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Local Connectivity of the Julia Set of Real Polynomials (after Levin and van Strien)

東大数理 沈維孝 (Weixiao Shen)

The talk is about the paper of G. Levin & S. van Strien under the same title (preprint of Inst. for Math, State Univ. of New York at Stony Brook # 1995/5)

We consider the map $f(z) = z^l + c_1$ with l an even positive integer and $c_1 \in \mathbb{R}$. Denote $J(f)$ the Julia set of f. We want to show:

Th. $J(f)$ is locally connected if and only if $J(f)$ is connected.

Remark: 1. By a result of Douady & Hubbard, $J(f)$ is locally connected if f has a periodic orbit which is attracting or parabolic.

2. In quadratic case, by Yoccoz's result, if f is not infinitely renormalizable, then the Theorem above holds.

So we may be mostly interested in the infinitely renormalizable case. On the other hand, the argument in non-renormalizable case in the paper is much more complicated. So I would like to take the infinitely renormalizable case for an example to explain the Main method of the paper.

<u>Def</u>. Suppose V is a symmetric interval around the critical point $c=0$. $s\in\mathbb{N}$ $s\geq 2$. $f^s: V\to V$ is called a (real) renormalization if $f^k(V)\cap V=\phi$. $f^s(V)\subset V$. $f^s(\partial V)\subset\partial V$. $1\leq k<s$

So we can write $V=(v,-v)$ with $f^s(v)=v$.

If f is infinitely renormalization, we can find two sequences $s(n)$. $q(n)$ of positive integers. s.t. $q(n)\geq 2$. $s(n+1)=s(n)\cdot q(n)$ and a sequence of symmetric intervals V_n, with $f^{s(n)}: V_n\to V_n$ a renormalization. Since f has no wandering interval. $|V_n|\to 0$.

<u>Def</u>. Let D_0, D be two ~~polyr~~ topological discs. and D_0 is compactly contained in D. A proper map $R: D_0\to D$ is called polynomial-like.

$K(R)=\{z\,|\,R^k(z)$ well defined for $\forall k\in\mathbb{N}\}$ is called the filled-in Julia set of the polynomial-like mapping R.

$J(R)=\partial K(R)$ is called the Julia set of R.

The Main method is to construct a sequence of polynomial-like mappings $f^{s(n)}: \Omega_n'\to\Omega_n$ which are an extension of the renormalization $f^{s(n)}: V_n\to V_n$ such that Ω_n has a fixed shape (and hence the diameter of Ω_n is comparable to $|V_n|$). Such a polynomial-like mapping $f^{s(n)}: \Omega_n'\to\Omega_n$

has the same Julia set with the polynomial-like mapping obtained from the corresponding Yoccoz puzzle. It follows that some critical puzzle piece lies in Ω_n and hence the Julia set is locally connected at the critical point. Finally, by using some argument, we show the local connectivity at other points.

Next, let us see how to extend a real renormalization to a polynomial-like mapping with the size of the outer domain comparable to the length of the renormalizable interval.

Step 1 "real bound"

Let $f^s: V \to V$ be a renormalization, then ∂V are nice (a point x is called nice if $\{f^n(x): n \in \mathbb{N}\} \cap (x, -x) = \phi$)

Denote $D_V := \{z \mid \exists k \in \mathbb{N}\ f^k(z) \in V\}$

and for $\forall z \in D_V$. $k(z) := \min\{k \in \mathbb{N} \mid f^k(z) \in V\}$ is called the transfer time.

Define $R_V: D_V \to V \quad z \mapsto f^{k(z)}(z)$, which is called the first return map to V.

Since ∂V are nice, the transfer time $k(z)$ is constant on each component of D_V.

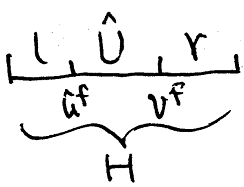
Denote $\hat{U}$ the component of D_V containing $c_1 = f(c)$, and ~~suppose~~ then $R_V \mid \hat{U} = f^{s-1}$.

Denote $V = (v, -v)$ with $f^s(v) = v$ and $\hat{U} = [\hat{u}^f, v^f)$ with $v^f = f(v)$.

Obviously $f^{s-1}: \hat{U} \to V$ is a diffeomorphism.

Let $H \supset \hat{U}$ be the maximal interval such that $f^{s-1}|H$ is monotone.

Let l, r be the left, right component of $H \setminus \hat{U}$.



Denote $L = f^s(l)$. $R = f^s(r)$

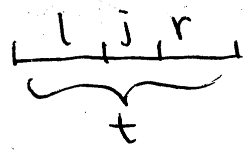
We have a "real bound"

Proposition ("real bound"): $|L| \geq 0.6\,|f(V)|$. $|R| \geq 0.5\,|f(V)|$

Remark: In the paper, the authors obtained $|L| \geq 0.6|f(V)|$ if f is not renormalizable of period $S/2$. However, the bound 0.6 still holds even though f is renormalizable of period $S/2$.

To obtain such a "real bound", the main tool is cross-ratio estimate and smallest interval argument.

Let $j \subset t$ be two intervals. l, r be the components of $t \setminus j$.



The cross-ratio $C(t, j)$ is defined by

$$C(t, j) = \frac{|t||j|}{|l||r|}$$

Lemma. Let $g: t \to \mathbb{R}$ be a monotone map and $Sg < 0$ where Sg is the Schwarzian derivative, i.e. $Sg = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2$

Then $C(gt, gj) \geq C(t, j)$.

We notice $Sf < 0$ and hence if $f^n|t$ is monotone then $Sf^n < 0$ on t

and hence $C(f^n t. f^n j) \geq C(t,j)$

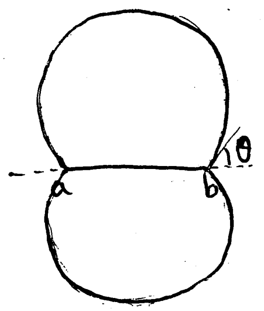
Then. for instance, we get a estimate $\frac{|r|}{|j|} \geq C^{-1}(t,j) \geq C^{-1}(f^n t. f^n j)$.

Step 2. "Complex bound"

For any interval T. denote $\mathbb{C}_T = \mathbb{C} \setminus (\mathbb{R} \setminus T)$

We notice if I is an interval with $c \notin I$. then $f: I \to f(I)$ is a diffeomorphism. and $(f|I)^{-1}$ has an extension $F: \mathbb{C}_{f(I)} \to \mathbb{C}_I$

Define $D(T;\theta) = \{ z \in \mathbb{C} \mid \angle azb < \theta \}$. where $T = (a,b)$. $\theta \in (0, \frac{\pi}{2}]$



D(T;θ)

i.e. $D(T;\theta)$ is the union of two discs which intersect the real line exactly in T and have an angle θ with the real line.

Fact: $\exists M(\theta) > 0$ depending only on θ s.t.

$$D(T;\theta) = \{ z \mid \rho_{\mathbb{C}_T}(z, T) < M(\theta) \}$$

where $\rho_{\mathbb{C}_T}$ denote the Poincaré metric on $\mathbb{C}_T$.

Hence by Schwarz lemma. $F(D(f(I);\theta)) \subset D(I;\theta)$

Denote $D_*(T) = D(T; \frac{\pi}{2})$. then $F(D_*(f(I)) \subset D_*(I)$

Since $f^{s-1}: \hat{U} \to V$ is a diffeomorphism, there exists $D' \subset D_*(\hat{U})$. such that $f^{s-1}: D' \to D_*(V)$ is conformal.

By an estimate of cross-ratio, we can prove:

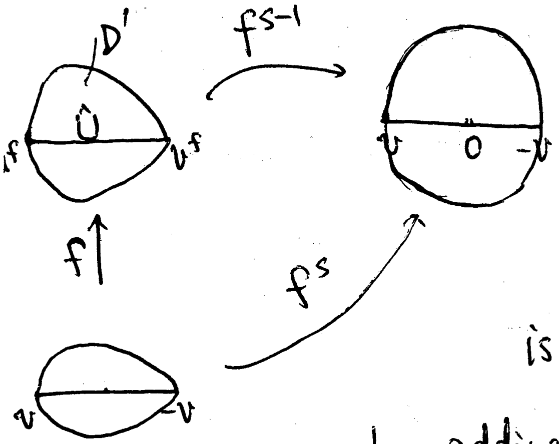
$$\frac{|\hat{u}^f - c_1|}{|v^f - c_1|} < K_\ell^*(y) = \frac{(1-1/\ell)^{\ell-1}}{\ell(1-y^{1/\ell})} \qquad \text{where } y = \frac{1}{1+\frac{|L|}{|f(V)|}}.$$

Since $|L| \geq 0.6|f(V)|$ ("real bound"),

$$\frac{|\hat{u}^f - c_1|}{|v^f - c_1|} < K_\ell^*(0.625) < 1 \qquad \text{when } \ell \geq 4.$$

And hence $f^{-1}(D_*(\hat{U})) \subset D_*(V)$, in particular $D = f^{-1}(D') \subset D_*(V)$.

and $\partial D \cap \partial D_*(V) = \partial V$



Then $f^s: D \to D_*(V)$ is "almost" polynomial-like. The only problem is the points v and $-v$.

Of course, we may assume that v is a repelling fixed point of f^s. And so by adding two small discs centered at ∂V to $D_*(V)$ we get a domain Ω, the component Ω' of $f^{-s}(\Omega)$ containing D is compactly contained in Ω. Then $f^s: \Omega' \to \Omega$ gives a polynomial-like mapping.

In fact, by more careful argument, we can even get a "complex bound". More precisely,

$\exists\ f^s: \Omega' \to \Omega$ polynomial-like mapping which is an extension of $f^s: V \to V$ such that the shape of Ω is fixed and the modulus of $\Omega \setminus \Omega'$ is bounded from below by a positive number depending only on $\deg f$.

The argument for $\ell=2$ is a little more complicated and we omit it here.

For non-renormalizable (or finitely renormalizable) case, we construct a generalized polynomial-like mapping defined below by extending some real map. After that, the proof is the same as in the infinitely renormalizable case. To make it possible, we need to assume that c is recurrent and $\omega(c)$ is minimal. The case when c is not recurrent is simple, but it seems that the argument for the case that $\omega(c) \ni c$ is not minimal in the paper is not sufficient.

Def. Let $D, D^0, D^1, \cdots, D^i$ be topological discs and the closure of D^j are mutually disjoint and contained in the interior of D. A holomorphic map $R: \cup D^j \to D$ is called ℓ-polynomial-like if

(1) $R|D^0$ is a proper map of degree ℓ.

(2) $R|D^j$ is conformal for $j=1, \cdots, i$.