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PERIODIC AND CRITICAL ORBITS OF ALGEBRAIC FUNCTIONS: TEICHMÜLLER SPACE OF AN ALGEBRAIC FUNCTION

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INTRODUCTION

An algebraic function is a plane algebraic curve $C \in \mathbb{C} \times \mathbb{C}$ or in $\mathbb{P} \times \mathbb{P}$. In this note we assume always that C is irreducible and the first and second projects of C are not constant maps.

Definition. *Two algebraic functions are equivalent* (respectively *topologically equivalent, quasi conformally equivalent* etc) if there exist projective linear (resp. topological, quasi conformal etc) equivalences ψ, ϕ of $\mathbb{P}$ such that

$$(\psi \times \phi)(C) = C'.$$

The main interest in this note is the topological rigidity defined as follows.

Definition. *An algebraic function C is topologically rigid* (respectively *weakly topologically rigid*) if C, C' are topologically equivalent then the equivalences ψ, ϕ are projective linear (resp. if C, C' are equivalent).

This problem was discussed in the papers [21], and already seen in a letter of Arnold to Il'yashenko [1] which motivated the Russian school to develop the theory of local complex dynamics independently of French works such as Ecalle[9].

An algebraic function is seen as a dynamical object as follows. We explain this by the following examples.

Example. *Let $C \subset \mathbb{P} \times \mathbb{P}$ be an ellipse defined by*

$$(x/a)^2 + (y/b)^2 = 1,$$

which is Riemann sphere embedded in $\mathbb{P} \times \mathbb{P}$. The first and second projections are branched coverings with two branch points. The monodromy of the projections extend to projective linear involutions f, g of the sphere, and the composite $g \circ f$ is also an involution.

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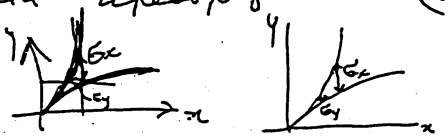
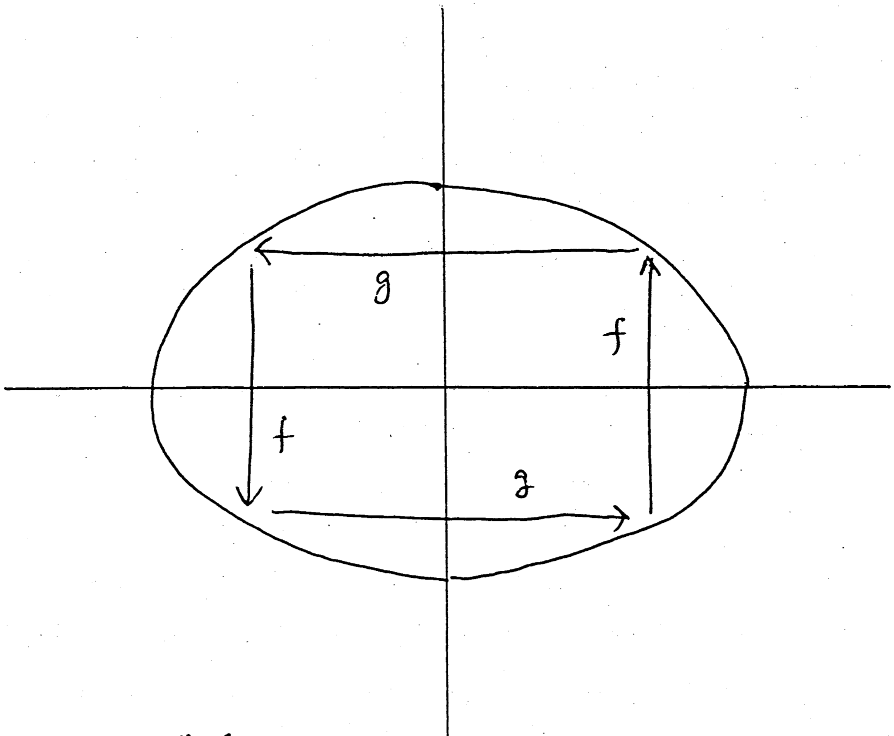
Юлик, меня вдруг очень остро интересовал вопрос о нормальной форме ростка $(\mathbb{C},0)\to(\mathbb{C},0)$ вида $t\mapsto t+t^2+\ldots$. Формальная н.ф. есть $t+t^2$.
Верно ли, что тут обычная расходимость?
И как с топологической приводимостью к н.ф.?
Я забыл совершенно — но Вы, по-моему, занимались (с Найшулем?) почти этим (или этим?).
Вопрос возникший в теории особенностей (приведение полукубической+... особенности преобразованиями $(X(x), Y(y))$.



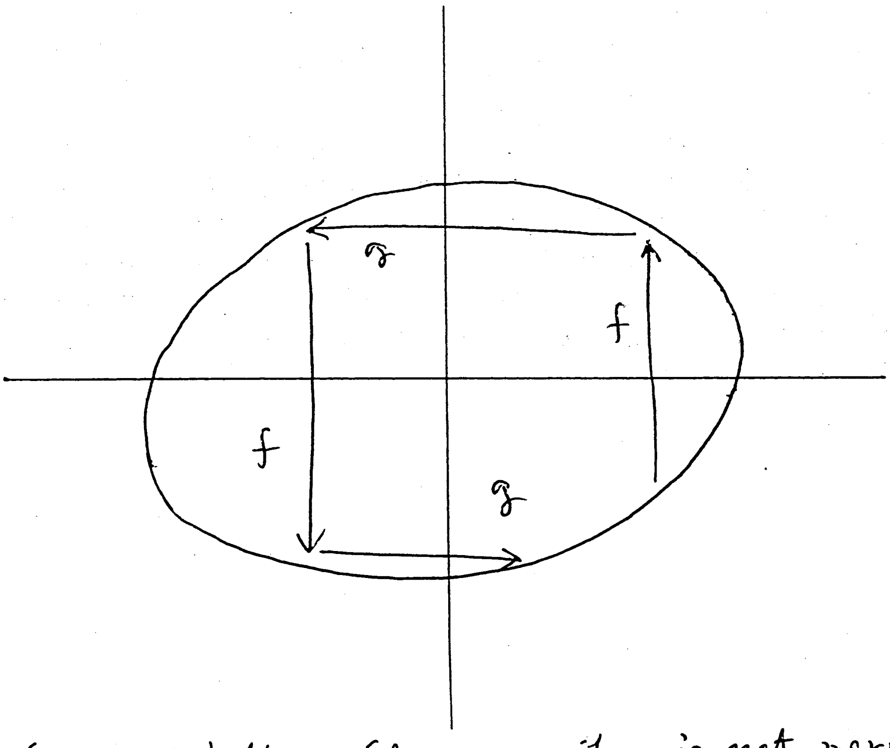
Здесь есть C^∞ приводимость к $(x+y)^2=(x-y)^3$, аналитическая же, видимо, неизвестна — и, кажется, отсутствует по изложенной выше причине (отображение полукубики в себя есть произведение двух инволюций, σx и σy). Если можно, оставьте мне ответ тут.

FIGURE 1. Copy of a letter from V. I. Arnold to Yu. S. Il'yashenko.

Copy of "Nonlinear Stokes Phenomena" by Il'yashenko. Advances in Soviet Math. Vol 14 AMS



$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. The composite $g \circ f$ is periodic.



Its perturbation. The composite is not periodic.

Next rotate the ellipse by an linear mapping with the fixed point the origin. Then the monodromy actions remain involutive, while, the composite $g \circ f$ is not involutive. The coordinate functions restricted to the quadratic curve is invariant functions of involutions acting on the Riemann sphere. And the classification of plane quadratic curves falls into that of the pairs of the involutions.

In general the monodromy actions of the first and second projections of a plane curve does not extend to a group action on the Riemann surface. So we introduce an alternative way to formulate the problem.

Definition. *The equivalence relation* $\sim$ on a plane algebraic curve $C \subset \mathbb{C} \times \mathbb{C}$ is generated by the following relations: $(x, y), (x', y') \in C$ are equivalent if $x = x'$ or $y = y'$.

Definition. *An orbit* $O(p)$ of a point $p \in C$ is the equivalence class of p. And a subset $K \subset C$ is invariant if it is a union of equivalence classes.

Clearly the orbit structure is topologically invariant. Our main interest in this note is

Question. *Assume generic orbits are dense. Then is C topologically rigid?*

Definition. *A point $p \in C$ is a critical point* if one of the followings holds.
1 C is non singular at p and the first or second projection is critical
2 C is singular at p

Definition. *The critical orbit of C* is the union of orbit of the critical points. And an algebraic function is critically finite if its critical orbit is finite.

Definition. *An algebraic function is an algebraic correspondence* if the source and target are identified. Two algebraic correspondences of $\mathbb{P}$ are equivalent if $\psi = \phi$.

In other words, an algebraic correspondence is a union of plane algebraic curve and the diagonal set $\Delta \subset \mathbb{P} \times \mathbb{P}$.

Example. *Assume C is defined by $(y - x^2 - c)(y - x) = 0$, which is the union of diagonal line $y = x$ and a parabola $y = x^2 + c$. By the equivalence relation $\sim$ the points (x, y) on the parabola are identified with those points (x, x) on the diagonal line. Therefore the first projection of the orbits are generated by the relations $x \sim x^2 + c$ and $x \sim -x$, which are the union of the forward orbit and its backward orbits. It is known that the cluster point set of any backward orbit is Julia set of the dynamics $x \to x^2 + c$.*

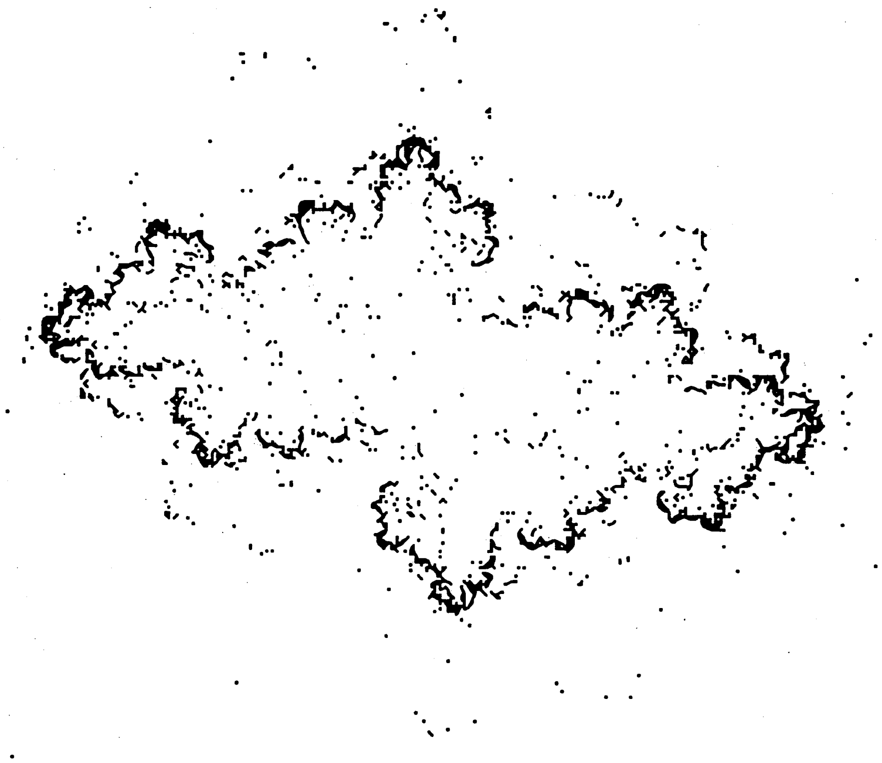
The following is a random orbit of the union of the diagonal set and a graph of the function $y = x^2 + 0.7 - 0.3i$, where $c = 0.7 - 0.3i$ and i stands for $\sqrt{-1}$.

The projection of a random orbit on the curve (Julia set)

$$(y - x^2 + 0.7 - 0.3i)(y - x) = 0$$

onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$.

After deformation of the defining equation $(y - x^2 + 0.7 - 0.3i)(y - x) = 0$, we can still see the Julia-like set.



The projection of a random orbit on the elliptic curve

$$(y - x^2 + 0.7 - 0.3i)(y - x) + 0.01 = 0$$

onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$.

Critically finite algebraic correspondences are classified by Bullet [6], and some other correspondences are studied by Bullet and Penrose [4].

Assume C is a critically finite algebraic function. Let $\pi : \mathbb{H} \to C -$ critical orbit be the universal covering. The composite $\pi_i \circ \pi$ with the i-th projection is the universal covering of the complement $\mathbb{P} - D_i$, D_i being π_i(critical orbit). Denote by G, H the fundamental group of the complement respectively for $i = 1, 2$. G, H act naturally on the upper half plane $\mathbb{H}$. Clearly the image of the quotient map $\mathbb{H} \to (\mathbb{P} - D_1) \times (\mathbb{P} - D_2) = \mathbb{P}/G \times \mathbb{P}/H$ is the complement of the critical orbit in the algebraic curve. The following theorem is easily seen.

Theorem 1. *The classification of critically finite algebraic functions is equivalent to the classification of the pair of Fuchsian groups G, H with only cusps and quotient spaces isomorphic to finitely punctured sphere by Möbius transformations. The indices of G, H over $G \cap H$ are respectively the degrees of the algebraic curve in y, x. An orbit of C corresponds to an orbit of the group K generated by G, H.*

Clearly the orbits are discrete if and only if G, H generate a discrete subgroup of $Aut(\mathbb{H})$. And then the group action is not topologically rigid, hence the algebraic function is not topologically rigid. In general the closure of the generated group is a Lie subgroup of $Aut(\mathbb{H})$. The only non topologically rigid connected Lie subgroup is the hyperbolic subgroup of dimension 1: hyperbolic, elliptic and parabolic. Therefore the quotient space by G , H are foliated annuli with finite modulus or foliated tori or a punctured disks, which are not isomorphic to a punctured sphere. Thus we obtain

Theorem 2. *Assume G, H generate a non discrete group, in other words, the orbits of C are not discrete. Then the algebraic function is topologically rigid and all orbits are dense.*

Next consider the non critically finite case. In this case the first and second projects of the critical orbit are countable but non finite sets. Let D_i denote the closure of the i-th project. Here we may apply the same argument as above by taking the universal coverings of the complement of the closures. The fundamental group of the complement is freely generated by countably many elements. The closure D_i contains the branch point set of the i-th projection, and if the singularity is complicated enough, D_i is a neighbourhood of the branch points. So the natural inclusion of the fundamental group of the complement of D_i to that of the branch point set may not be surjective.

From Fuchsian groups to Pseudo group actions.

The local structure of algebraic function at a critical point $p \in C$ is interpreted into a pseudogroup action on the Riemann surface C with the fixed point p.

Let p be a singular point of C of the second type: projections of C are singular at p and the local multiplicity of the first and second projections are respectively d, e. Let $t \in \mathbb{C}$ be a local coordinate of the curve centered at p. Then the monodromy actions f, g of the first and second projections are respectively order d, e and generate a pseudogroup of diffeomorphisms of open neighbourhoods of 0 in the t space. The orbit of p under this pseudogroup Γ_p generated by f, g is contained in the equivalence class $O(p)$ of the algebraic function.

Let G_p be the group of germs of elements of Γ_p.

Proposition 3. *If C is critically finite, the group G_p is commutative for all critical points p.*

Problem. *Assume G_p is commutative at a critical point p. Then is C critically finite.*

Definition. *The basin* $B_{\Gamma_p} \subset \mathbb{C}$ of p for the pseudogroup Γ_p ($B_p \subset C$ for an algebraic function C) is the set of those q such that the topological closure of the equivalence class ($O(q)$ of dynamics of C) contains p.

It is easy to see

Proposition 4. *If G_p is not commutative, the basins $B_{\Gamma_p} \subset B_p$ are neighbourhoods of p.*

On the basin B_p, the dynamics of C is seen by the pseudo group action Γ_p.

Example. *Here regard the singularity of plane curve in the letter of Arnold. The first and second projections are branched covering of multiplicity 2 at the origin. The monodromy actions of the projections are idempotent and generate a subgroup $Aut(\mathbb{C}, 0)$ of germs of holomorphic diffeomorphisms of parameter $t \in \mathbb{C}$. The classification of the cusp singularities is equivalent to that of the group with generators. This was studied by Voronin [26].*

Now recall a result on the structure of non solvable pseudo groups. Let Γ be a pseudo group of diffeomorhisms of open neighbourhoods of $0 \in \mathbb{C}$. Assume B_Γ is an open neighbourhood of 0.

Definition. *The separatrix* $\Sigma(\Gamma)$ for Γ is a closed real semianalytic subset of B_Γ, which possesses the following properties.

(1) $\Sigma(\Gamma)$ is invariant under Γ and smooth off p,

(2) The germ of $\Sigma(\Gamma)$ at p is holomorphically diffeomorphic to a union of $0 \in \mathbb{C}$ and some branches of the real analytic curve $\text{Im } z^k = 0$ for some k,

(3) Any orbit is dense or empty in each connected component of $B_\Gamma - \Sigma(\Gamma)$,

(4) Any orbit is dense or empty in each connected component of $\Sigma(\Gamma) - p$.

Local separatrix theorem [22]. *If a pseudogroup Γ is non-solvable, then the basin B_Γ is a neighbourhood of $0 \in \mathbb{C}$ and Γ admits the separatrix $\Sigma(\Gamma)$.*

The above density of orbits propagates to the basin B_p of the algebraic function.

Definition. *The separatrix* Σ_p of a critical point of an algebraic function $p \in C$ is a closed real semianalytic subset of B_p, which possesses the following properties.

(1) Σ_p is invariant under the dynamics of C and smooth off p,

(2) The germ of Σ_p at p is empty or holomorphically diffeomorphic to a union of $0 \in \mathbb{C}$ and some branches of the real analytic curve $\text{Im } z^k = 0$ for some k,

(3) Any orbit is dense or empty in each connected component of $B_p - \Sigma_p$,

(4) Any orbit is dense or empty in each connected component of $\Sigma_p - p$.

From the local separatrix theorem we obtain immediately

The separatrix Theorem. *Given an algebraic function and a non solvable critical point $p \in C$, there exists the separatrix $\Sigma_p \subset B_p$.*

However the global structure of the separatrix is not known. One of the most important problems would be

Problem. *Study the boundary of the basin $B_p \subset C$ (if $\bar{B}_p = C$).*

Finally we give the only known example of a plane curve singularity for which the local dynamics is solvable.

Example. *Let $a, b \in \mathbb{C}$ be distinct and p, q positive and coprime integers. Let $X(t) = (t-a)^p, Y(t) = (t-b)^q$. The image of the map $C(t) = (X(t), Y(t)) : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ is an algebraic curve with only critical point at ∞ ($= \infty \times \infty$). At the ∞ the first and second projections are respectively p and q sheeted branched coverings. The monodromy actions are periodic of order p, q and generate a solvable group of length 2.*

Proof of local separatrix theorem and Ergodicity.

Let Γ be a pseudogroup of diffeomorphisms $f : U_f, 0 \to f(U_f), 0$ of open neighbourhoods of the origin in $\mathbb{C}$. Assume that the germ Γ_0 of Γ is non-solvable. Then Γ contains diffeomorphisms f, g with Taylor expansions

$$f(z) = z + az^{i+1} + \cdots, \quad g(z) = z + bz^{j+1} + \cdots, \qquad a, b \neq 0,\ i < j.$$

and the commutator

$$[f, g](z) = z + cz^{k+1} + \cdots, \qquad c \neq 0, j < k.$$

Let $\lambda_n = n^{(j-i)/i}$. Define the vector field χ on the set (basin of f) B_f of those z for which $f^{(n)}(z) \to 0$ as $n \to \infty$ by

$$\chi = \lim_{n \to \infty} \lambda_n \left\{ f^{(-n)} g f^{(n)} - id \right\} \partial / \partial z$$

Define the vector field ζ on $B_g - 0$ similarly replacing f, g with g and another h.

Theorem 5. *The vector field χ is invariant under df and induces a linear vector field on each Ecalle-Voronin cylinder for f.*

Similarly, the vector field ζ induces a linear vector field on each cylinder of g.

It is seen that

Lemma 6. *$[\chi, \zeta]$ is non constant.*

By construction of χ, ζ we obtain

Theorem 7. *The set of uniform convergence limits on any compact set (Geometric limit) of Γ contains the real flow of χ. More precisely, the sequence $f^{(-n)} g^{(m)} f^{(n)}$ as $m, n \to \infty$ converges to the time $t \in \mathbb{R}$ map of χ choosing m, n in a certain manner.*

Definition. *Let Γ' be a pseudogroup. We say that Γ and Γ' are topologically equivalent* (respectively *holomorphically equivalent*) if there exists a homeomorphism (resp. holomorphic diffeomorphism) $h : U, 0 \to h(U), 0$ of open neighbourhoods of the origin such that $U_f \subset U, U_g \subset h(U)$ for $f \in \Gamma, g \subset \Gamma'$ and a bijection $\phi : \Gamma \to \Gamma'$, which induces a group isomorphism of Γ_0 to Γ'_0 such that $U_{\phi(f)} = h(U_f)$ and $h \circ f = \phi(f) \circ h$ hold for $f \in \Gamma$. We call h a *linking homeomorphism* (resp. *linking diffeomorphism*).

Corollary 8. *If χ, ζ are $\mathbb{R}$-linearly independent at a $z \in B_f \cap B_g - 0$, any orbit is dense or empty on a neighbourhood of z.*

From which a part of Local separatrix theorem follows. Also From Theorem 7 it follows

Corollary 9. *The closure of Γ-orbits are invariant under the flows of the vector fields χ, ζ. The real vector fields χ, ζ defined above are real-time-preservingly invariant under topological equivalence of pseudo groups.*

From which we obtain

Topological rigidity theorem [22]. *Assume that pseudogroups Γ, Γ' are topologically equivalent and the germs Γ_0, Γ'_0 are non-solvable. Then the restriction of the linking homeomorphism $h : B_\Gamma \to B_{\Gamma'}$ is a holomorphic (respectively anti-holomorphic) diffeomorphism if h is orientation preserving (resp. reversing).*

Now by using the above method, we can prove the separatrix theorem and even a more strong theorem as follows.

Definition. *The pseudogroup Γ is finitely ergodic* if any subset $A \subset U$ of a component $U \subset B_\Gamma - \Sigma_\Gamma$ has a positive measure, then $O(A)$ has full measure in $O(U)$. Similarly the dynamics is ergodic on the basin B_p is a subset A of a component U of $B_p - \Sigma_p$ has positive measure then $O(A)$ has full measure in $O(U)$.

Ergodicity Theorem. *If a pseudo group Γ_p is non solbavle, then the dynamics is finitely ergodic on the basin B_p.*

Teichmüller space of an algebraic function.

Teichmüller space is a canonical subject in the study of the complex structure of holomorphic dynamics as we as Riemann surfaces as introduced in [20]. Here we introduce the definition.

Definition. *A periodic orbit of length ℓ* of an algebraic function is a chain of points $p_i = (x_i, y_i), i = 1, \dots, \ell$ such that $x_i = x_{i+1}$ or $y_i = y_{i+1}$ for all i and non trivial: it does not retract to a point by replacing the subwords $p_1 p_2 \cdots p_k$ with equal x coordinate (or y) coordinate. These p_i are called *periodic points.*

Definition. *Denote by the closure of set of periodic points and critical orbit by $\hat{J}$.*

By definition $\hat{J}$ is a closed invariant set, and on the complement $C - \hat{J}$ the first and second projections restrict to coverings onto open subset $\mathbb{P} - \pi(\hat{J})$ of the first and second $\mathbb{P}$.

Let O be one of connected components of $C - \hat{J}$. The dynamics on O is defined as follows. Let $U_i \subset \mathbb{P}, U_j \subset \mathbb{P}, i, j = 1, \dots,$ be the first and second projects of the orbit $O(O) = \cup O_k$ of O.

Definition. *The fundamental groups* $G_i = \pi(U_i), H_j = \pi(V_j)$ *act naturally on the universal cover* $\mathbb{H}_k$ *of* O_k*. Also deck transformations of the first and second projections lift to isomorphisms to identify those* $\mathbb{H}_k$*, which are unique up to* G_i, H_j*. Denote by* K *the Fuchsian group acting on an* $\mathbb{H}_k$ *generated by those* G_i, H_j *and the composites of the isomorphisms along cycles from* $\mathbb{H}_k$ *to itself. The group* K *is independent of* $\mathbb{H}_k$*.*

Definition. *A connected component* O_k *of* $C - \hat{J}$ *is a* ***discrete component*** if $K \subset Aut(\mathbb{H})$ is a discrete subgroup. The other components are called ***foliated components.***

To a non discrete component a similar argument to that for critically finite algebraic functions applies. If the closure of K is of dimension 2 or 3, the group action is topologically rigid. If of dimension 1, the orbits of the closure give a foliation of $\mathbb{H}$ by curves, and the components O_k are the quotients of $\mathbb{H}$ which are foliated by the closure of the orbits. Therefore those components are either annuli or the punctured disc, and the foliations are invariant under the rotations.

Theorem 10 [20].
1 The foliated discs (Siegel discs) are topologically rigid in weak sense.
2 The foliated annuli (Herman rings) are not topologically rigid: The Teichmüller space of the orbit of O_k *is*

$$Teich(\mathbb{H}, K) = \mathbb{H}$$

The definition of the Teichmüller space of the subgroup K is defined by McMullen-Sullivan [20].

First we give the definition of Teichmüller space of C. A quasi conformal mapping of algebraic functions C to C' is a product of quasi conformal mappings $\psi \times \phi$ of the sphere such that

$$(\psi \times \phi)(C) = C'$$

Denote by $Def(C)$ the set of quasi conformal mappings of C to C' (possibly C). The restriction of the Beltrami differencial of $\psi \times \phi$ to C is invariant under monodromy action of the first and second projections. By the fundamental theorem of Teichmüller theory, $Def(C)$ corresponds to the space $M_1(C)$ of invariant Beltrami differentials with essential sup-norm < 1.

Denote by $QC(C)$ the group of quasi conformal mappings of C to its self, and by $QC_0(C)$ the subgroup of those quasi conformal mappings, which are isotopic to identity relative to the ideal boundary. These groups acts naturally on $Def(C)$ by pull back by composite.

Definition. *Teichmüller space of* C *is*

$$Teich(C) = Def(C)/QC_0(C)$$

and the modular group is

$$Mod(C) = QC(C)/QC_0(C)$$

Clearly $Mod(C)$ acts on $Teich(C)$. The quotient $Teich(C)/Mod(C)$ is studied by McMullen-Sullivan [20] for rational functions.

The Teichmüller theory of algebraic functions is almost parallel to that of algebraic correspondence. To see this define an algebraic correspondence of $\mathbb{P}_x \cup \mathbb{P}_y$ by the union of $C \in \mathbb{P}_x \times \mathbb{P}_y$ and its transpose $C^t \subset \mathbb{P}_y \times \mathbb{P}_x$. It is easy to see the various notions coincide with each other: in fact Teichmüller space of the union as an algebraic correspondence is

$$Teich(C \cup C^t) = Teich(C) \times Teich(C^t) = Teich(C) \times Teich(C),$$

$Teich(C), Teich(C^t)$ being Thichmüller space of algebraic functions.

Theorem 11. *Let O_k be a connected component of $C - \hat{J}$. Then the Teichmüller space of the dynamics on the orbit of O_k is*

$$Teich(O(O_k)) = Teich(K),$$

where K is the Fuchsian group defined above.

Now we interpret a basic result in the paper [20].

Decomposition Theorem.

$$Teich(C) = M_1(\hat{J}) \times \Pi\ Teich(O_k^{fol}) \times \Pi\ Teich(O_\ell^{dis})$$

where $M_1(\hat{J})$ is the space of invariant Beltrami differentials on $\hat{J}$ and O_k^{fol} (O_ℓ^{dis}) runs over the set of all foliated (discrete) components of $C - \hat{J}$.

The second and third components are already described above, and Π denote the restricted product with bounded essential sup-norm.

Conjecture by McMullen-Sullivan [20]. *For a correspondense defined by a rational function, which is not covered by the dynamics $z \to nz$ on a tolus $\mathbb{C}/\Lambda$ by a semi conjugacy $p : \mathbb{C}/\Lambda \to \hat{\mathbb{C}}$ such that $p(-z) = p(z)$, an invariant Beltrami differential is trivial:*

$$M_1(\hat{J}) = 0$$

An invariant Beltrami defferential (invariant ellips field) on $\hat{J}$ defines a measurable line field by the long direction, which is a base of $M_1(\hat{J})$ over the space of invariant functions on $\hat{J}$. The invariant line field may be supported on an ergodic component of $\hat{J}$, which is invariant under the dynamics of C and does not intersect with the critical orbit. The above is equivalent to

No invariant line field Conjecture [20]. *For a correspondense defined by a rational function, which is not covered by the dynamics $z \to nz$ on a tolus $\mathbb{C}/\Lambda$, there exists no invariant measurable line field on $\hat{J}$.*

Although the conjecture remains open, our Topologically rigidity theorem gives an affirmative answer to this conjecture in some other cases. Namely

No invariant line field Theorem. *There is no invariant measurable line field on the basin of non solvable critical points.*

Here we give an elementary proof of the theorem. Loray [19], Belliart-Liousse-Loray [3] and Wirtz [14] proved

Density Theorem. *On the basin of a non solvable pseudo group* Γ, *periodic points are dense.*

We give the local density of periodic points at a point $q \in B_{\Gamma_p} - \Sigma(\Gamma_p)$. Define

$$\phi_{s,t} = \text{ext } -t'\chi \circ \text{ext } -s'\zeta \circ \text{ext } t\chi \circ \text{ext } s\zeta$$

with the holomorhic vector fields χ, ξ defined before, choosing a real (s', t') close to (s, t) so that $\phi_{s,t}(q) = q$. Approximate it by a sequence of elements f_i of Γ uniformly convergent on a compact neighbourhood of q. Since $\phi_{s,t}$ has a non real and non nutral derivative at generic point, f_i has a fixed point close to q with the derivative close to that of $\phi_{s,t}$ for any sufficiently large i. Clearly those fixed points of f_i are periodic points of Γ_p.

Corollary 12. *On the basin* B_p *of a non solvable critical point* $p \in C$ *of an algebraic function, periodic points are dense.*

Problem. *1 Find infinitely many periodic orbits in the basin of a non solvable prudo group.*
2 Find a periodic point in the basin of a non solvable prudo group, where the stabilizer is not $\mathbb{Z}$.

This argument shows also non existence of invariant line fields on the set of periodic points, which is dense in the basis B_p.
In order to show non existence of measurable invariant line field on B_p, we use

Tangential Ergodicity Theorem. *On the* S^1*-bundle* $PT\ (B_p - \Sigma_p)$, *the pseudo group* $d\Gamma$ *lifted from* Γ *is finitely ergodic.*

It is easy to see that this implies

Theorem 13. *All section of* $PT\ (B_p - \Sigma_p)$ *invariant under the dynamics of algebraic function* C *is not measurable.*

Similar argument may apply to a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Let J denote the Julia set of the function f. It is known (c.f. [7]) that $J \subset \hat{J}$ and $\hat{J} - J$ is of measure 0.
Recall the following classical result [7].

Theorem 14. *Julia set is (completely) invariant, and all invariant subsets are dense. On* J *periodic points are dense and* J *contains all repeling periodic points.*

And also

Theorem 15. *Let* $z \in J$ *and* U *be a neighbourhood. Then the forword image of* U *under* f^n *for an* $n \geq 0$ *contains* J.

Now we see

Proposition 16. *Assume the dynamics* df *restricted to the fiber* $PT\mathbb{C}_E$ *on an ergodic comoponent* E *of* J *is ergodic. Then there exists no invariant measurable line field on* E.

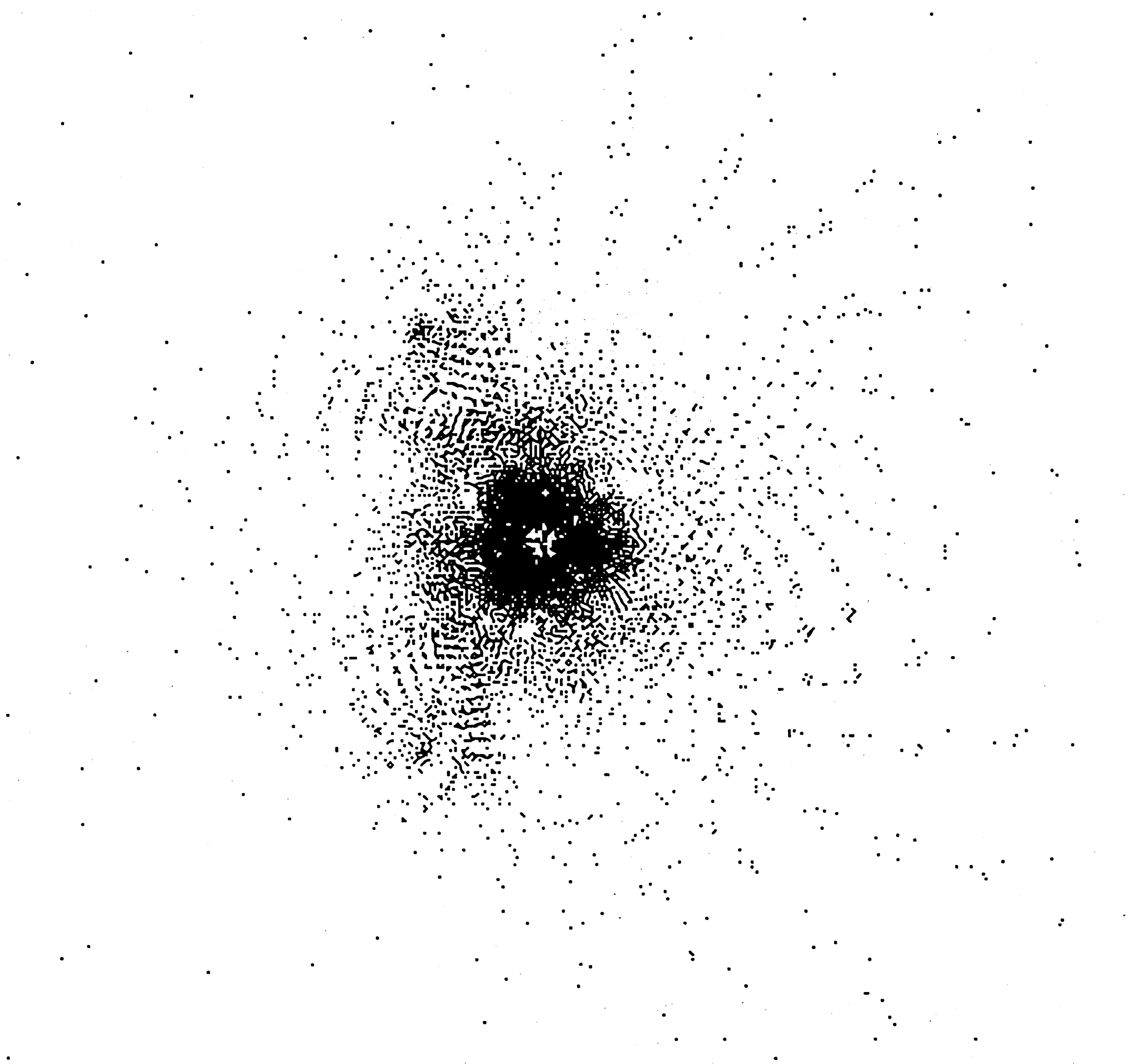
The tangential ergodicity is rather difficult to see, however we see

Proposition 17. *Assume there exists a periodic point $p \in J$ where the linear term of the return map $df^{(n)}(p)$ has the argument $\theta\pi$, θ being non integer. Then all invariant sections of $PT\hat{\mathbb{C}}$ defined on an invariant subset of J are everywhere discontinuous.*

For simplicity assume that there is a periodic point $z \in J$ and the linear term $df^n(z)$ has a irrational argument. Let U be a neighbourhood of a $p \in PT\hat{\mathbb{C}}_z$. Then the union $\bigcup_{m=1,\dots,k} df^{mn}(U)$ contains the fiber over J by the irrationality and the obove theorem on the dynamics on J. This implies that for any point $q \in PT\hat{\mathbb{C}}_J$, the itterated preimage $df^{-mn}(q)$ accumulates to p. By the density of the grand orbit $O(z)$ of z in J, we see that the grand orbit $O(p) \subset PT\hat{\mathbb{C}}$ is dense in $PT\hat{\mathbb{C}}_J$. This implies

Corollary 18. *If θ is irrational, then all (grand) orbits are dense on $PT\mathbb{C}_J$, and in particular a graph of an ingariant line field is dense in $PT\mathbb{C}_J$.*

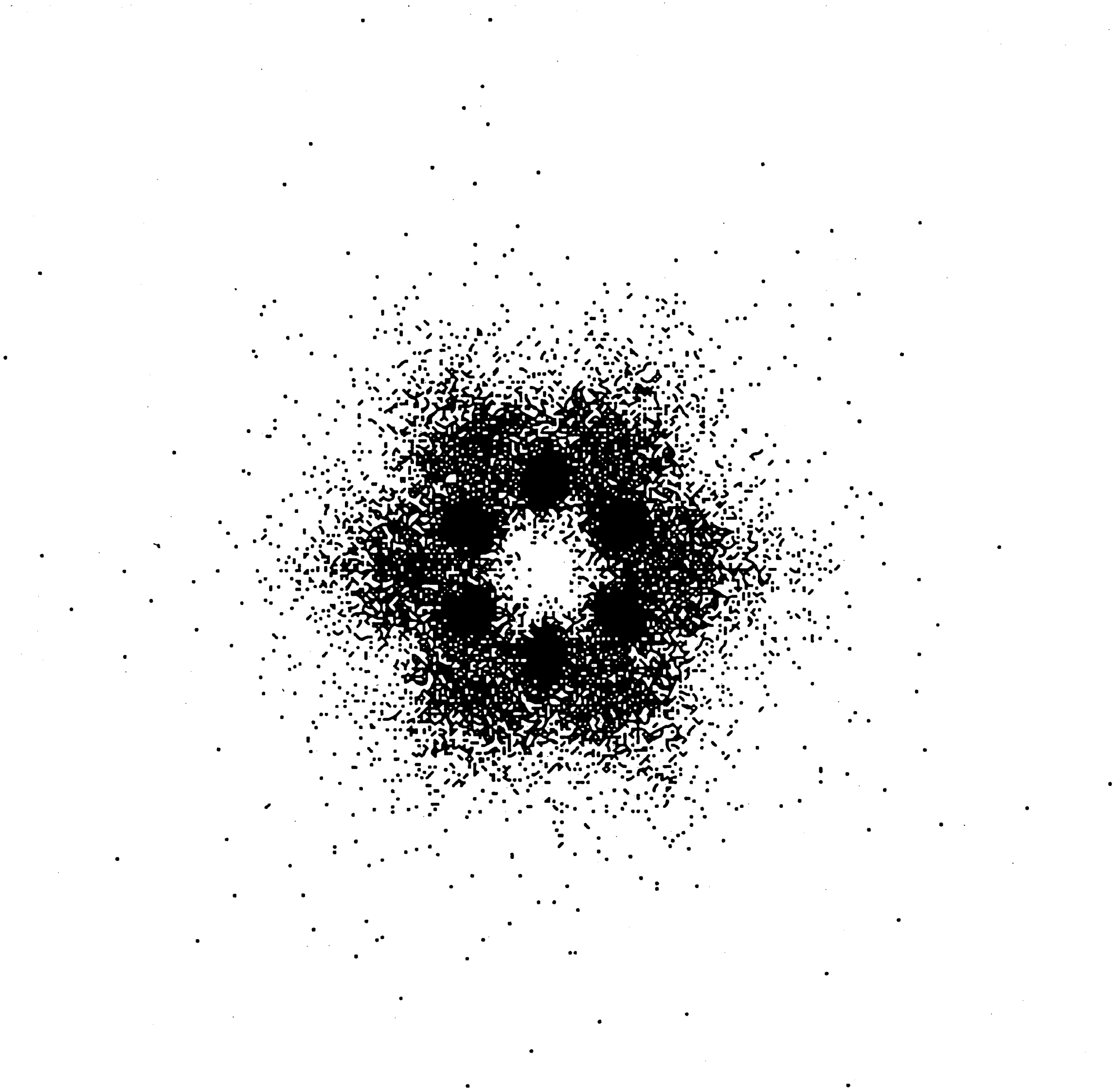
In the following we present some generic random orbits of the dynamics on the various algebraic curves.



The projection of an orbit on the curve

$$x^3 + x^2 + 0.1xy + y^2 + x = 0$$

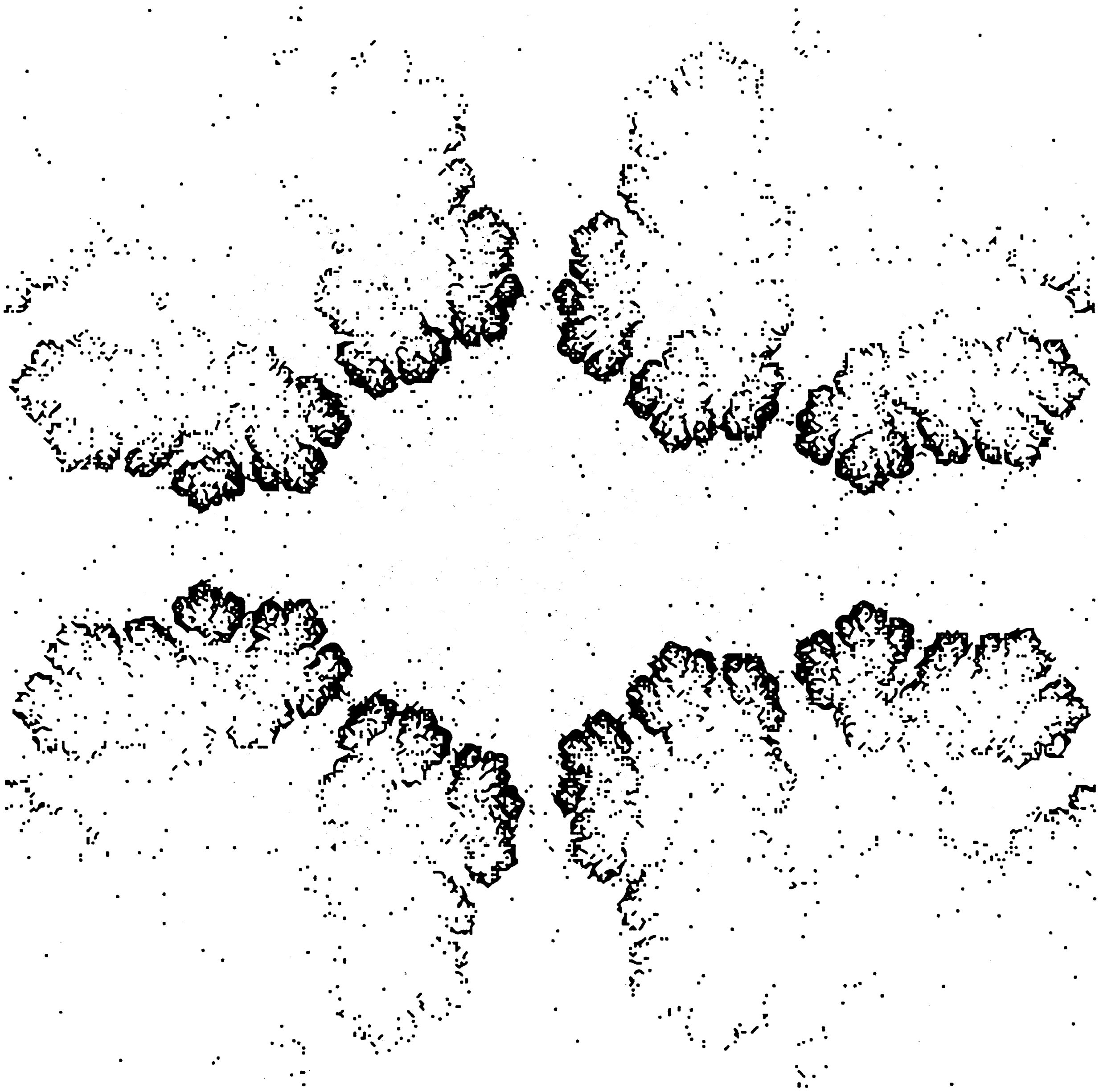
onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$. This is a perturbation of $x^3 + y_2 = 0$, for which the group generated is $\mathbb{Z}_6$.



The projection of an orbit on the curve

$$x^2y + y^3 + 1 = 0$$

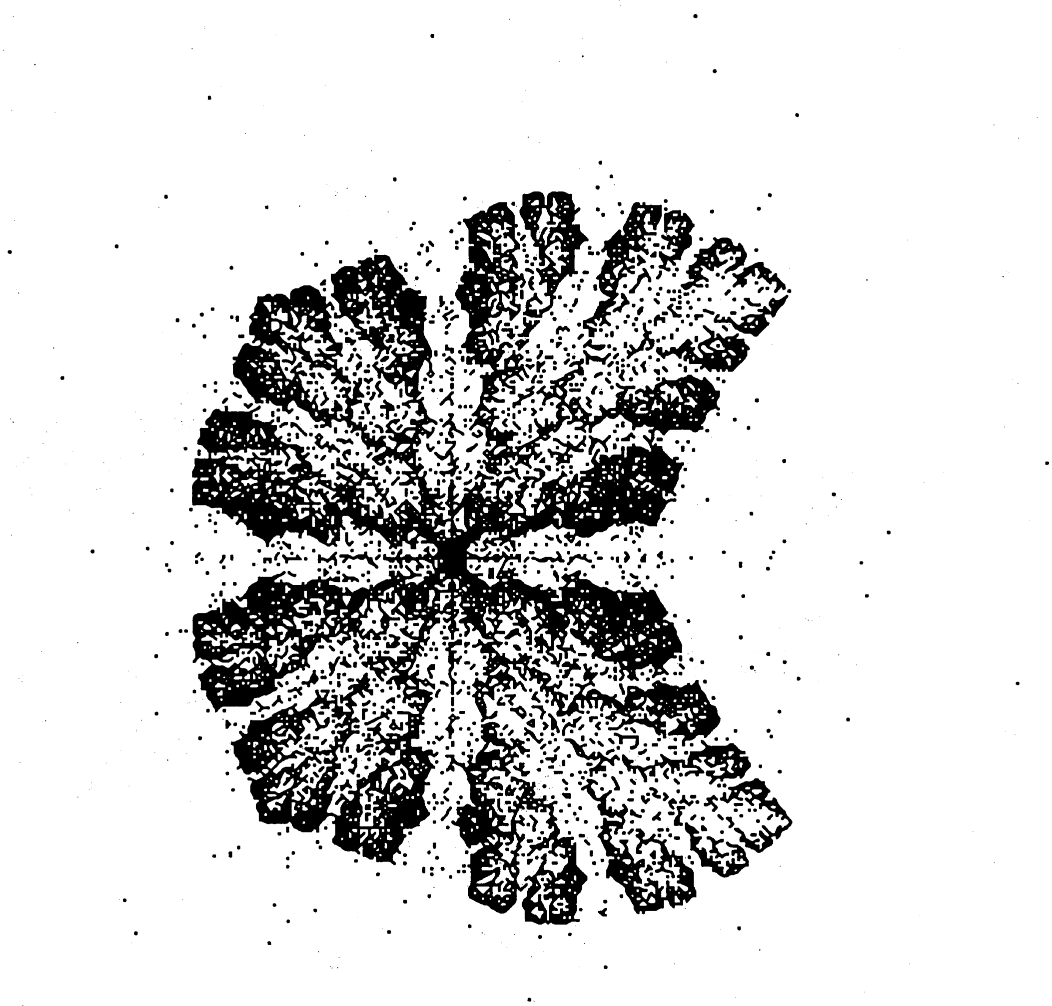
onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$. This is also very symmetric.



The projection of an orbit on the curve

$$(y - x^2 + 0.3)(y - x) - 0.3 = 0$$

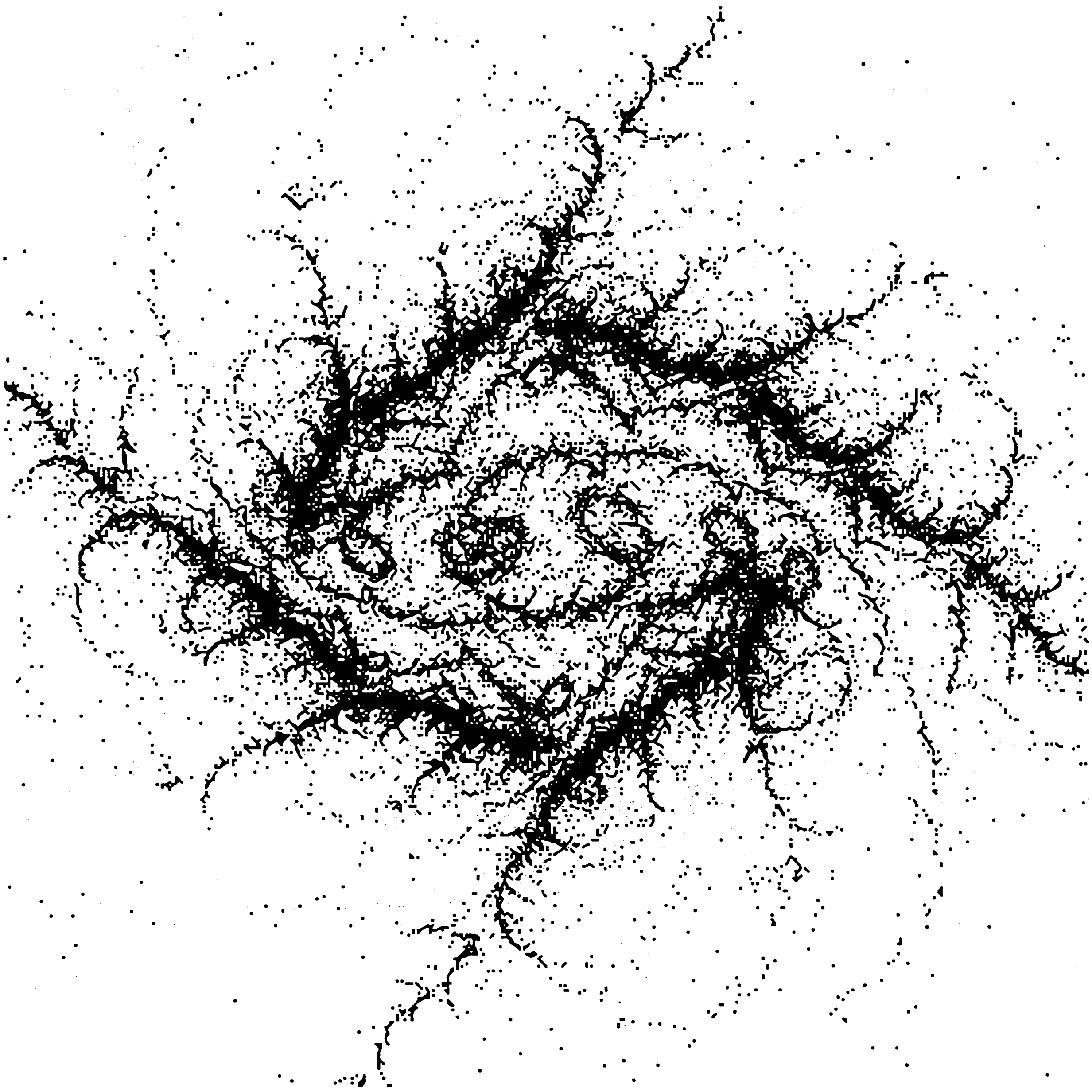
onto the x-plane. A defoemation of Julia set for $c = -0.3$.



The projection of an orbit on the curve

$$(y - x^2 + 0.3)(y - x) - 0.3 = 0$$

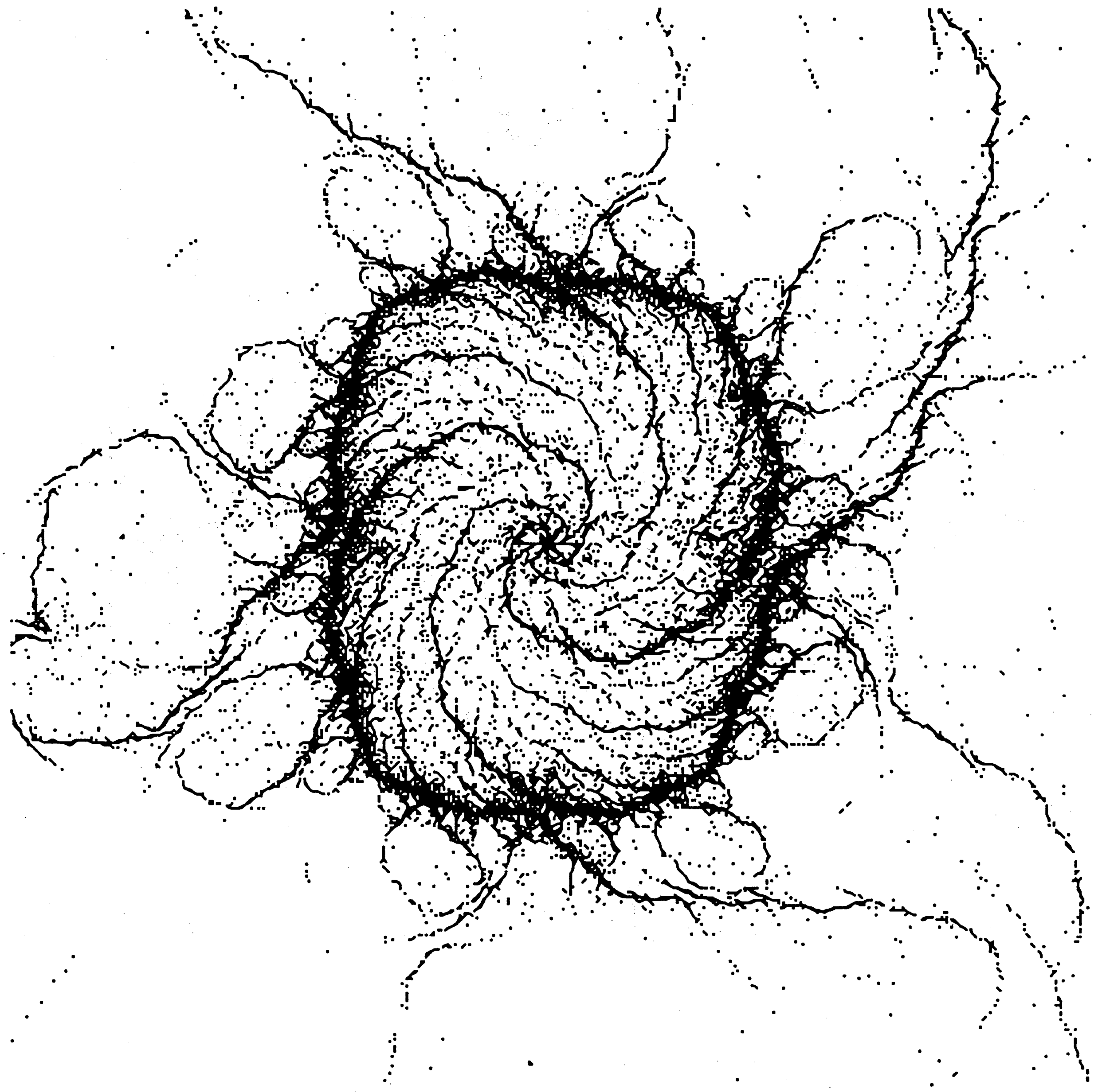
onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$. The +-shaped part centered at 0 seems to be a bug.



The projection of an orbit on the curve

$$(y - x^2 + 0.3)(y - x) + 0.1iy^2 = 0$$

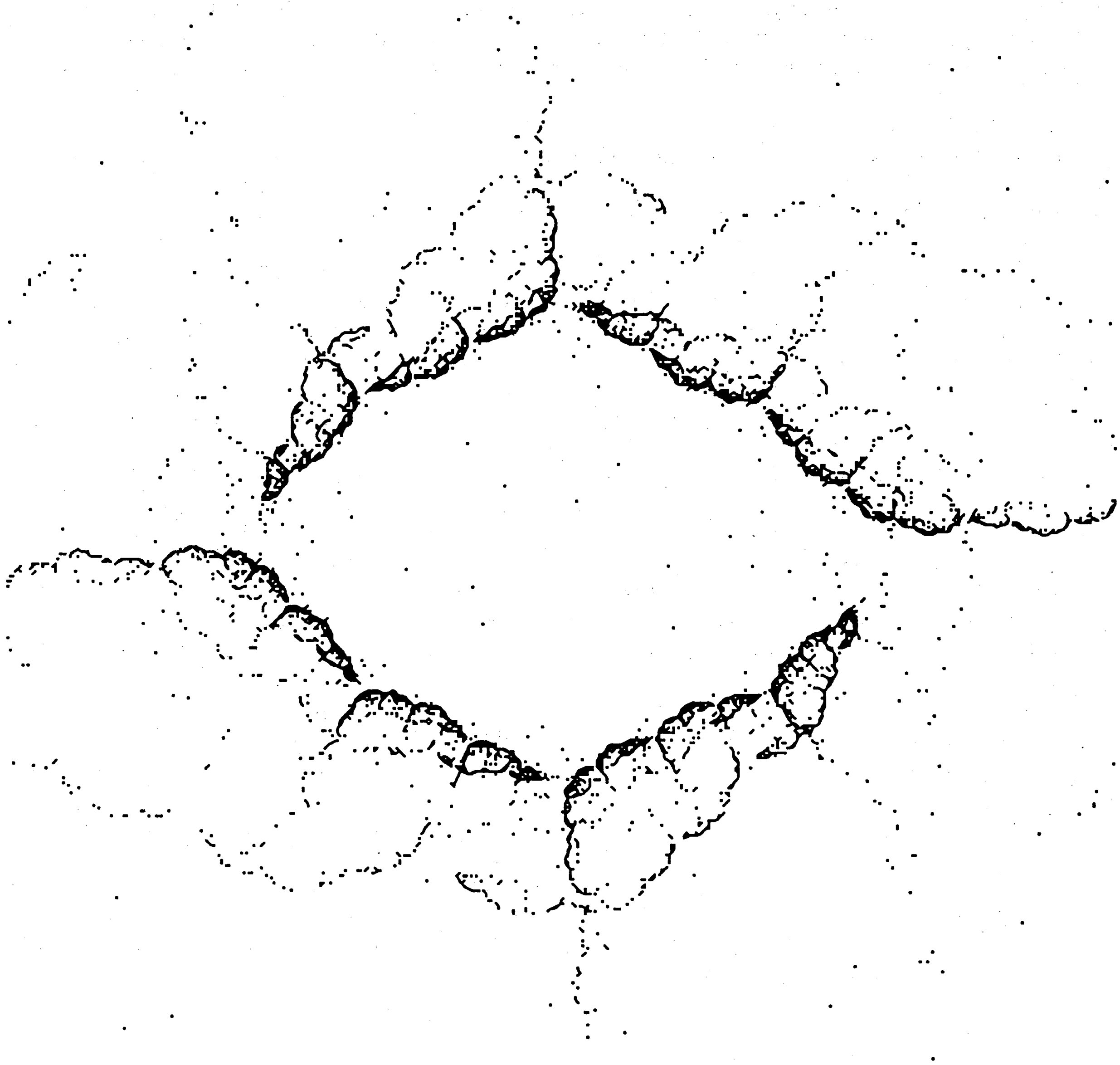
onto the x-plane. Another deformation of Julia set for $c = -0.3$. The orbits seem to be sense.



The projection of an orbit on the curve

$$(y - x^2 + 0.3)(y - x) + 0.1iy^2 = 0$$

onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$. The center is a super attractive critical point, and "real" non random orbits are foliation-like.



The projection of an orbit on the curve

$$(y - x^2 + 0.3)(y - x) - 0.03 + 0.03i = 0$$

onto the x-plane.

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