monotone* if

$$\langle F(x) - F(x'), x - x' \rangle \geq 0 \quad \text{for all } \; x, x' \in \Re^n,$$

strictly monotone if

$$\langle F(x) - F(x'), x - x' \rangle > 0 \quad \text{for all } \; x, x' \in \Re^n \; (x \neq x'),$$

and *strongly* (or *uniformly*) *monotone* with modulus $\mu \, (> 0)$ if

$$\langle F(x) - F(x'), x - x' \rangle > \mu \|x - x'\|^2 \quad \text{for all } \; x, x' \in \Re^n.$$

Clearly any strongly monotone mapping is strictly monotone and any strictly monotone mapping is monotone. The following relations are well known [39, Theorem 5.4.3]: Let F be continuously differentiable. Then F is monotone if and only if the Jacobian matrix $\nabla F(x)$ is positive semidefinite for all x; F is strictly monotone if $\nabla F(x)$ is positive definite for all x; and F is strongly monotone if and only if $\nabla F(x)$ is uniformly positive definite, which is equivalent to saying that the minimum eigenvalues of the symmetric matrices $\nabla F(x) + \nabla F(x)^{\top}$ are bounded away from zero on $\Re^n$, where $\top$ denotes transposition. Throughout the paper, we suppose that F is continuously differentiable everywhere.

2. REFORMULATIONS OF VIP

A real-valued function f on $\Re^n$ is called a *merit function* for the VIP (1.1) if the set of global minimizers of f on a given set $X \subset \Re^n$ coincides with the set of solutions to the VIP, where X is usually the set S involved in the VIP itself or the entire space $\Re^n$. If f is a merit function for the VIP, then the VIP can be reformulated as the following optimization problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X. \end{array} \tag{2.1}$$

Desirable properties the merit function f is expected to have may be listed as follows:

◇ f is differentiable.

- ◇ Any stationary point or local minimum of f on X is also a global minimum of f on X.
- ◇ f provides a global (respectively, local) error bound for the VIP, i.e., for any given point x, the distance from x to the solution set of the VIP can be bounded by the value of $f(x)$ multiplied by some positive constant on the whole space $\Re^n$ (respectively, on some neighborhood of the solution set).

2.1. Classical gap functions

The first merit function for the VIP is the *gap function* introduced by Auslender [2] and Hearn [19]. This function is defined by

$$g(x) = \sup\{\langle F(x), x - y\rangle \mid y \in S\}. \tag{2.2}$$

If the set S is bounded, then the function g is finite everywhere. Otherwise, there may exist a point x such that $g(x) = +\infty$.

It is not difficult to see that the gap function g has the following properties:

- ◇ $g(x) \geq 0$ for all $x \in S$.
- ◇ $g(x) = 0$ and $x \in S$ if and only if x is a solution to the VIP.

These two properties imply that the VIP (1.1) is equivalent to the optimization problem

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to} & x \in S. \end{array} \tag{2.3}$$

Hence g is a merit function for the VIP. Note that this equivalence does not require any monotonicity assumption on the mapping F.

Now let

$$\Omega(x) = \arg\max\{\langle F(x), x - y\rangle \mid y \in S\}.$$

Then the function g is differentiable at x if and only if

$$\nabla F(x)y = \nabla F(x)y' \quad \text{for all } y, y' \in \Omega(x). \tag{2.4}$$

If (2.4) holds, the gradient of g at x is actually given by

$$\nabla g(x) = F(x) - \nabla F(x)(y - x)$$

for any $y \in \Omega(x)$. Clearly, the condition (2.4) is satisfied whenever the set $\Omega(x)$ is a singleton. As pointed out by Auslender [2, p. 139], the latter condition is met when the set S is "strongly" convex in the sense that, for all $x, x' \in S$ such that $x \neq x'$ and for all $\lambda \in (0,1)$, there exists a constant $r > 0$ such that

$$\| \lambda x + (1-\lambda)x' - z \| \leq r \quad \Longrightarrow \quad z \in S.$$

Unfortunately, the strong convexity of S is too restrictive a condition to be met in many practical applications. For instance, it is never satisfied by a polyhedral convex set. Thus we cannot in general expect that the gap function g is differentiable everywhere.

Since the gap function g yields an optimization problem equivalent to the VIP, it is quite natural to try to solve the VIP by iteratively decreasing the function g. Under the monotonicity assumption on F, Hearn [19] and Marcotte [33] claim that, if $\nabla g(x)$ exists and $g(x) > 0$, then the vector $d = y - x$ with $y \in \Omega(x)$ is a descent direction of g at x. However, since g is hardly differentiable everywhere as mentioned above, the Frank-Wolfe method that uses d and g as the search direction and the merit function, respectively, does not seem to be applicable to a general VIP. Taking into account the nondifferentiability of g, Marcotte [33] proposed a bundle-type descent algorithm for the gap function g and proved its convergence under monotonicity assumption on F. On the other hand, Marcotte and Dussault [34] proposed using the gap function g as a merit function to globalize the Newton method that solves at each iteration the following linearized VIP: Given x^k, find a point $z \in S$ such that

$$\langle \bar{F}(z, x^k), y - z \rangle \geq 0 \quad \text{for all } y \in S, \tag{2.5}$$

where $\bar{F}(\cdot, x^k) : \Re^n \to \Re^n$ is the affine mapping defined by

$$\bar{F}(z, x^k) = F(x^k) + \nabla F(x^k)^\top (z - x^k). \tag{2.6}$$

Assuming the monotonicity of F and the compactness of S, Marcotte and Dussault [34] showed that the vector $d^k = \bar{z}(x^k) - x^k$ with $\bar{z}(x^k)$ being a solution to the linearized VIP (2.5) is a feasible descent direction of g at x^k and that the Newton method with exact line search on g is globally convergent to a solution of the VIP (1.1). Moreover, assuming the strong monotonicity of F, the polyhedral convexity of S and the strict complementarity condition at the solution, they showed that the method is quadratically convergent. Another iterative method that uses the gap function g as a merit function has also been proposed by the same authors [35] for the VIP (1.1) in which the constraint set S is a compact polyhedral convex set specified by a system of linear inequalities.

The following "dual" gap function has also been used to reformulate the VIP (1.1):

$$h(x) = \sup\{\langle F(y), x - y \rangle \mid y \in S\}. \tag{2.7}$$

When F is monotone, it follows from the definition of functions g and h that

$$g(x) \geq h(x) \quad \text{for all } x \in \Re^n, \tag{2.8}$$

because the monotonicity of F implies

$$\langle F(x), x - y \rangle \geq \langle F(y), x - y \rangle \quad \text{for all } x, y \in \Re^n.$$

Since $g(x) = 0$ and $x \in S$ if and only if x solves the VIP (1.1) as mentioned above, and since $h(x) \geq 0$ clearly holds for each $x \in S$, the inequality (2.8) indicates that $h(x) = 0$ and $x \in S$ hold for any solution x of the VIP (1.1), provided that F is monotone. Namely, when F is monotone, any solution x to (1.1) is also a solution of the problem

$$\begin{array}{ll} \text{minimize} & h(x) \\ \text{subject to} & x \in S. \end{array} \tag{2.9}$$

Moreover, when F is strictly monotone, the unique solution of (1.1), if exists, uniquely minimizes the function h over S. Notice that the function h is a convex function, which may take value $+\infty$ unless some additional assumption like compactness is imposed on S. The evaluation of h, however, generally requires maximization of a nonconcave function, whereas g can be evaluated by maximizing a linear function over S. Though convex, the function h may not be differentiable in general. Nguyen and Dupuis [38] proposed a special cutting plane method for minimizing the function h. (Some properties of the dual gap function h are summarized in [29].)

2.2. Regularized gap functions

Whether or not the general VIP can be reformulated as a differentiable optimization problem had long been an open problem until Fukushima [13] and Auchmuty [1] independently and almost simultaneously gave an affirmative answer to it. Fukushima [13] considered the *regularized gap function* defined by

$$f_\alpha(x) = \max_{y \in S} \left\{ \langle F(x), x - y \rangle - \frac{\alpha}{2} \|y - x\|^2 \right\}, \tag{2.10}$$

where α is a positive parameter. (The Euclidean norm used in the definition (2.10) may be replaced by any norm induced by a positive definite symmetric matrix [13]. We adopt the above definition here only for simplicity of notation.) Since the definition (2.10) of f_α consists of maximizing a strongly concave function over a closed convex set, the supremum is always attained and hence the function f_α is finite-valued everywhere. This is in contrast with the gap function g, which may take value $+\infty$ somewhere. Moreover, since the maximizer on the right-hand side of (2.10) is unique, the function f_α is guaranteed to be differentiable at any point. Specifically, if we denote the unique maximizer by $y_\alpha(x)$, then the gradient of f_α is given by

$$\nabla f_\alpha(x) = F(x) - [\nabla F(x) - \alpha I](y_\alpha(x) - x). \tag{2.11}$$

Note that

$$y_\alpha(x) = \Pi_S[x - \alpha^{-1} F(x)], \tag{2.12}$$

where Π_S denotes the projection onto S. It is well known that x solves the VIP (1.1) if and only if $x = \Pi_S[x - \alpha^{-1} F(x)]$. Since the function f_α can be expressed as

$$f_\alpha(x) = \frac{\alpha}{2} \left\{ \|\alpha^{-1} F(x)\|^2 - \|y_\alpha(x) - (x - \alpha^{-1} F(x))\|^2 \right\}, \tag{2.13}$$

it is not difficult to deduce from (2.12) and (2.13) that, similar to the gap function g, the regularized gap function f_α possesses the following properties:

- ◇ $f_\alpha(x) \geq 0$ for all $x \in S$.
- ◇ $f_\alpha(x) = 0$ and $x \in S$ if and only if x solves the VIP (1.1).

Therefore the VIP (1.1) is equivalent to

$$\begin{array}{ll} \text{minimize} & f_\alpha(x) \\ \text{subject to} & x \in S. \end{array} \tag{2.14}$$

This fact implies that f_α serves as a merit function for the VIP. A remarkable feature of the regularized gap function f_α is of course that it is continuously differentiable. In general, the function f_α need not be convex even though the mapping F has some desirable properties like the strong monotonicity. (This is also the case for the gap function g.) So the function f_α may have local minima or stationary points which do not minimize $f_\alpha(x)$ over S globally, i.e., do not solve the VIP (1.1). Since most of the iterative minimization methods are only guaranteed to converge to local minima or stationary points, it is important to know when a local minimum or a stationary point of f_α over S is actually a global minimum. In this regard, Fukushima [13] proved the following result:

◇ If the Jacobian $\nabla F(x)$ is positive definite on S, then any stationary point of (2.14) solves the VIP (1.1).

Moreover, Taji, Fukushima and Ibaraki [52] showed that, if F is strongly monotone with modulus $\mu > 0$ and if $\alpha < 2\mu$, then the following error bound is obtained for the VIP (1.1):

$$\|x - x^*\| \leq \sqrt{\frac{f_\alpha(x)}{\mu - \alpha/2}} \qquad \text{for all } x \in S, \tag{2.15}$$

where x^* is the unique solution of the VIP (1.1).

The minimization formulation (2.14) naturally suggests that a globally convergent algorithm for solving the VIP (1.1) be developed on the basis of the merit function f_α. Fukushima [13] showed that, if $\nabla F(x)$ is positive definite on S, then the vector

$$d^k = y_\alpha(x^k) - x^k$$

is a feasible descent direction of f_α at each x^k. A descent algorithm that utilizes the vector d^k as a search direction and performs line search for the merit function f_α was developed in [13]. Note that, in this 'first-order' method, we need not evaluate the gradient $\nabla f_\alpha(x)$. This could be desirable because the evaluation of $\nabla f_\alpha(x)$ requires the evaluation of $\nabla F(x)$, which may be expensive in some practical problems. If one is willing to use $\nabla F(x)$, then it is possible to develop a 'second-order' method of Newton type. Specifically, Taji, Fukushima and Ibaraki [52] showed that the vector

$$d^k = \bar{z}(x^k) - x^k,$$

where $\bar{z}(x^k)$ is a solution of the linearized VIP (2.5), is a feasible descent direction of f_α at each x^k, provided that F is strongly monotone with modulus μ and α is chosen small enough to satisfy $\alpha < 2\mu$. Note that, under the strong monotonicity assumption, the solution $\bar{z}^k$ of the linearized VIP (2.5) exists uniquely. It was shown in [52] that this algorithm is globally convergent under the above-mentioned assumptions and that the rate of convergence is quadratic under additional assumptions such as the polyhedral convexity of S and the strict complementarity at the solution. This algorithm shares much in common with the Newton method of Marcotte and Dussault [34], which uses the gap function g as a merit function. It is worth mentioning, however, that using the regularized gap function f_α enables us to adopt inexact line search with Armijo rule [52].

Similar but more general classes of regularized gap functions have been considered by several authors [1, 29, 43, 55, 60, 61]. Those functions are typically derived by replacing the quadratic term in the definition (2.10) of f_α by a more general convex term. In particular, Larsson and Patriksson [29] recently presented a unified framework of regularized gap functions derived from the class of Auchmuty's merit functions [1]. Error bound results that generalize the condition (2.15) have been established in [29, 61]. Descent methods, which are extensions of the one given in Fukushima [13], have been proposed in [29, 43, 55, 60, 61]. In particular, Zhu and Marcotte [60] proposed a modified descent algorithm which uses varying parameter α in the regularized gap function f_α. A remarkable feature of the latter algorithm is that its convergence requires only monotonicity of F (see also [43]).

By definition, the evaluation of the regularized gap function f_α requires computing the projection of a point x onto the closed convex set S. In general, however, this is not an easy task unless S has a certain tractable structure such as polyhedral convexity. To cope with this difficulty, Taji and Fukushima [50, 51] recently proposed a modification of the regularized gap function f_α by way of polyhedral approximations to S, and showed that many of the favorable properties of f_α are transmitted to the modified regularized gap function.

2.3. The D-gap function

The equivalent optimization formulations discussed so far are all constrained problems, each of which is to minimize a certain merit function over the feasible set S of the original VIP. Recently Yamashita and Fukushima [57] considered further applying the Moreau-Yosida regularization to some gap functions. The resulting functions, which may not be easy to evaluate in general, certainly possess nice theoretical properties. Aside from the differentiability, those functions are shown to provide global error bounds for the VIP under the strong monotonicity only.

More recently, Peng [44] considered the function

$$M_\alpha(x) = f_\alpha(x) - f_{1/\alpha}(x), \tag{2.16}$$

where f_α is the regularized gap function defined by (2.10) and α is a parameter such that $0 < \alpha < 1$, and showed the following results:

- $\diamond$ $M_\alpha(x) \geq 0$ for all $x \in \Re^n$.
- $\diamond$ $M_\alpha(x) = 0$ if and only if x solves the VIP (1.1).

Thus the VIP (1.1) is equivalent to the unconstrained minimization problem

$$\begin{array}{ll} \text{minimize} & M_\alpha(x) \\ \text{subject to} & x \in \Re^n. \end{array} \tag{2.17}$$

By definition, it is clear that the differentiability of M_α is inherited from f_α. In fact, the function M_α shares much in common with f_α. Among others, the following result is very important.

◇ If the Jacobian $\nabla F(x)$ is positive definite for each x, then any stationary point of (2.17) solves the VIP (1.1).

Moreover, Peng [44] proved that M_α satisfies the following inequalities:

$$(1-\alpha)\|r_1(x)\|^2 \leq M_\alpha(x) \leq \frac{1-\alpha}{\alpha}\|r_1(x)\|^2 \quad \text{for all } x \in \Re^n, \tag{2.18}$$

where r_1 is the "natural" residual for the VIP defined by $r_1(x) = x - y_1(x)$. The inequalities (2.18) in particular imply that the function $\sqrt{M_\alpha}$ provides an error bound for the VIP if $\|r_1\|$ does. For the VIP, it is known that the latter is true if F is strongly monotone and Lipschitz continuous on $\Re^n$ [40]. In [45], it is shown that, in the case of the CP, the function M_α is reduced to a merit function called the implicit Lagrangian [32] (see §3.2). Thus M_α may be regarded as a generalization of the implicit Lagrangian for the CP to the VIP.

The results obtained in [44, 45] have further been extended by Yamashita, Taji and Fukushima [59], who considered the function

$$g_{\alpha\beta}(x) = f_\alpha(x) - f_\beta(x), \tag{2.19}$$

where f_α and f_β are regularized gap functions with parameters α and β such that $0 < \alpha < \beta$. (In [59], the functions f_α and f_β are defined by (2.10) with the quadratic term replaced by a more general convex term.) The function $g_{\alpha\beta}$ is called the *D-gap function*, where "D" stands for the word "difference". Similar to the function M_α, the following properties hold for the D-gap function $g_{\alpha\beta}$:

◇ $g_{\alpha\beta}(x) \geq 0$ for all $x \in \Re^n$.

◇ $g_{\alpha\beta}(x) = 0$ if and only if x solves the VIP (1.1).

Therefore the VIP (1.1) is equivalent to the unconstrained minimization problem

$$\begin{array}{ll} \text{minimize} & g_{\alpha\beta}(x) \\ \text{subject to} & x \in \Re^n. \end{array} \tag{2.20}$$

The function $g_{\alpha\beta}$ is differentiable whenever F is differentiable. Also we have the following result:

◇ If the Jacobian $\nabla F(x)$ is positive definite for each x, then any stationary point of (2.20) solves the VIP (1.1).

Moreover, it is shown in [59] that $g_{\alpha\beta}$ satisfies the following inequalities:

$$\frac{\beta-\alpha}{2}\|x - y_\beta(x)\|^2 \leq g_{\alpha\beta}(x) \leq \frac{\beta-\alpha}{2}\|x - y_\alpha(x)\|^2 \quad \text{for all } x \in \Re^n,$$

where $y_\alpha(x)$ is the unique maximizer in the definition (2.10) of f_α and $y_\beta(x)$ is defined similarly for f_β. Also it is shown in [59] that, if F is strongly monotone and if either F is Lipschitz continuous or S is compact, then the function $\sqrt{g_{\alpha\beta}}$ provides an error bound for the VIP.

Although the D-gap function $g_{\alpha\beta}$ is not twice differentiable, it is possible to apply a Newton-type method to minimize it. Such an attempt has recently been made in [48].

The results obtained in [59] and [48] have been further refined by Kanzow and Fukushima [26] for the special case of the VIP, where the feasible set S is the box given by

$$S = \prod_{i=1}^{n} [l_i, u_i] \tag{2.21}$$

with $l_i \in [-\infty, +\infty)$ and $u_i \in (-\infty, +\infty]$ such that $l_i \leq u_i$ for all $i = 1, \cdots, n$. Note that, in the box constrained case, the vectors $y_\gamma(x), \gamma = \alpha, \beta$, can be written componentwise as

$$\begin{aligned} y_{\gamma,i}(x) &= \mathrm{Proj}_{[l_i,u_i]}\left(x_i - \frac{1}{\gamma}F_i(x)\right) \\ &= \mathrm{mid}\left\{x_i - \frac{1}{\gamma}F_i(x), l_i, u_i\right\} \qquad i = 1, \ldots, n, \end{aligned} \tag{2.22}$$

The Cartesian structure of the problem enables us to slightly refine the above-mentioned result on stationary points of (2.20) as follows:

- ◇ If the Jacobian $\nabla F(x)$ is a P-matrix for each x, then any stationary point of (2.20) solves the VIP (1.1) with the box constraints S given by (2.21).

Moreover we have the following results on the boundedness of level sets of the D-gap function $g_{\alpha\beta}$.

- ◇ Let S be given by (2.21). If F is a uniform P-function, then the level sets $\mathcal{L}(c) := \{x \in \Re^n \mid g_{\alpha\beta}(x) \leq c\}$ are bounded for all $c \geq 0$. If the set S is bounded, then $\mathcal{L}(c)$ are also bounded for all $c \geq 0$.

Note that the latter half of this result does not depend on the box structure of S.

The error bound result with the D-gap function for the general VIP can be refined for the box constrained VIP

- ◇ Let S be given by (2.21) and F be a uniform P-function. If either F is Lipschitz continuous or S is bounded, then there exists a constant $c > 0$ such that

  $$\|x - x^*\| \leq c\sqrt{g_{\alpha\beta}(x)},$$

  holds for all $x \in \Re^n$, where x^* is the unique solution of the VIP (1.1).

It is not difficult to observe that the D-gap function $g_{\alpha\beta}$ is semismooth. However, by some reasons, it is not necessarily easy to deal with generalized Hessians of $g_{\alpha\beta}$ in the ordinary sense. Kanzow and Fukushima [26] developed a Newton-type algorithm for minimizing $g_{\alpha\beta}$, which uses an approximation to the generalized Hessian. The resulting algorithm is easily implemented and is shown to have global and superlinear convergence properties.

3. REFORMULATIONS OF CP

Since the CP is a special case of the VIP such that the feasible set S is the nonnegative orthant in $\Re^n$, any merit function for the VIP can in principle be used as a merit function for the CP. For example, the regularized gap function yields an equivalent optimization problem with simple nonnegativity constraints on the variables. Moreover, remarkable progress has recently been made in the study of merit functions that lead to unconstrained optimization reformulations of the CP. In this section, we first review optimization formulations with simple bound constraints and then discuss two important classes of merit functions that lead to unconstrained optimization reformulations of the CP.

3.1. Optimization formulations with simple bound constraints

The regularized gap function f_α for the VIP can be specialized to the CP as follows:

$$\begin{aligned} f_\alpha(x) &= \langle F(x), x - [x - \frac{1}{\alpha}F(x)]_+\rangle - \frac{\alpha}{2}\|[x - \frac{1}{\alpha}F(x)]_+ - x\|^2 \\ &= \langle F(x), x\rangle + \frac{\alpha}{2}\left(\|[x - \frac{1}{\alpha}F(x)]_+\|^2 - \|x\|^2\right), \end{aligned} \tag{3.1}$$

where $[z]_+$ denotes the vector with components $\max(z_i, 0), i = 1, \cdots, n$. The CP is equivalent to the optimization problem

$$\begin{array}{ll} \text{minimize} & f_\alpha(x) \\ \text{subject to} & x \geq 0. \end{array} \tag{3.2}$$

Because the function f_α can easily be evaluated and the optimization problem (3.2) has the simple bound constraints on the variables only, globally convergent algorithms that utilize the regularized gap function, originally developed for the VIP, may be specialized to the CP in an effective manner [49].

For the LCP (1.3), Friedlander, Martínez and Santos [11] considered the optimization problem

$$\begin{array}{ll} \text{minimize} & \rho\|Mx + q - z\|^2 + (\langle x, z\rangle)^p \\ \text{subject to} & x \geq 0, \ \ z \geq 0, \end{array} \tag{3.3}$$

where $\rho > 0$ and $p > 1$ are arbitrary constants. Clearly this problem is equivalent to the LCP (1.3) in the sense that a global optimal solution of (3.3) with zero objective value is a solution of the LCP and vice versa. It was shown in [11] that, under some condition, which is implied by the positive semidefiniteness of M, any stationary point of problem (3.3) is a solution of the LCP (1.3). Using a similar idea, the same authors [12] proposed a bound constrained optimization reformulation of a linearly constrained VIP.

For the CP (1.2), Moré [36] considered the following optimization problem, which is similar to the problem (3.3) for the LCP:

$$\begin{array}{ll} \text{minimize} & \|F(x) - z\|^2 + \sum_{i=1}^{n}(x_i z_i)^2 \\ \text{subject to} & x \geq 0, \ \ z \geq 0. \end{array} \tag{3.4}$$

In particular, it was shown in [36] that any stationary point of problem (3.4) comprises a solution of the CP (1.2) under a regularity condition, which is satisfied, for example, if $\nabla F(x)$ is a positive definite matrix or an M-matrix. Moré [36] also proposed a trust region method for solving (3.4) and discussed its global and superlinear convergence properties.

Yet another bound constrained optimization formulation of the CP (1.2) is the following problem:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} [\min(x_i, F_i(x))]^2 \\ \text{subject to} & x \geq 0. \end{array} \tag{3.5}$$

It is clear that this problem is equivalent to the CP (1.2). It was shown in [41] that, under a regularity condition, a stationary point of (3.5) is a solution to the CP (1.2). Unlike the optimization reformulations mentioned in this subsection, the objective function in problem (3.5) is not differentiable. Nevertheless, the algorithms proposed by Pang and Gabriel [41, 15] are shown to have global and superlinear convergence properties under some regularity assumptions.

3.2. Implicit Lagrangian

Mangasarian and Solodov [32] introduced the following merit function, which is now commonly called the *implicit Lagrangian*, for the CP:

$$\begin{aligned} M_\alpha(x) = \langle F(x), x\rangle + \frac{\alpha}{2}\Big(& \|[x - \frac{1}{\alpha}F(x)]_+\|^2 - \|x\|^2 \\ & + \|[F(x) - \frac{1}{\alpha}x]_+\|^2 - \|F(x)\|^2 \Big), \end{aligned} \tag{3.6}$$

where α is a parameter such that $0 < \alpha < 1$. They showed that

- $\diamond$ $M_\alpha(x) \geq 0$ for all $x \in R^n$.
- $\diamond$ $M_\alpha(x) = 0 \iff x$ solves the CP (1.2).

Therefore the CP (1.2) is equivalent to the unconstrained optimization problem

$$\begin{array}{ll} \text{minimize} & M_\alpha(x) \\ \text{subject to} & x \in \Re^n. \end{array} \tag{3.7}$$

Comparing the definitions (3.1) and (3.6), one may notice some similarity between the regularized gap function and the implicit Lagrangian. In fact, these two functions can be treated in a unified manner [54]. As observed by Peng and Yuan [45], in particular, the implicit Lagrangian M_α can be represented as the difference of two regularized gap functions (see (2.16)). Like f_α, the implicit Lagrangian M_α is differentiable and the gradient of M_α can be expressed as

$$\begin{aligned} \frac{1}{\alpha}\nabla M_\alpha(x) &= \left(I - \frac{1}{\alpha}\nabla F(x)\right)\left([x - \frac{1}{\alpha}F(x)]_+ - x\right) \\ &\quad + \left(\nabla F(x) - \frac{1}{\alpha}I\right)\left([F(x) - \frac{1}{\alpha}x]_+ - F(x)\right). \end{aligned}$$

Yamashita and Fukushima [56] proved the following result:

◇ If $\nabla F(x)$ is positive definite for every $x \in \Re^n$, then any stationary point of M_α is a solution to the CP (1.2).

This result was recently improved by Jiang [20], who showed that the positive definiteness of $\nabla F(x)$ can be replaced by the weaker condition that $\nabla F(x)$ is a P-matrix.

Concerning error bounds for the CP, Luo et al. [30] showed that there exist $\kappa > 0$ and $\delta > 0$ such that

$$M_\alpha(x) \geq \kappa \operatorname{dist}(x, X^*)^2 \quad \text{for all } x \in \{x \mid M_\alpha(x) \leq \delta\},$$

where X^* is the solution set of the CP. Moreover, Yamashita and Fukushima [56] showed that, if F is strongly monotone and Lipschitz continuous, then the following inequality holds for some $\kappa > 0$:

$$M_\alpha(x) \geq \kappa \|x - x^*\|^2 \quad \text{for all } x \in \Re^n,$$

where x^* denotes the unique solution to the CP. Note that the last inequality in particular implies that, if F is strongly monotone and Lipschitz continuous, then the level sets of the implicit Lagrangian M_α are bounded. More on the error bound results for the CP, the reader may refer to [54].

To solve a differentiable CP, we may directly apply any gradient-based algorithm to the equivalent minimization problem (3.7) (see [32]). As an alternative to the gradient-based algorithms, a simple descent method that does not require evaluating $\nabla F(x)$ has been proposed in [56].

3.3. NCP functions

Recently the following function $\Psi : \Re^n \to \Re$ has attracted the attention of many researchers in the field of complementarity problems:

$$\Psi(x) = \frac{1}{2} \sum_{i=1}^{n} \varphi(x_i, F_i(x))^2, \tag{3.8}$$

where $\varphi : \Re^2 \to \Re$ is defined by

$$\varphi(a, b) = \sqrt{a^2 + b^2} - a - b. \tag{3.9}$$

This function was first introduced by Fischer [7] but attributed to Burmeister. As can be seen easily, the function φ satisfies

$$\varphi(a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ ab = 0, \tag{3.10}$$

so that the CP is equivalent to the system of equations

$$\Phi(x) = 0, \tag{3.11}$$

where $\Phi(x) = (\varphi(x_1, F_1(x)), \varphi(x_2, F_2(x)), \cdots, \varphi(x_n, F_n(x)))^\top$. Hence the CP can be cast as the following unconstrained optimization problem:

$$\begin{array}{ll} \text{minimize} & \Psi(x) \\ \text{subject to} & x \in \Re^n. \end{array} \tag{3.12}$$

Alternatively, one may obtain another reformulation of the CP by considering the augmented system of equations.

$$\hat{\Phi}(x, z) = 0, \tag{3.13}$$

where $\hat{\Phi}(x, z) = (F(x) - z, \varphi(x_1, z_1), \varphi(x_2, z_2), \cdots, \varphi(x_n, z_n))^{\mathsf{T}}$. This is clearly equivalent to (3.11) and yields the following unconstrained optimization problem:

$$\begin{array}{ll} \text{minimize} & \hat{\Psi}(x, z) \\ \text{subject to} & x \in \Re^n, \quad z \in \Re^n, \end{array} \tag{3.14}$$

where

$$\hat{\Psi}(x, z) = \frac{1}{2}\left(\|F(x) - z\|^2 + \sum_{i=1}^{n} \varphi(x_i, z_i)^2\right).$$

Any function satisfying the condition (3.10) may be called an *NCP-function* [10]. Besides the above-mentioned function φ, various NCP functions are known to date (see, e.g., [23, 24, 27]). In this section, however, we will confine ourselves to the NCP function φ defined by (3.9), because it enjoys many favorable properties in the complementarity problem. First it is noted that Ψ is differentiable. (Note that $\varphi(x_i, F_i(x))$ is not differentiable at a point x such that $x_i = F_i(x) = 0$.) The gradient of Ψ is given by

$$\nabla\Psi(x) = [D_a(x) + \nabla F(x) D_b(x)]\ \Phi(x), \tag{3.15}$$

where $D_a(x)$ and $D_b(x)$ are diagonal matrices with nonpositive diagonal elements $a_i(x)$ and $b_i(x)$, $i = 1, \cdots, n$, respectively, which are given by

$$(a_i(x), b_i(x)) = \begin{cases} \left(\frac{x_i}{\sqrt{x_i^2 + F_i(x)^2}} - 1, \frac{F_i(x)}{\sqrt{x_i^2 + F_i(x)^2}} - 1\right), & \text{if } (x_i, F_i(x)) \neq (0, 0) \\ (\xi_i - 1, \eta_i - 1), & \text{if } (x_i, F_i(x)) = (0, 0) \end{cases}$$

for some ξ_i and η_i such that $\xi_i^2 + \eta_i^2 \leq 1$.

Geiger and Kanzow [16] proved the following results:

- ◇ If $\nabla F(x)$ is positive semidefinite for every $x \in \Re^n$, then any stationary point of Ψ is a solution to the CP (1.2).
- ◇ If F is strongly monotone, then the level sets of Ψ are bounded.

Comparing these with the corresponding results for the implicit Lagrangian, we see that the NCP function Ψ enjoys these properties under weaker assumptions than the implicit Lagrangian. (In [56], a counter-example is provided to show that the positive definiteness of $\nabla F(x)$ cannot be replaced even by the strict monotonicity of F in order to guarantee that any stationary point of M_α solves the CP .) Facchinei and Soares [5] pointed out that the monotonicity and the strong monotonicity assumptions in the above results may further be replaced by the weaker conditions that F is a P_0-function and that F is a uniformly P-function (see also [4]). Tseng [53] studied growth properties of various NCP functions including Ψ and $\hat{\Psi}$.

As mentioned above, the functions Φ and $\hat{\Phi}$ are not differentiable. In spite of this fact, a number of Newton-type methods have been developed to solve the nonsmooth equation reformulations (3.11) and (3.13) for the LCP [8, 9] and the CP [4, 5, 6, 21, 24, 58]. A key property for such algorithms to converge superlinearly is the semismoothness of the functions Φ and $\hat{\Phi}$. The semismoothness of vector functions was introduced in [46] and used in the analysis of generalized Newton methods for nonsmooth equations [46, 47] (see also [22]). In the present case, the function Φ is actually shown to be semismooth [5, 21]. A direction-finding problem in a Newton-type method for (3.11) may therefore be given as

$$V_k d = -\Phi(x^k), \tag{3.16}$$

where V_k is an $n \times n$ matrix chosen from the generalized Jacobian $\partial\Phi(x^k)$. If a solution d^k of (3.16) is obtained, then the next iterate x^{k+1} may be determined by line search for the merit function Ψ. A possible trouble with this approach is that the matrix V_k is not necessarily nonsingular, and hence the linear equation (3.16) may not have a solution, even though the function F is monotone. In order to guarantee the nonsingularity of V_k, we need to assume that $\nabla F(x)$ is positive definite or F is a uniform P-function [21]. A remedy for this trouble is to use the steepest descent direction for the merit function Ψ when the system (3.16) is not solvable or d^k does not afford a direction of sufficient decrease for Ψ [4, 5, 6, 21]. Another cure is to slightly perturb V_k to obtain a positive definite matrix, so that the modified direction-finding subproblem is always consistent [8, 9, 58]. Under appropriate conditions, those methods are globally convergent to a solution of the LCP or the CP. Moreover, it can be shown that, under some regularity assumptions, they have a superlinear or quadratic convergence property.

As shown above, the NCP function Ψ turns out to be very useful in solving the CP. Continuing effort is still under way to develop a merit function with better properties in regard to its local and global behaviors [31, 28].

4. CONCLUDING REMARKS

In this article, we have reviewed the recent developments of merit functions for-jdvi2kps the VIP and CP. In closing, we point out that some of those results can be extended to more general classes of problems. For example, for the generalized complementarity problem (GCP)

$$F(x) \geq 0, \quad G(x) \geq 0, \quad \langle F(x), G(x) \rangle = 0 \tag{4.1}$$

with $F : \Re^n \to \Re^n$ and $G : \Re^n \to \Re^n$, Tseng, Yamashita and Fukushima [54] showed that the regularized gap function f_α and the implicit Lagrangian M_α can be derived in a unified manner, and extended a number of basic results about those merit functions for the CP. (In [54], the GCP (4.1) is considered under a more general setting where the vector inequality is defined in terms of a closed convex cone and its polar cone.) For the GCP (4.1), Kanzow and Fukushima [25] also extended various results known for the merit function Ψ with the NCP function φ defined by (3.9) for the CP. On the other hand, Giannessi [17] considered the quasi-variational inequality problem (QVIP), which is to find a point $x \in S(x)$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \text{for all } y \in S(x), \tag{4.2}$$

where $S(x)$ is given by

$$S(x) = \{y \in Y(x) \,|\, c(x,y) \in C\},$$

with a point-to-set mapping $Y(x) : \Re^n \to \Re^n$, a function $c : \Re^n \times \Re^n \to \Re^m$ and a closed convex cone $C \subseteq \Re^m$. When $Y(x)$ and $c(x,y)$ are independent of x, the QVIP (4.2) reduces to an ordinary VIP. In [17], attempts were made to extend some gap functions for the ordinary VIP to the QVIP (4.2).

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