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Evaluation of Investment Opportunity under Entry and Exit Decisions

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Abstract: We study the evaluation of the project which includes the entry and exit decisions to invest the production plant. We assume that the price process P_t of the production goods follows a geometric Brownian motion. Then we show the explicit evaluation formula for the discounted present value from the project when the entry-exit is controlled by the trigger prices. Furthermore we prove the optimality of the trigger control strategies to maximize the discounted present value from the project. Finally we studied the multiple entry-exit model originally proposed by Dixit and show the analytical closed form solution for the Dixit's valuation problem.

Keywords: Optimal Stopping Problem, First Passage Time, Net Present Value, Entry-Exit Model.

1 Introduction

We study the evaluation of the project which includes the entry and exit decisions to invest the production plant. To start the production activity, we have to pay the initial investment cost which amounts to $I_+ > 0$. Once the production plant is activated, it continues to makes a fixed amount of goods by the constant production cost $C > 0$ until the investor exit from the project. We assume that the price process P_t of the production goods follows a geometric Brownian motion :

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t,$$

where W_t is a standard Brownian motion. To stop the production, we have to pay the terminal investment cost which amounts to $I_- > 0$. The basic problem is to derive the optimal entry and exit strategies (τ_+^*, τ_-^*) which attain the maximum discounted present value from the project investment :

$$\sup_{0\leq\tau_+\leq\tau_-} E[-e^{-r\tau_+}I_+ + \int_{\tau_+}^{\tau_-}(P_t - C)e^{-rt}dt - e^{-r\tau_-}I_-|P_0 = P].$$

This problem is first considered by Brennan and Schwartz [1] and evolutionary studied by Dixit and Pindyck [2, 3, 4] to the case of multiple entry-exit model. In this paper, we studied the same type of model proposed by Dixit [2]. Dixit derived a system of differential equations for the project valuation functions using the no arbitrage argument. Then he derived the semi-closed form solutions for the project valuation functions. However his approach is not sufficient to get the valuation functions explicitly. To

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avoid the ambiguity of his approach, we used the probabilistic analysis to evaluate the cash flow from the project. This enables us not only to derive the explicit forms of the valuation functions but also we can prove the optimality of the simple entry-exit desision rule by the constant trigger prices.

This paper is organized as follows. In section 2, we view the exit problem as the stopping time problem and derive the optimal stopping time which maximize the net present value of the existing project. In Section 3, we generalize the formulation to include the decision for the entry timing and completely characterize the optimal solution in this situation. Finally in Section 4, we treat the multiple entry-exit model originally proposed by Dixit and show the analytical closed form solution for the Dixit's valuation problem.

2 Exit Problem

First we assume that the investor has already activated the project and so he can decide only the exiting timing from the production activity. At time 0, the project is active and the production state is $+$. The problem is to derive the optimal exiting strategy τ_-^* which attains the maximum discounted present value :

$$\sup_{0\leq\tau_-} E[\int_0^{\tau_-}(P_t-C)e^{-rt}dt-e^{-r\tau_-}I_-|P_0=P].$$

To solve this problem, we consider the simple strategy that stops the production activity when the price process hits the inactivation trigger price P_-. For the notational convenience, let

$$V_+(P,P_-)=E[\int_0^{\tau_{P_-}}(P_+-C)e^{-rt}dt-e^{-r\tau_{P_-}}I_-|P_0=P], \tag{2.1}$$

where

$$\tau_{P_-}=\inf\{t\geq 0; P_t\leq P_-\}.$$

Hereafter we assume the following condition.

Assumption 2.1

$$r>\mu \quad \textit{and} \quad \frac{C}{r}>I_-. \tag{2.2}$$

This condition must be satisfied so that the exit becomes reasonable. That is,

$$\begin{aligned} E_P[\int_0^\infty e^{-rt}(P_t-C)dt] &= \int_0^\infty e^{-rt}\,(Pe^{\mu t}-C)\,dt \\ &= \lim_{t\to\infty}\left[\frac{P}{\mu-r}(e^{(\mu-r)t}-1)+\frac{C}{r}(e^{-rt}-1)\right] \\ &= \begin{cases} \frac{P}{r-\mu}-\frac{C}{r}>-I_-, & \text{if} \quad r>\mu, \\ \infty, & \text{if} \quad r\leq\mu. \end{cases} \end{aligned}$$

Hence if Assumption 2.1 is violated, exit from the investment opportunity is not rational to maximize the net present value from the project. Under this condition, we can derive the net present value of the cash flow from the existing project when the stopping time is given by the first hitting time of the fixed price level.

Theorem 2.2 *Under Assumption 2.1,*

$$V_+(P, P_-) = \frac{P}{r-\mu} - \frac{C}{r} - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2 + 2\eta}}, \tag{2.3}$$

where

$$\begin{cases} \nu_+ &=& \frac{\mu}{\sigma^2} + \frac{1}{2}, \\ \nu_- &=& \frac{\mu}{\sigma^2} - \frac{1}{2}, \\ \eta &=& \frac{r-\mu}{\sigma^2} > 0. \end{cases}$$

Proof. By definition,

$$V_+(P, P_-) = E_P[\int_0^{\tau_{P_-}} (P_t - C)e^{-rt}dt] - E_P[e^{-r\tau_{P_-}}]I_-. \tag{2.4}$$

Each expectation terms are computed as follows. Let $\theta_- = \frac{\mu}{\sigma} - \frac{\sigma}{2}$. We can easily check

$$\tau_{P_-} = \inf\{t; W_t + \theta_- t \le x_-\}, \qquad \text{where} \quad x_- = \frac{1}{\sigma}\log\frac{P^-}{P} < 0.$$

Let

$$f_{x_-}(t) = \lim_{\Delta \to 0}\frac{P[t \le \tau_{P_-} \le t + \Delta]}{\Delta} = \frac{-x_-}{\sqrt{2\pi t^3}}e^{-\frac{1}{2t}(x_- - \theta_- t)^2}.$$

Then

$$\begin{aligned} &\int_0^\infty e^{-rt} f_{x_-}(t)dt \\ =\ & \frac{1}{\sqrt{2\pi}}e^{x_- \theta_- + \sqrt{\theta_-^2 + 2r}\cdot x_-}\int_0^\infty e^{-\frac{1}{2}\left(\frac{x_-}{\sqrt{t}} + \sqrt{(\theta_-^2 + 2r)t}\right)^2}\frac{\partial}{\partial t}\left(\frac{x}{\sqrt{t}} + \sqrt{(\theta_-^2 + 2r)t}\right)dt \\ &+ \frac{1}{\sqrt{2\pi}}e^{x_- \theta_- - \sqrt{\theta_-^2 + 2r}\cdot x_-}\int_0^\infty e^{-\frac{1}{2}\left(\frac{x_-}{\sqrt{t}} - \sqrt{(\theta_-^2 + 2r)t}\right)^2}\frac{\partial}{\partial t}\left(\frac{x}{\sqrt{t}} - \sqrt{(\theta_-^2 + 2r)t}\right)dt \\ =\ & e^{(\theta_- + \sqrt{\theta_-^2 + 2r})x_-}. \end{aligned}$$

This means that

$$E_P[e^{-r\tau_{P_-}}] = \left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2 + 2\eta}}. \tag{2.5}$$

On the other hand,

$$\begin{aligned} &E_P[\int_0^{\tau_{P_-}} (P_t - C)e^{-rt}dt] \\ =\ & \int_0^\infty E_P[P_t e^{-rt}1\{t < \tau_{P_-}\}]dt - CE_P[\int_0^{\tau_{P_-}} e^{-rt}dt] \\ =\ & \int_0^\infty E_P[P_t e^{-rt}1\{t < \tau_{P_-}\}]dt - \frac{C}{r}\left(1 - E_P[e^{-r\tau_{P_-}}]\right) \\ =\ & \int_0^\infty E_P[P_t e^{-rt}1\{t < \tau_{P_-}\}]dt - \frac{C}{r}\left(1 - \left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2 + 2\eta}}\right). \end{aligned} \tag{2.6}$$

Let

$$\begin{aligned}
g_{t,\theta}(x,y) &= \lim_{\Delta\to 0}\frac{P[x\le \min_{0\le u\le t} W_u+\theta u\le x+\Delta_x,\ y\le W_t+\theta t\le y+\Delta_y]}{\Delta_x\Delta_y}\\
&= \frac{-2(2x-y)}{\sqrt{2\pi t^3}}e^{-\frac{(2x-y+\theta t)^2}{2t}+2x\theta},\\
\theta_+ &= \frac{\mu}{\sigma}+\frac{\sigma}{2}.
\end{aligned}$$

Since

$$t<\tau_{P_-}\Leftrightarrow \min_{0\le u\le t}(W_u+\theta_- u)>\frac{1}{\sigma}\log\frac{P_-}{P},$$

we have,

$$\begin{aligned}
&E_P[P_te^{-rt}1\{t<\tau_{P_-}\}]\\
&= E_P[P_te^{-rt}1\{\min_{0\le u\le t}(W_u+\theta_- u)>\frac{1}{\sigma}\log\frac{P_-}{P}\}]\\
&= \int_{x_-}^{\infty}\int_{x_-}^{y\wedge 0}Pe^{\sigma y-rt}g_{t,\theta_-}(x,y)dxdy\\
&= \int_{x_-}^{\infty}\int_{x_-}^{y\wedge 0}Pe^{\sigma y-rt}\frac{-2(2x-y)}{\sqrt{2\pi t^3}}e^{-\frac{(2x-y+\theta_- t)^2}{2t}+2x\theta_-}dxdy\\
&= \frac{P}{\sqrt{2\pi t}}\int_{x_-}^{\infty}\int_{x_-}^{y\wedge 0}e^{\sigma y-rt+\theta_- y-\frac{\theta_-^2}{2}t-\frac{1}{2t}(2x-y)^2}\frac{\partial}{\partial x}\left(-\frac{1}{2t}(2x-y)^2\right)dxdy\\
&= \frac{P}{\sqrt{2\pi t}}\int_{x_-}^{0}e^{\theta_+ y-(r+\frac{\theta_-^2}{2})t}\int_{x_-}^{y}\frac{\partial}{\partial x}\left(e^{-\frac{1}{2t}(2x-y)^2}\right)dxdy+\frac{P}{\sqrt{2\pi t}}\int_{0}^{\infty}e^{\theta_+ y-(r+\frac{\theta_-^2}{2})t}\int_{x_-}^{0}\frac{\partial}{\partial x}\left(e^{-\frac{1}{2t}(2x-y)^2}\right)dxdy\\
&= \frac{P}{\sqrt{2\pi t}}\int_{x_-}^{0}e^{\theta_+ y-(r+\frac{\theta_-^2}{2})t}(e^{-\frac{y^2}{2t}}-e^{-\frac{1}{2t}(2x_--x_-)^2})dy+\frac{P}{\sqrt{2\pi t}}\int_{0}^{\infty}e^{\theta_+ y-(r+\frac{\theta_-^2}{2})t}(e^{-\frac{y^2}{2t}}-e^{-\frac{1}{2t}(2x_--y)^2})dy\\
&= \frac{P}{\sqrt{2\pi t}}\int_{x_-}^{\infty}e^{-\frac{1}{2t}(y^2-2\theta_+ yt+(2\mu+\theta_-^2)t^2)}e^{-(r-\mu)t}dy-\frac{P}{\sqrt{2\pi t}}\int_{x_-}^{\infty}e^{-\frac{1}{2t}((2x_--y)^2-2\theta_+ yt+(2\mu+\theta_-^2)t^2)}e^{-(r-\mu)t}dy\\
&= Pe^{-(r-\mu)t}\frac{1}{\sqrt{2\pi t}}\int_{x_-}^{\infty}e^{-\frac{1}{2t}(y-\theta_+ t)^2}dy-Pe^{-(r-\mu)t+2x_-\theta_+}\frac{1}{\sqrt{2\pi t}}\int_{x_-}^{\infty}e^{-\frac{1}{2t}(y-(2x_-+\theta_+))^2}dy\\
&= Pe^{-(r-\mu)t}\left[\Phi\left(\frac{-x_-+\theta_+ t}{\sqrt{t}}\right)-e^{2x_-\theta_+}\Phi\left(\frac{x_-+\theta_+ t}{\sqrt{t}}\right)\right].
\end{aligned}$$

This together with Assumption 2.1 yields

$$\begin{aligned}
&\int_0^{\infty}E_P[P_te^{-rt}1\{t<\tau_{P_-}\}]dt\\
&= \int_0^{\infty}Pe^{-(r-\mu)t}\left[\Phi\left(\frac{-x_-+\theta_+ t}{\sqrt{t}}\right)-e^{2x_-\theta_+}\Phi\left(\frac{x_-+\theta_+ t}{\sqrt{t}}\right)\right]dt\\
&= \left(-\frac{P}{r-\mu}e^{-(r-\mu)t}\right)\left(\Phi\left(\frac{-x_-+\theta_+ t}{\sqrt{t}}\right)-e^{2x_-\theta_+}\Phi\left(\frac{x_-+\theta_+ t}{\sqrt{t}}\right)\right)\Bigg|_0^{\infty}\\
&\quad+\frac{P}{r-\mu}\int_0^{\infty}e^{-(r-\mu)t}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{-x_-+\theta_+ t}{\sqrt{t}}\right)^2}\frac{\partial}{\partial t}\left(\frac{-x_-+\theta_+ t}{\sqrt{t}}\right)dt\\
&\quad-\frac{P}{r-\mu}\int_0^{\infty}e^{-(r-\mu)t+2x_-\theta_+}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_-+\theta_+ t}{\sqrt{t}}\right)^2}\frac{\partial}{\partial t}\left(\frac{x_-+\theta_+ t}{\sqrt{t}}\right)dt\\
&= \frac{P}{r-\mu}-\frac{P}{r-\mu}\int_0^{\infty}e^{-(r-\mu)t}\frac{-x_-}{\sqrt{2\pi t^3}}e^{-\frac{1}{2}\left(\frac{x_--\theta_+ t}{\sqrt{t}}\right)^2}dt\\
&= \frac{P}{r-\mu}\left\{1-\left(\frac{P_-}{P}\right)^{\nu_++\sqrt{\nu_+^2+2\eta}}\right\}. \qquad (2.7)
\end{aligned}$$

Here the last equality follows from (2.5) with $\mu := \mu + \sigma^2$ and $r := r - \mu$. Then we get (2.3) from (2.4) through (2.7). $\square$

Equation (2.3) can be derived by the no arbitrage argument when we consider the convenience yield for the products. Let us fix the exit trigger price P_- and denote the value function by $V_+(P) = V_+(P, P_-)$. Consider the portfolio of $-V_+'(P_t)$ products' stock, one unit of project investment in active state $V_+(P_t)$ and $V_+'(P_t)P_t - V_+(P_t)$ riskless asset. The total portfolio value X_t is 0. The return from the portfolio is :

$$\begin{aligned} dX_t &= -V_+'(P_t)(dP_t + P_t(r-\mu)dt) + dV_+(P_t) + (P_t - C)dt + (V_+'(P_t)P_t - V_+(P_t))rdt \\ &= [\frac{1}{2}V_+''(P_t)P_t^2\sigma^2 + \mu V_+'(P_t)P_t + P_t - C - V_+(P_t)r]dt. \end{aligned}$$

Here we have assumed that the investor gets the convenience yield rate $r - \mu > 0$ from the products' stock investment. Under the no arbitrage condition, the riskless return must be 0. Hence we get the following differential equation for the arbitrage free value function :

$$\frac{1}{2}\sigma^2 P^2 V_+''(P) + \mu P V_+'(P) - rV_+(P) = C - P. \tag{2.8}$$

(2.8) is Euler type non-homogeneous differential equation whose general solution is given by :

$$V_+(P) = C_1 P^{-\nu_- + \sqrt{\nu_+^2 + 2\eta}} + C_2 P^{-\nu_- - \sqrt{\nu_+^2 + 2\eta}} + \frac{P}{r-\mu} - \frac{C}{r}. \tag{2.9}$$

By the basic property of the value function $V_+(P)$, we have the following boundary conditions.

$$\lim_{P \to \infty} \frac{|V_+(P)|}{P} < \infty, \quad V_+(P_-) = -I_-.$$

These conditions mean

$$C_1 = 0, \;\; C_2 = \left(\frac{C}{r} - \frac{P_-}{r-\mu} - I_-\right) P_-^{\nu_- + \sqrt{\nu_+^2 + 2\eta}}.$$

Thus we get the function $V_+(P)$ given by (2.3).

Next we shall consider the optimal trigger price for the exit problem. From (2.3),

$$\begin{aligned} &\frac{\partial V_+(P, P_-)}{\partial P_-} \\ &= \frac{1}{P}\left(\frac{C}{r} - I_-\right)(\nu_- + \sqrt{\nu_+^2 + 2\eta})\left(\frac{P_-}{P}\right)^{\nu_- - 1 + \sqrt{\nu_+^2 + 2\eta}} - \frac{1}{r-\mu}(\nu_+ + \sqrt{\nu_+^2 + 2\eta})\left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2 + 2\eta}} \\ &\begin{cases} \geq 0, & \text{if } P_- \leq P_-^*, \\ \leq 0, & \text{if } P_- \geq P_-^*, \end{cases} \end{aligned}$$

where

$$P_-^* = (r-\mu)\left(\frac{C}{r} - I_-\right)\frac{\nu_- + \sqrt{\nu_+^2 + 2\eta}}{\nu_+ + \sqrt{\nu_+^2 + 2\eta}}. \tag{2.10}$$

From Assumption 2.1, we have $0 < P_-^* < C - rI_-$. Also the optimal exit trigger price P_-^* does not depend on the initial price level. Therefore we can define the optimal value function by

$$\begin{aligned} V_+^*(P) &= V_+(P, P_-^*) \\ &= \frac{P}{r-\mu} - \frac{C}{r} + \left(\frac{\frac{C}{r} - I_-}{\nu_+ + \sqrt{\nu_+^2 + 2\eta}}\right)^{\nu_+ + \sqrt{\nu_+^2+2\eta}} \left(\frac{r-\mu}{P}(\nu_- + \sqrt{\nu_+^2 + 2\eta})\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \end{aligned} \tag{2.11}$$

From the definition, we can easily check that

$$V_+^*(P_-^*) = V_+(P_-^*, P_-^*) = -I_-.$$

Let us define the optimal value function by $V_+^*(P) = V_+^*(P_-^*)$ for $P \leq P_-^*$. Then we can show the following property.

Lemma 2.3 *Under Assumption 2.1,*

$$V_+^*(P) \geq -I_-, \quad \forall \quad P \geq 0. \tag{2.12}$$

Proof. It is clear that (2.12) holds for $0 \leq P \leq P_-^*$. So we assume that $P \geq P_-^*$. From (2.3),

$$\begin{aligned} &\frac{\partial V_+(P, P_-^*)}{\partial P} \\ &= \frac{1}{r-\mu} + \left(\frac{P_-^*}{r-\mu} + I_- - \frac{C}{r}\right) \frac{\nu_- + \sqrt{\nu_+^2 + 2\eta}}{P} \left(\frac{P_-^*}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \\ &= \frac{1}{r-\mu}\left(1 - \left(\frac{P_-^*}{P}\right)^{\nu_+ + \sqrt{\nu_+^2+2\eta}}\right) \geq 0, \quad \text{if} \quad P \geq P_-^*. \end{aligned} \tag{2.13}$$

Hence we have

$$V_+(P, P_-^*) \geq V(P_-^*, P_-^*) = V^*(P_-^*) = -I_-, \quad \forall \quad P \geq P_-^*. \quad \square$$

Now we shall show the optimality of the exit strategy which is given by the first hitting time for P_-^*.

Theorem 2.4 *Under Assumption 2.1,*

$$\sup_{0 \leq \tau} E_P\left[\int_0^\tau e^{-rt}(P_t - C)dt - e^{-r\tau} I_-\right] = E_P\left[\int_0^{\tau_-^*} e^{-rt}(P_t - C)dt - e^{-P\tau_-^*} I_-\right] = V_+^*(P), \tag{2.14}$$

where

$$\begin{aligned} \tau_-^* &= \inf\{t \geq 0; P_t \leq P_-^*\}, \\ P_-^* &= (r-\mu)\left(\frac{C}{r} - I_-\right) \frac{\nu_- + \sqrt{\nu_+^2 + 2\eta}}{\nu_+ + \sqrt{\nu_+^2 + 2\eta}} \in (0, C - rI_-). \end{aligned}$$

Proof. Let $Y_t \triangleq e^{-rt}V_+^*(P_t \vee P_-^*)$. From (2.8) and Ito's lemma,

$$dY_t = \begin{cases} e^{-rt}V_t^{*'}(P_t)P_t\sigma dW_t - e^{-rt}(P_t - C)dt, & \text{if } P_t > P_-^*, \\ -rY_t dt, & \text{if } P_t < P_-^*. \end{cases}$$

Then from the generalized Ito's Lemma [7],

$$\begin{aligned} dY_t &= [e^{-rt}V_+^{*'}(P_t)P_t\sigma dW_t - e^{-rt}(P_t - C)dt]1\{P_t > P_-^*\} \\ &\quad -rY_t dt 1\{P_t < P_-^*\} + e^{-rt}V_+^{*'}(P_-^* + 0)d\Lambda_t(P_-^*), \end{aligned} \tag{2.15}$$

where $\Lambda_t(x)$ denotes the local time of Semi-martingale P_t. Rewritting (2.15) in the stochastic integral form, we have

$$\begin{aligned} Y_t = Y_0 &+ \int_0^t e^{-ru}V_+^{*'}(P_u)P_u\sigma 1\{P_u > P_-^*\}dW_u - \int_0^t e^{-ru}(P_u - C)1\{P_u > P_-^*\}du \\ &- \int_0^t rY_u 1\{P_u < P_-^*\}du + \int_0^t e^{-ru}V_+^{*'}(P_-^* + 0)d\Lambda_u(P_-^*). \end{aligned}$$

Then

$$\begin{aligned} &Y_t + \int_0^t e^{-ru}(P_u - C)du \\ &= Y_0 + \int_0^t e^{-ru}V_+^{*'}(P_u)P_u\sigma 1\{P_u > P_-^*\}dW_u + \int_0^t e^{-ru}(P_u - C)1\{P_u \le P_-^*\}du \\ &\quad - \int_0^t rY_u 1\{P_u < P_-^*\}du + \int_0^t e^{-ru}V_+^{*'}(P_-^* + 0)d\Lambda_u(P_-^*) \\ &= Y_0 + \int_0^t e^{-ru}V_+^{*'}(P_u)P_u\sigma 1\{P_u > P_-\}dW_u + \int_0^t e^{-ru}(rI_- + P_u - C)1\{P_u < P_-^*\}du \\ &\quad + \int_0^t e^{-ru}(P_-^* - C)1\{P_u = P_-^*\}du + \int_0^t e^{-ru}V_+^{*'}(P_-^* + 0)d\Lambda_u(P_-^*) \\ &\le Y_0 + \int_0^t e^{-ru}V_+^{*'}(P_u)P_u\sigma 1\{P_u > P_-\}dW_u + \int_0^t e^{-ru}V_+^{*'}(P_-^* + 0)d\Lambda_u(P_-^*) \\ &= Y_0 + \int_0^t e^{-ru}V_+^{*'}(P_u)P_u\sigma 1\{P_u > P_-\}dW_u. \end{aligned}$$

Here the inequality follows from $P_-^* < C - rI_-$ and the last equality follows from $V_+^{*'}(P_-^* + 0) = 0$. Thus for any stopping time τ,

$$E_P[Y_\tau + \int_0^\tau e^{-rt}(P_t - C)dt] \le Y_0 + E_P[\int_0^\tau e^{-ru}V_+^{*'}(P_t)P_t\sigma 1\{P_t > P_-^*\}dW_t].$$

Furthermore from the uniform integrability of the stochastic integral,

$$E_P[\int_0^\tau e^{-ru}V_+^{*'}(P_t)P_t\sigma 1\{P_t > P_-\}dW_t] = 0.$$

Then the following inequality holds for any stopping time τ.

$$E_P[e^{-r\tau}V_+^*(P_\tau) + \int_0^\tau e^{-r\tau}(P_t - C)dt] \le V_+^*(P). \tag{2.16}$$

From Lemma 2.3, $V_+^*(P) \geq -I_-$. This together with (2.16) yields,

$$\begin{aligned} & E_P[\int_0^\tau e^{-rt}(P_t - C)dt - e^{-r\tau} I_-] \\ \leq \ & E_P[\int_0^\tau e^{-rt}(P_t - C)dt + e^{-r\tau} V_+^*(P_\tau)] \\ \leq \ & V_+^*(P) = E_P[\int_0^{\tau_-^*} e^{-rt}(P_t - C)dt - e^{-r\tau_-^*} I_-]. \end{aligned}$$

Since τ is arbitrary, this implies

$$\sup_\tau E_P[\int_0^\tau e^{-rt}(P_t - C)dt - e^{-r\tau} I_-] \leq V_+^*(P). \tag{2.17}$$

On the other hand, from the definition,

$$\sup_\tau E_P[\int_0^\tau e^{-rt}(P_t - C)dt - e^{-r\tau} I_-] \geq V_+^*(P) = E_P[\int_0^{\tau_-^*} e^{-rt}(P_t - C)dt - e^{-r\tau_-^*} I_-]. \tag{2.18}$$

From (2.17) and (2.18), we arrive at (2.14). □

3 Entry-Exit Problem

Next we shall generalize the flexibility of the investment for the production plant so that the investor can decide not only the exiting timing but also the entering timing to the production activity. At time 0, the project is inactive and the production state is $-$. Our problem is to derive the entering-exiting strategy (τ_+, τ_-) which attains:

$$\sup_{0 \leq \tau_+ \leq \tau_-} E[-e^{-r\tau_+} I_+ + \int_{\tau_+}^{\tau_-} (P_t - C)e^{-rt}dt - e^{-r\tau_-} I_- | P_0 = P]. \tag{3.1}$$

To solve the problem, we consider the simple strategy that starts (stops, respectively) the production activity when the price process hits the activation(inactivation) trigger price P_+ (P_-). For the notational convenience, let

$$V_-(P, P_+, P_-) = E[-e^{-r\tau_{P_+}} I_+ + \int_{\tau_{P_+}}^{\tau'_{P_-}} (P_t - C)e^{-rt}dt - e^{-r\tau'_{P_-}} I_- | P_0 = P], \tag{3.2}$$

where

$$\begin{cases} \tau_{P_+} = \inf\{t; P_t \geq P_+\}, \\ \tau'_{P_-} = \inf\{t \geq \tau_{P_+}; P_t \leq P_-\}. \end{cases}$$

From the strong Markov property and Theorem 2.4, we can rewrite (3.1) as follows.

$$\sup_{0 \leq \tau_+} E[e^{-r\tau_+}(V_+^*(P_{\tau_+}) - I_+) | P_0 = P],$$

where $V_+^*(P)$ is defined by (2.11). Let $P \le P_t$ and

$$
\begin{aligned}
V_-^*(P,P_+) &= E[e^{-r\tau_{P_+}}(V_+^*(P_+)-I_+)|P_0=P], \\
\tau_{P_+} &= \inf\{t\ge 0;\ P_t \ge P_+\}.
\end{aligned}
$$

Since

$$\tau_{P_+} = \inf\{t;\ W_t+\theta_- t \ge x_+\}, \quad \text{where} \quad x_+ = \frac{1}{\sigma}\log\frac{P_+}{P} > 0,$$

we get

$$E_P[e^{-r\tau_{P_+}}] = e^{(\theta_- - \sqrt{\theta_-^2+2r})x_+} = \left(\frac{P_+}{P}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}. \tag{3.3}$$

Hence from (2.11) and (3.3),

$$
\begin{aligned}
&V_-^*(P,P_+) \\
&= (V_+^*(P_+)-I_+)E_P[e^{-r\tau_{P_+}}] \\
&= \left[\frac{P_+}{r-\mu}-\frac{C}{r}+\left(\frac{\frac{C}{r}-I_-}{\nu_+ + \sqrt{\nu_+^2+2\eta}}\right)^{\nu_+ + \sqrt{\nu_+^2+2\eta}}\left(\frac{r-\mu}{P_+}(\nu_- + \sqrt{\nu_+^2+2\eta})\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} - I_+\right] \\
&\quad \times \left(\frac{P_+}{P}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}} \\
&= \left[\begin{array}{l} \left(\frac{P_+}{r-\mu}-\frac{C}{r}-I_+\right)P_+^{\nu_- - \sqrt{\nu_+^2+2\eta}} + \left(\frac{\frac{C}{r}-I_-}{\nu_+ + \sqrt{\nu_+^2+2\eta}}\right)^{\nu_+ + \sqrt{\nu_+^2+2\eta}} \\ \times \left((r-\mu)(\nu_- + \sqrt{\nu_+^2+2\eta})\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} P_+^{-2\sqrt{\nu_+^2+2\eta}} \end{array}\right] P^{-\nu_- + \sqrt{\nu_+^2+2\eta}}.
\end{aligned} \tag{3.4}
$$

As shown earlier, equation (3.4) can be derived by the no arbitrage argument. Let us fix the entry trigger prices P_+ and denote the value function by $V_-(P) = V_-^*(P,P_+)$. Consider the portfolio of $-V_-'(P_t)$ products' stock, one unit of project investment in inactive state $V_-(P_t)$ and $V_-'(P_t)P_t - V_-(P_t)$ riskless asset. The total portfolio value X_t is 0. The return from the portfolio is :

$$
\begin{aligned}
dX_t &= -V_-'(P_t)(dP_t + P_t(r-\mu)dt) + dV_-(P_t) + (V_-'(P_t)P_t - V_-(P_t))rdt \\
&= [\frac{1}{2}V_-''(P_t)P_t^2\sigma^2 + \mu V_-'(P_t)P_t - V_-(P_t)r]dt.
\end{aligned}
$$

Here notice that we can obtain no profit from sales of product since the project's state is inactive. Then we get the following differential equation for the arbitrage free value function :

$$\frac{1}{2}\sigma^2P^2V_-''(P) + \mu PV_-'(P) - rV_-(P_t) = 0. \tag{3.5}$$

(3.5) is Euler type homogeneous differential equation whose general solution is given by :

$$V_-(P) = C_1P^{-\nu_- + \sqrt{\nu_+^2+2\eta}} + C_2P^{-\nu_- - \sqrt{\nu_+^2+2\eta}}. \tag{3.6}$$

By the basic property of the value function $V_-(P)$, we have the following boundary conditions.

$$\lim_{P\to 0}|V_-(P)| < \infty, \quad V_-(P_+) = V_+^*(P_+) - I_+,$$

where $V_+^*(\cdot)$ is given by (2.11). These conditions yield

$$C_2 = 0, \quad C_1 = (V_+^*(P_+) - I_+)P_+^{\nu_- - \sqrt{\nu_+^2+2\eta}}.$$

Thus we get the function $V_-(P)$ given by (3.4).

Next we shall consider the optimal trigger price for the entry-exit problem.

Theorem 3.1 *Under Assumption 2.1 and Condition*

$$\frac{\sigma^2}{2r} \cdot \frac{C - rI_-}{C + rI_+}\left(\nu_+ + 2\eta - \sqrt{\nu_+^2 + 2\eta}\right) < \left(\frac{1}{2}\left(1 - \frac{\nu_+}{\sqrt{\nu_+^2 + 2\eta}}\right)\right)^{\frac{1}{\nu_+ + \sqrt{\nu_+^2+2\eta}}}, \tag{3.7}$$

then

$$\begin{aligned} V_-^*(P) &\triangleq \max_{P_+ \geq P} V_-^*(P, P_-) \\ &= E_P[-e^{-r\tau_+^*}I_+ + \int_{\tau_+^*}^{\tau_-^{*\prime}} e^{-rt}(P_t - C)dt - e^{-r\tau_-^{*\prime}}I_-] \\ &= \begin{cases} V_-^*(P, P_+^*), & \text{if } P < P_+^*, \\ V_+^*(P) - I_+, & \text{if } P \geq P_+^*, \end{cases} \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \tau_+^* &= \inf\{t \geq 0 \ ; \ P_t \geq P_+^*\}, \\ \tau_-^{*\prime} &= \inf\{t \geq \tau_+^* \ ; \ P_t \leq P_-^*\}, \\ V_-^*(P, P_+^*) &= \left[\frac{1}{r-\mu}\frac{\nu_+ + \sqrt{\nu_+^2 + 2\eta}}{2\sqrt{\nu_+^2 + 2\eta}}P_+^{*\,\nu_+ - \sqrt{\nu_+^2+2\eta}} - \left(\frac{C}{r} + I_+\right)\frac{\nu_- + \sqrt{\nu_+^2 + 2\eta}}{2\sqrt{\nu_+^2 + 2\eta}}P_+^{*\,\nu_- - \sqrt{\nu_+^2+2\eta}}\right] \\ &\quad \times P^{-\nu_- + \sqrt{\nu_+^2+2\eta}}. \end{aligned} \tag{3.9}$$

P_-^* is given by (2.14) and P_+^* is the larger solution of the following equation :

$$\begin{aligned} f(x) &\triangleq \frac{-1}{r-\mu}(\sqrt{\nu_+^2 + 2\eta} - \nu_+)x^{\nu_+ + \sqrt{\nu_+^2+2\eta}} + (\sqrt{\nu_+^2 + 2\eta} - \nu_-)(\frac{C}{r} + I_+)x^{\nu_- + \sqrt{\nu_+^2+2\eta}} \\ &\quad -2\sqrt{\nu_+^2 + 2\eta}\left(\frac{\frac{C}{r} - I_-}{\nu_+ + \sqrt{\nu_+^2 + 2\eta}}\right)^{\nu_+ + \sqrt{\nu_+^2+2\eta}}\left((r-\mu)(\nu_- + \sqrt{\nu_+^2 + 2\eta})\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \\ &= 0, \end{aligned} \tag{3.10}$$

which always exists in $(C + rI_+, \infty)$ under Condition (3.7). If (3.7) is not satisfied, $V_-^*(P) = V_+^*(P) - I_+$.

Proof. From (3.4), we have

$$\frac{\partial V_-^*(P, P_+)}{\partial P_+}$$

$$
\begin{aligned}
&= \frac{P^{-\nu_-+\sqrt{\nu_+^2+2\eta}}}{P_+^{1+2\sqrt{\nu_+^2+2\eta}}}\left[\begin{array}{c}\frac{1}{r-\mu}(\nu_+-\sqrt{\nu_+^2+2\eta})P_+^{\nu_++\sqrt{\nu_+^2+2\eta}}-\left(\nu_--\sqrt{\nu_+^2+2\eta}\right)\left(\frac{C}{r}+I_+\right)P_+^{\nu_-+\sqrt{\nu_+^2+2\eta}} \\ -2\sqrt{\nu_+^2+2\eta}\left(\frac{\frac{C}{r}-I_-}{\nu_++\sqrt{\nu_+^2+2\eta}}\right)^{\nu_++\sqrt{\nu_+^2+2\eta}}\left((r-\mu)(\nu_-+\sqrt{\nu_+^2+2\eta})\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}\end{array}\right] \\
&= \frac{P^{-\nu_-+\sqrt{\nu_+^2+2\eta}}}{P_+^{1+2\sqrt{\nu_+^2+2\eta}}}f(P_+),
\end{aligned}
$$

where $f(x)$ is defined by (3.10). Since

$$
\begin{aligned}
f(0) &= 2\sqrt{\nu_+^2+2\eta}\left(\frac{\frac{C}{r}-I_-}{\nu_++\sqrt{\nu_+^2+2\eta}}\right)^{\nu_++\sqrt{\nu_+^2+2\eta}}\left((r-\mu)(\nu_-+\sqrt{\nu_+^2+2\eta})\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}} < 0, \\
f'(x) &= \frac{2}{\sigma^2}x^{\nu_--1+\sqrt{\nu_+^2+2\eta}}(C+rI_+-x)\begin{cases}\geq 0, & \text{if } 0\leq x\leq C+rI_+, \\ <0, & \text{if } x\geq C+rI_+,\end{cases}
\end{aligned}
$$

$f(\cdot)$ attains its maximum value at $x^*=C+rI_+$ which is given by

$$
\begin{aligned}
&f(x^*) \\
&= \frac{1}{r-\mu}\left(\nu_+-\sqrt{\nu_+^2+2\eta}\right)x^{*\,\nu_++\sqrt{\nu_+^2+2\eta}}-\left(\frac{C}{r}+I_+\right)(\nu_--\sqrt{\nu_+^2+2\eta})x^{*\,\nu_-+\sqrt{\nu_+^2+2\eta}}+f(0) \\
&= \frac{\sigma^2(\sqrt{\nu_+^2+2\eta}-\nu_-)(\sqrt{\nu_+^2+2\eta}-\nu_+)}{2r(r-\mu)} \\
&\quad\times\left[(C+rI_+)^{\nu_++\sqrt{\nu_+^2+2\eta}}-\frac{2\sqrt{\nu_+^2+2\eta}}{\sqrt{\nu_+^2+2\eta}-\nu_+}\left(\frac{\sigma^2}{2}\left(\frac{C}{r}-I_-\right)(\nu_++2\eta-\sqrt{\nu_+^2+2\eta})\right)^{\nu_++\sqrt{\nu_+^2+2\eta}}\right].
\end{aligned}
$$

Therefore $f(x^*)>0$ if and only if Condition (3.8) holds. In this case, $\max_{P_+\geq P}V_-^*(P,P_+)$ is attained at $P_+=P_-^*$ or $P_+=P$. Now we need the following property to show the optimality of P_+^*.

Lemma 3.2 *Under Assumption 2.1 and Condition (3.7),*

$$
V_+^*(P)-I_+<V_-^*(P),\quad \forall\ 0\leq P<P_+^*. \tag{3.11}
$$

Proof. Let $G(P)=V_+^*(P)-I_+-V_-^*(P,P_+^*)$. Then $G(P_+)=0$ and

$$
\begin{aligned}
&G'(P) \\
&= V_+^{*\prime}(P)-V_-^{*\prime}(P,P_+^*) \\
&= \frac{1}{r-\mu}+\frac{1}{2\sqrt{\nu_+^2+2\eta}}\left\{\begin{array}{c}\frac{1}{r-\mu}(\nu_++2\eta-\sqrt{\nu_+^2+2\eta})\left(\frac{P_+^*}{P}\right)^{\nu_++\sqrt{\nu_+^2+2\eta}}\left(1-\left(\frac{P_+^*}{P}\right)^{-2\sqrt{\nu_+^2+2\eta}}\right) \\ +\left(\frac{C}{r}+I_+\right)(2\nu_-+2\eta+1)\left(\frac{P_+^*}{P}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}\frac{1}{P}\left(1-\left(\frac{P_+^*}{P}\right)^{-2\sqrt{\nu_+^2+2\eta}}\right)\end{array}\right\} \\
&\geq \frac{1}{r-\mu}>0,\quad\text{for }\ P<P_+^*.
\end{aligned}
$$

Hence we have

$$
G(P)<G(P_+^*)=0,\qquad \forall P<P_+^*,
$$

which is equivalent to (3.11). □

On the other hand, if Condition (3.8) is not satisfied, $\max_{P_+ \geq P} V_-^*(P, P_+)$ is attained at $P_+ = P$ since $\frac{\partial V_-^*(P,P_+)}{\partial P_+} < 0$ for all $P_+ \geq P$. □

Now we shall show the optimality of the entry strategy which is given by the first hitting time for P_+^*.

Theorem 3.3 *Under Assumption 2.1,*

$$\begin{aligned} & \sup_{0 \leq \tau_+ \leq \tau'_-} E_P[-e^{-r\tau_+} I_+ + \int_{\tau_+}^{\tau'_-} (P_t - C)e^{-rt}dt - e^{-r\tau'_-} I_-] \\ = \ & E_P[-e^{-r\tau_+^*} I_+ + \int_{\tau_+^*}^{\tau_-^{*\prime}} e^{-rt}(P_t - C)dt - e^{-r\tau_-^{*\prime}} I_-] = V_-^*(P), \end{aligned} \tag{3.12}$$

where τ_+^*, $\tau_-^{*\prime}$ and $V_-^*(P)$ are given by (3.9).

Proof. From the strong Markov property of diffusion processes, we have

$$\sup_{0 \leq \tau_+ \leq \tau'_-} E_P[-e^{-r\tau_+} I_+ + \int_{\tau_+}^{\tau'_-} (P_t - C)e^{-rt}dt - e^{-r\tau'_-} I_-] = \sup_{0 \leq \tau_+} E_P[e^{-r\tau_+}(V_+^*(P_{\tau_+}) - I_+)].$$

Then equation (3.12) is satisfied if

$$\sup_{0 \leq \tau_+} E_P[e^{-r\tau_+}(V_+^*(P_{\tau_+}) - I_+)] = V_-^*(P). \tag{3.13}$$

We assume that Condition (3.7) is satisfied. Even if this condition is not satisfied, we can prove the result by mimicking the argument below for $V_-^*(P) = V_+^*(P) - I_+$. Let $Y_t \triangleq e^{-rt}V_-^*(P_t)$. From (3.5) and Ito's lemma,

$$dY_t = \begin{cases} e^{-rt}V_-^{*\prime}(P_t)P_t\sigma dW_t, & \text{if } P_t < P_+^*, \\ e^{-rt}V_+^{*\prime}(P_t)P_t\sigma dW_t - e^{-rt}(P_t - C)dt + re^{-rt}I_+dt, & \text{if } P_t > P_+^*. \end{cases}$$

Then from the generalized Ito's lemma [7],

$$\begin{aligned} dY_t \ = \ & [e^{-rt}V_+^{*\prime}(P_t)P_t\sigma dW_t - e^{-rt}(P_t - C)dt]1\{P_t > P_+^*\} \\ & + e^{-rt}V_-^{*\prime}(P_t)P_t\sigma dW_t 1\{P_t > P_+\} + e^{-rt}(V_+^{*\prime}(P_+) - V_-^{*\prime}(P_+^*))d\Lambda_t(P_+^*). \end{aligned} \tag{3.14}$$

Rewritting (3.14) in the stochastic integral form, we get

$$\begin{aligned} Y_t \ = \ & Y_0 + \int_0^t e^{-ru}V_+^{*\prime}(P_u)P_u\sigma 1\{P_t > P_+^*\}dW_u - \int_0^t e^{-ru}(P_u - C)1\{P_t > P_+^*\}du \\ & + \int_0^t e^{-ru}V_-^{*\prime}(P_u)P_u\sigma 1\{P_t < P_+^*\}dW_u + \int_0^t e^{-ru}(V_+^{*\prime}(P_+ + 0) - V_-^{*\prime}(P_+ - 0))d\Lambda_u(P_+^*) \\ \leq \ & Y_0 + \int_0^t e^{-ru}V_+^{*\prime}(P_u)P_u 1\{P_u > P_+\}dW_u + \int_0^t e^{-ru}V_-^{*\prime}(P_u)P_u 1\{P_u > P_+^*\}dW_u \\ & + \int_0^t e^{-ru}(V_+^{*\prime}(P_+ + 0) - V_-^{*\prime}(P_+ - 0))d\Lambda_u(P_+^*) \\ = \ & Y_0 + \int_0^t e^{-ru}V_+^{*\prime}(P_u)P_u 1\{P_u > P_+\}dW_u + \int_0^t e^{-ru}V_-^{*\prime}(P_u)P_u 1\{P_u > P_+^*\}dW_u. \end{aligned}$$

Here the inequality follows from $P_+^* > C + rI_+$ and the last equality follows from $V_+^{*\prime}(P_+^*) = V_-^{*\prime}(P_+^*)$. Thus for any stopping time τ,

$$E_P[Y_\tau] \le Y_0 + E_P[\int_0^\tau e^{-ru}V_+^{*\prime}(P_u)P_u 1\{P_u > P_+\}dW_u + \int_0^\tau e^{-ru}V_-^{*\prime}(P_u)P_u 1\{P_u > P_+^*\}dW_u].$$

Furthermore from the uniform integrability of stochastic integrals,

$$E_P[\int_0^\tau e^{-ru}V_+^{*\prime}(P_u)P_u 1\{P_u > P_+\}dW_u + \int_0^\tau e^{-ru}V_-^{*\prime}(P_u)P_u 1\{P_u > P_+^*\}dW_u] = 0.$$

Then the following inequality holds for any stopping time τ.

$$E_P[e^{-r\tau}V_-^*(P_\tau)] \le V_-^*(P). \tag{3.15}$$

From Lemma 3.2, $V_+^*(P) - I_+ < V_-^*(P)$ for $0 \le P \le P_+^*$. This together with (3.8) and (3.15) yields

$$E_P[e^{-r\tau}(V_+^*(P_\tau) - I_+)] \le E_P[e^{-r\tau}V_-^*(P_\tau)] \le V_-^*(P). \tag{3.16}$$

Since τ is arbitrary, we get

$$\sup_{0 \le \tau_+} E_P[e^{-r\tau_+}(V_+^*(P_{\tau_+}) - I_+)] \le V_-^*(P). \tag{3.17}$$

On the other hand, from the definition,

$$\sup_{0 \le \tau_+} E_P[e^{-r\tau_+}(V_+^*(P_{\tau_+}) - I_+)] \ge E_P[e^{-r\tau_+^*}(V_+^*(P_{\tau_+^*}) - I_+)] = V_-^*(P). \tag{3.18}$$

From (3.13), (3.17) and (3.18), we arrive at (3.12). □

4 Multiple Entry-Exit Problem

In this section, we consider the evaluation of the project when the investor can activate / inactivate the project many times under the constant entry and exit costs. At time 0, the project is active or inactive and the production state is $x \in \{+, -\}$. Our problem is to evaluate the sequential entering-exiting strategy $\{\tau_+^{(k)}, \tau_-^{(k)}\ ;\ k \ge 1\}$ which attains the maximum discounted present value :

$$\sup_{0 \le \cdots \le \tau_+^{(k)} \le \tau_-^{(k)} \le \tau_+^{(k+1)} \cdots} E_P[\sum_{k=1}^{\infty}\left(-e^{-r\tau_{P_+}^{(k)}}I_+ 1\{x = - \text{ or } k \ge 2\} + \int_{\tau_{P_+}^{(k)}}^{\tau_{P_-}^{(k)}} e^{-rt}(P_t - C)dt - e^{-r\tau_{P_-}^{(k)}}I_-\right)]. \tag{4.1}$$

Especially we consider the sequential simple strategy that starts (stops, respectively) the production activity when the price process hit the activation (inactivation) trigger price P_+ (P_-). For the notational convenience, let

$$V_+(P; P_+, P_-) \quad = \quad E_P[\int_0^{\tau_{P_-}^{(1)}} e^{-rt}(P_t - C)dt - e^{-r\tau_{P_-}^{(1)}}I_-$$

$$+\sum_{k=2}^{\infty} e^{-r\tau_{P_+}^{(k)}}\left(-I_+ + \int_{\tau_{P_+}^{(k)}}^{\tau_{P_-}^{(k)}} e^{-r(t-\tau_{P_+}^{(k)})}(P_t - C)dt - e^{-r(\tau_{P_-}^{(k)}-\tau_{P_+}^{(k)})}I_-\right)],$$

$$V_-(P;P_+,P_-) = E_P[\sum_{k=1}^{\infty} e^{-r\tau_{P_+}^{(k)'}}\left(-I_+ + \int_{\tau_{P_+}^{(k)'}}^{\tau_{P_-}^{(k)'}} e^{-rt}(P_t - C)dt - e^{-r\tau_{P_-}^{(k)'}}I_-\right)],$$

where

$$\tau_{P_+}^{(k)} = \begin{cases} \inf\{t \geq \tau_{P_-}^{(k-1)} \ ; \ P_t \geq P_+\}, & \text{if } k \geq 2, \\ 0, & \text{if } k = 1, \end{cases}$$

$$\tau_{P_-}^{(k)} = \inf\{t \geq \tau_{P_+}^{(k)} \ ; \ P_t \leq P_-\}, \ \text{for } k \geq 1,$$

$$\tau_{P_+}^{(k)'} = \inf\{t \geq \tau_{P_-}^{(k-1)'} \ ; \ P_t \geq P_+\}, \ \text{for } k \geq 1,$$

$$\tau_{P_-}^{(k)'} = \begin{cases} \inf\{t \geq \tau_{P_+}^{(k)'} \ ; \ P_t \leq P_-\}, & \text{if } k \geq 1, \\ 0, & \text{if } k = 0. \end{cases}$$

Then we have the following value functions for this multiple entry-exit model.

Theorem 4.1 *Under Assumption 2.1,*

$$\begin{aligned} & V_+(P;P_+,P_-) \\ = \ & \frac{P}{r-\mu} - \frac{C}{r} - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \\ & + \left(\frac{P_-}{P}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}} \frac{\left(\frac{P_+}{P_-}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}}{1-\left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}}\left(\frac{P_+}{r-\mu} - \frac{C}{r} - I_+ - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P_+}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}\right) \\ = \ & \frac{P}{r-\mu} - \frac{C}{r} + \left(-\frac{P_-}{r-\mu} - I_- + \frac{C}{r} + V_-(P_-;P_+,P_-)\right)\left(\frac{P_-}{P}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} & V_-(P;P_+,P_-) \\ = \ & \frac{\left(\frac{P_+}{P}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}}{1-\left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}}\left(\frac{P_+}{r-\mu} - \frac{C}{r} - I_+ - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P_+}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}\right) \\ = \ & (V_+(P_+;P_+,P_-) - I_+)\left(\frac{P_+}{P}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}. \end{aligned} \tag{4.3}$$

Proof. By the definition,

$$\begin{aligned} & V_+(P;P_+,P_-) \\ = \ & E_P[\int_0^{\tau_{P_-}^{(1)}} e^{-rt}(P_t - C)dt - e^{-r\tau_{P_-}^{(1)}}I_-] \\ & + \sum_{k=2}^{\infty} E_P[e^{-r\tau_{P_+}^{(k)}}]E[-I_+ + \int_{\tau_{P_+}^{(k)}}^{\tau_{P_-}^{(k)}} e^{-r(t-\tau_{P_+}^{(k)})}(P_t - C)dt - e^{-r(\tau_{P_-}^{(k)}-\tau_{P_+}^{(k)})}I_- | P_{\tau_{P_+}^{(k)}}]. \end{aligned} \tag{4.4}$$

From the strong Markov property and (2.3),

$$E[-I_+ + \int_{\tau_{P_+}^{(k)}}^{\tau_{P_-}^{(k)}} e^{-r(t-\tau_{P_+}^{(k)})}(P_t - C)dt - e^{-r(\tau_{P_-}^{(k)}-\tau_{P_+}^{(k)})}I_- | P_{\tau_{P_+}^{(k)}}]$$

$$\begin{aligned}
&= E_{P_+}[-I_+ + \int_0^{\tau_{P_-}} e^{-rt}(P_t - C)dt - e^{-r\tau_{P_-}} I_-] \qquad (4.5)\\
&= \frac{P_+}{r-\mu} - \frac{C}{r} - I_+ - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P_+}\right)^{\nu_- + \sqrt{\nu_+^2 + 2\eta}}\\
&= A^*.
\end{aligned}$$

Also from (2.3),

$$E_P[\int_0^{\tau_{P_-}^{(1)}} e^{-rt}(P_t - C)dt - e^{-r\tau_{P_-}^{(1)}} I_-] = \frac{P}{r-\mu} - \frac{C}{r} - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}}. \qquad (4.6)$$

Using (2.5), (3.3) and the strong Markov property,

$$\begin{aligned}
E_P[e^{-r\tau_{P_+}^{(k)}}] &= E_P[e^{-r(\tau_{P_+}^{(k)} - \tau_{P_-}^{(k)} + \tau_{P_-}^{(k-1)} - \tau_{P_+}^{(k-1)} + \cdots + \tau_{P_+}^{(2)} - \tau_{P_-}^{(1)} + \tau_{P_-}^{(1)})}]\\
&= E_P[e^{-r\tau_{P_-}}]E_{P_-}[e^{-r\tau_{P_+}}]\left(E_{P_+}[e^{-r\tau_{P_-}}]E_{P_-}[e^{-r\tau_{P_+}}]\right)^{k-2}\\
&= \left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \left(\frac{P_+}{P_-}\right)^{\nu_- + 3\sqrt{\nu_+^2+2\eta}} \left(\frac{P_-}{P_+}\right)^{2k\sqrt{\nu_+^2+2\eta}}. \qquad (4.7)
\end{aligned}$$

Substituting (4.5) through (4.7) into (4.4), we get

(4.4)

$$\begin{aligned}
&= \frac{P}{r-\mu} - \frac{C}{r} - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P_+}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}}\\
&\quad + \left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \left(\frac{P_+}{P_-}\right)^{\nu_- + 3\sqrt{\nu_+^2+2\eta}} A^* \sum_{k=2}^{\infty} \left(\frac{P_-}{P_+}\right)^{2k\sqrt{\nu_+^2+2\eta}}\\
&= \frac{P}{r-\mu} - \frac{C}{r} - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} + \frac{\left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \left(\frac{P_+}{P_-}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}}{1 - \left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}} A^*,
\end{aligned}$$

which implies the first equality of (4.2). By the same way,

$$\begin{aligned}
&V_-(P; P_+, P_-)\\
&= \sum_{k=1}^{\infty} E_P[e^{-r\tau_{P_+}^{(k)'}}]E[-I_+ + \int_{\tau_{P_+}^{(k)'}}^{\tau_{P_-}^{(k)'}} e^{-r(t - \tau_{P_-}^{(k)'})}(P_t - C)dt - e^{-r(\tau_{P_-}^{(k)'} - \tau_{P_+}^{(k)'})} I_- | P_{\tau_{P_+}^{(k)'}}]. \qquad (4.8)
\end{aligned}$$

Using (2.5), (3.3) and the strong Markov property,

$$\begin{aligned}
E_P[e^{-r\tau_{P_+}^{(k)'}}] &= E_P[e^{-r(\tau_{P_+}^{(k)'} - \tau_{P_-}^{(k-1)'} + \tau_{P_-}^{(k-1)'} - \tau_{P_+}^{(k-1)'} + \cdots + \tau_{P_+}^{(1)'})}]\\
&= E_P[e^{-r\tau_{P_+}^{(1)'}}](E_{P_+}[e^{-r\tau_{P_-}}]E_{P_-}[e^{-r\tau_{P_+}}])^{k-1}\\
&= \left(\frac{P_+}{P}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}} \left(\frac{P_-}{P_+}\right)^{2(k-1)\sqrt{\nu_+^2+2\eta}}. \qquad (4.9)
\end{aligned}$$

Substituting (4.5) and (4.9) into (4.8), we get

$$E[-I_+ + \int_{\tau_{P_+}^{(k)'}}^{\tau_{P_-}^{(k)'}} e^{-r(t - \tau_{P_-}^{(k)'})}(P_t - C)dt - e^{-r(\tau_{P_-}^{(k)'} - \tau_{P_+}^{(k)'})} I_- | P_{\tau_{P_+}^{(k)'}}]$$

$$= \left(\frac{P_+}{P}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}} \left(\frac{P_-}{P_+}\right)^{-2\sqrt{\nu_+^2+2\eta}} A^* \sum_{k=1}^{\infty} \left(\frac{P_-}{P_+}\right)^{2k\sqrt{\nu_+^2+2\eta}}$$

$$= \frac{\left(\frac{P_+}{P}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}}{1 - \left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}} A^*,$$

which implies the first equality of (4.3). The second equalities of (4.2) and (4.3) can be derived from the first equalities. □

Corollary 4.2 *$V_-(P; P_+, P_-)$ and $V_+(P; P_+, P_-)$ has the following relationship.*

$$\begin{cases} V_-(P_+; P_+, P_-) + I_+ &= V_+(P_+; P_+, P_-), \\ V_+(P_-; P_+, P_-) + I_- &= V_-(P_-; P_+, P_-). \end{cases}$$

Proof. From (4.2),

$$\begin{aligned}
&V_+(P_+; P_+, P_-) \\
&= \frac{P_+}{r-\mu} - \frac{C}{r} - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P_+}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \\
&\quad + \left(\frac{P_-}{P_+}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \frac{\left(\frac{P_+}{P_-}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}}{1 - \left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}} \left(\frac{P_+}{r-\mu} - \frac{C}{r} - I_+ - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P_+}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}}\right) \\
&= \left(1 - \left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}\right) V_-(P_+; P_+, P_-) + I_+ + \left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}} V_-(P_+; P_+, P_-) \\
&= V_-(P_+; P_+, P_-) + I_+.
\end{aligned}$$

The last equality follows from (4.3). By the same way from (4.3) and (4.2),

$$\begin{aligned}
&V_-(P_-; P_+, P_-) \\
&= \frac{\left(\frac{P_+}{P_-}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}}}{1 - \left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}} \left(\frac{P_+}{r-\mu} - \frac{C}{r} - I_+ - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\frac{P_-}{P_+}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}}\right) \\
&= V_+(P_-; P_+, P_-) + I_-. \quad \square
\end{aligned}$$

Next we shall show the optimal trigger prices to activate or inactivate the project. Here we study the value function for the active project, that is $V_+(P; P_+, P_-)$. From the first order condition for the optimality of P_+,

$$\begin{aligned}
&\frac{\partial V_+(P; P_+, P_-)}{\partial P_+} \\
&= \left(\frac{P_-}{P}\right)^{\nu_- + \sqrt{\nu_+^2+2\eta}} \left.\frac{\partial V_-(P_-; P_+, P_-)}{\partial P_+}\right|_{P=P_-}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{1}{P_+}\left(\frac{P_-}{P}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}}{1-\left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}}\left(\left(\nu_- + \sqrt{\nu_+^2+2\eta}\right)\left(\frac{P_+}{r-\mu} - \frac{C}{r} - I_+\right) - 2\sqrt{\nu_+^2+2\eta}\ \ V_-(P_+;P_+,P_-) + \frac{P_+}{r-\mu}\right) \\
&= 0.
\end{aligned}
$$

Also from the optimality of P_-,

$$
\begin{aligned}
&\frac{\partial V_+(P;P_+,P_-)}{\partial P_-} \\
&= \left(\frac{P_-}{P}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}} \left(\begin{array}{c} \frac{-\frac{1}{r-\mu}(\nu_+ + \sqrt{\nu_+^2+2\eta}) - \frac{1}{P_-}(I_- - \frac{C}{r})(\nu_- + \sqrt{\nu_+^2+2\eta})}{1-\left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}} \\ + \frac{\left(\frac{P_+}{P_-}\right)^{\nu_- - \sqrt{\nu_+^2+2\eta}} \frac{2}{P_-}\sqrt{\nu_+^2+2\eta}\left(\frac{P_+}{r-\mu} - \frac{C}{r} - I_+ - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\cdot\left(\frac{P_-}{P_+}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}\right)}{\left(1-\left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}\right)^2} \end{array} \right) \\
&= \frac{\frac{1}{P_-}\left(\frac{P_-}{P}\right)^{\nu_-+\sqrt{\nu_+^2+2\eta}}}{1-\left(\frac{P_-}{P_+}\right)^{2\sqrt{\nu_+^2+2\eta}}}\left(2\sqrt{\nu_+^2+2\eta}\ \ V_-(P_-;P_+,P_-) - \frac{P_-}{r-\mu} - (\nu_- + \sqrt{\nu_+^2+2\eta})\left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\right) \\
&= 0.
\end{aligned}
$$

Then we arrive at the following necessary conditions for the optimal trigger prices P_+^* and P_-^*.

$$
2\sqrt{\nu_+^2+2\eta}\ \ V_-(P_+;P_+,P_-) - \frac{P_+}{r-\mu} - \left(\frac{P_+}{r-\mu} - I_+ - \frac{C}{r}\right)\left(\nu_- + \sqrt{\nu_+^2+2\eta}\right) = 0, \tag{4.10}
$$

$$
2\sqrt{\nu_+^2+2\eta}\ \ V_-(P_-;P_+,P_-) - \frac{P_-}{r-\mu} - \left(\frac{P_-}{r-\mu} + I_- - \frac{C}{r}\right)\left(\nu_- + \sqrt{\nu_+^2+2\eta}\right) = 0. \tag{4.11}
$$

Here notice that the optimality conditions of P_+ and P_- for $V_-(P;P_+,P_-)$ are also result in (4.10) and (4.11). This property is consistent with the optimality of the entry-exit strategy which is expressed by the constant trigger prices P_+^*, P_-^*. In fact, we can prove the optimality of this stopping strategy by mimicking the argument shown in Sections 2 and 3, iteratively.

Finally we sketch how to get equations (4.2), (4.3) and the optimal trigger price conditions (4.10), (4.11) from the no arbitrage and *smooth pasting conditions* [3]. Let us fix the entry and exit trigger prices P_+, P_- and denote the value function by $V_+(P) = V_+(P;P_+,P_-)$ and $V_-(P) = V_-(P;P_+,P_-)$. From the arbitrage free condition for the active or inactive project, $V_+(P)$ and $V_-(P)$ must satisfy the differential equations (2.8) and (3.4). By the basic property of the value function $V_+(P)$, we have the following boundary conditions.

$$
\lim_{P\to\infty}\frac{|V_+(P)|}{P} < \infty, \quad V_+(P_-) = V_-(P_-) - I_-.
$$

Substituting this condition into the general solution (2.9), the we get (4.2) in the second form. The boundary conditions for $V_-(P)$ are given by

$$
\lim_{P\to 0}|V_+(P)| < \infty, \quad V_-(P_+) = V_+(P_+) - I_+.
$$

This together with (3.6) yields (4.3) in the second form. Furthermore the smooth pasting conditions for the optimal trigger prices are :

$$V'_+(P_+) = V'_-(P_+), \tag{4.12}$$

$$V'_+(P_-) = V'_-(P_-). \tag{4.13}$$

From equations (4.2) and (4.3), we can easily check that conditions (4.12) and (4.13) are actually corresponding to (4.10) and (4.11). Thus equations (4.2), (4.3), (4.10) and (4.11) give the analytical solution form for the valuation problem of entry-exit model which is solved by Dixit [2] numerically.

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