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Heat convection of compressible fluid

Dedicated to the sixtieth birthday of Professor Hideo Kawarada

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1. Mathematical Formulation

We consider the heat convection problem for the compressible viscous and heat conducting fluid in the layer between the upper plane $X_n = 0$ and the lower plane $X_n = d$, where we use the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ in $\mathbf{R}^n$, $n = 2$ or 3 , and $\mathbf{e}_n$ is considered in the vertically downward direction. Using the velocity of fluid : $\mathbf{u}' = (u'_1, u'_2, ..., u'_n)$, temperature : θ' , density : ρ', the governing equations of motion are

$$
\begin{aligned}
\frac{\partial \rho'}{\partial t'} + \nabla' \cdot (\rho' \mathbf{u}') &= 0 , && (1)\\
\rho'(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}') &= -\nabla' p' + g\rho' \mathbf{e}_n + \mu \Delta' \mathbf{u}' + \frac{1}{3}\mu \nabla'(\nabla' \cdot \mathbf{u}') , && (2)\\
\rho' c_v(\frac{\partial \theta'}{\partial t'} + \mathbf{u}' \cdot \nabla' \theta') &= \kappa \Delta' \theta' + 2\mu \mathbf{D}' : \mathbf{D}' - \frac{2}{3}\mu(\nabla' \cdot \mathbf{u}')^2 \quad . && (3)
\end{aligned}
$$

Here p' is the pressure , $g\mathbf{e}_n$ the acceleration of gravity, μ viscosity, κ heat conduction coefficient, c_v specific heat at constant volume, $\mathbf{D}'$ the deformation tensor. We assume the equation of state for the ideal gas :

$$
p = R_* \rho \theta \quad , \tag{4}
$$

where $R_* = c_p - c_v$ is the gas constant and c_p specific heat at constant pressure . Let the temperatures on the boundaries be given as

$$\theta' = T_u \quad \text{at} \quad X_n = 0 \quad \text{and} \quad \theta' = T_l \quad \text{at} \quad X_n = d \ , \tag{5}$$

where $0 < T_u < T_l$, and $0 < \beta_0 = (T_l - T_u)/d$ is the constant gradient of the temperature across the layer. Then the equilibrium solution $\mathrm{s}_0 = (\rho_0, \mathrm{u}_0, \theta_0)$ is the purely heat conducting one and is given by

$$\mathrm{u}_0 \ = \ 0 \ , \quad \theta_0 \ = \ \beta_0 x_n' \ , \quad \rho_0 \ = \ \frac{P}{R_* \beta_0} {x_n'}^m \ , \tag{6}$$

where

$$x_i' = X_i \ , \quad 1 \leq i \leq n-1 \ , \quad x_n' = \frac{T_u}{\beta_0} + X_n \ ,$$

P is an integration constant and m is the polytropic index :

$$m \ = \ \frac{g}{R_* \beta_0} - 1 \ . \tag{7}$$

We consider the perturbation to the equilibrium solution in the following dimensionless form :

$$\mathrm{u}' = \tilde{\mathrm{u}} \ , \quad \rho' = \tilde{\rho} + \rho_0 \ , \quad T' = \tilde{\theta} + \theta_0 \ .$$

Defining Prandtl number and other constants as follows :

$$\mathcal{P}_r = \frac{c_v \mu}{\kappa} \ , \quad \beta = \beta_0 - \frac{g}{c_p} \ , \quad b = \frac{\beta}{\beta_0} \quad \text{and} \quad \gamma = \frac{\mathrm{c_p}}{\mathrm{c_v}} \ ,$$

we introduce dimensionless variables

$$t = At' \ , \quad \mathrm{u} = B\tilde{\mathrm{u}} \ , \quad \theta = C\tilde{\theta} \ , \quad \rho = D\tilde{\rho} \ , \quad x_i = \frac{x_i'}{d} \ , \ 1 \leq i \leq n \quad ,$$

where

$$A = \frac{R_*\beta_0\mu}{\mathcal{P}_r P d^{m+2}}\,, \quad B = \left(\frac{\mathcal{P}_r}{gdb\gamma}\right)^{\frac{1}{2}}\,, \quad C = \frac{1}{\beta_0 d}\,, \quad D = \frac{R_*\beta_0}{Pd^m}\,.$$

Then the layer becomes $\Omega = \{\ z_0 \le x_n = z \le z_0 + 1\}$, where $z_0 = \frac{T_u}{\beta_0 d}$. We also define the Rayleigh number for the upper plane $z = z_0$

$$\mathcal{R}_a(z_0) = \mathcal{R}^2 z_0^{2m-1} = \frac{P^2\beta R_* c_p (m+1)^3 d^{2m+3}}{g^2\mu\kappa} z_0^{2m-1}\,.$$

Thus we obtain the dimensionless system for the perturbation :

$$\frac{\partial\rho}{\partial t} + \mathcal{R}\,\nabla\cdot(z^m \mathbf{u}) = N_1\,, \tag{8}$$

$$\frac{1}{\mathcal{P}_r} z^m \frac{\partial \mathbf{u}}{\partial t} - \Delta\mathbf{u} - \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}) = -\frac{\mathcal{R}}{b\gamma(m+1)}\left\{z^{m+1}\nabla(z^{-m}\rho) + \nabla(z^m\theta)\right\} + N_2\,, \tag{9}$$

$$z^m\frac{\partial\theta}{\partial t} - \Delta\theta + \mathcal{R}\, z^m u_n = -\mathcal{R}(\gamma-1)z^{m+1}\nabla\cdot\mathbf{u} + N_3\,, \tag{10}$$

where the nonlinear terms are the followings

$$N_1 = -\mathcal{R}\,\nabla\cdot(\rho\mathbf{u})\,, \tag{11}$$

$$N_2 = -\tfrac{1}{\mathcal{P}_r}\rho\tfrac{\partial\mathbf{u}}{\partial t} - \tfrac{\mathcal{R}}{\mathcal{P}_r}\rho\mathbf{u}\cdot\nabla\mathbf{u} - \tfrac{\mathcal{R}}{\mathcal{P}_r}z^m\mathbf{u}\cdot\nabla\mathbf{u} - \tfrac{\mathcal{R}}{b\gamma(m+1)}\nabla(\rho\theta)\,, \tag{12}$$

$$N_3 = -\rho\tfrac{\partial\theta}{\partial t} - \mathcal{R}\,\rho\mathbf{u}\cdot\nabla\theta - \mathcal{R}\,z^m\mathbf{u}\cdot\nabla\theta - \mathcal{R}\,\rho u_n - \mathcal{R}(\gamma-1)(\rho+z^m)\theta\nabla\cdot\mathbf{u} - \mathcal{R}(\gamma-1)z\rho\,\nabla\cdot\mathbf{u} + \tfrac{2gb\gamma}{\beta_0 c_v}\mathbf{D}:\mathbf{D} - \tfrac{2gb\gamma}{3\beta_0 c_v}(\nabla\cdot\mathbf{u})^2\,. \tag{13}$$

The system is formulated and treated by [7] , [3] , [4] , [6] , [1] and many others. Here we consider the instability and introduce a method to obtain the critical Rayleigh number as a computer assisted proof.

2. The linearized problem and the stability

We consider the linearized problem

$$\frac{\partial \rho}{\partial t} = -\mathcal{R}\nabla\cdot(z^m \mathbf{u}), \tag{14}$$

$$\frac{1}{\mathcal{P}_r} z^m \frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}) = -\frac{\mathcal{R}}{b\gamma(m+1)}\{z^{m+1}\nabla(z^{-m}\rho) + \nabla(z^m\theta)\}, \tag{15}$$

$$z^m \frac{\partial \theta}{\partial t} - \Delta\theta = -\mathcal{R}\, z^m u_n - \mathcal{R}(\gamma-1)\, z^{m+1}\nabla\cdot\mathbf{u}, \tag{16}$$

in the strip $\Omega = \{z_0 \le z = x_n \le z_0 + 1\}$, where the slip boundary condition for the velocity and Dirichlet boundary condition for the temperature are assumed

$$\frac{\partial u_i}{\partial z} = 0, \ i = 1, \cdots, n-1, \qquad u_n = \theta = 0$$
$$\text{at} \quad \mathrm{z} = \mathrm{z}_0 \quad \text{and} \quad \mathrm{z} = \mathrm{z}_0 + 1. \tag{17}$$

We can treat also the Dirichlet boundary condition for the velocity. We consider the solution $(\rho, \mathbf{u}, \theta)$ in the strip Ω , which is periodic in x_i with period ℓ_i, $1 \le i \le n-1$ and satisfies the boundary condition.

Theorem 1 *If Rayleigh number is small, then the solution* $s_0 = (\rho_0, \mathbf{u}_0, \theta_0)$ *is asymptotically energy stable.*

See Coscia and Padula [1] for the slip boundary condition and Pyi Aye [6] for the Dirichlet boundary condition.

3. Eigenvalue problems for the linearized system

We notice that main dimensionless parameters are Rayleigh number, Prandtl number, polytropic index m, periodicity ℓ_i and the shallowness of the layer which is proportinal to $1/z_0$. We want to consider the instability of the purely heat conducting state, which is given by the critical Rayleigh number when we change Rayleigh number, ℓ_i and z_0.

The eigenvalue problem for the linearized system is the following :

$$\lambda\rho = -\mathcal{R}\nabla\cdot(z^m\mathbf{u}) + f\,, \tag{18}$$

$$\frac{\lambda}{\mathcal{P}_r}z^m\mathbf{u} - \Delta\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}) = -\frac{\mathcal{R}}{b\gamma(m+1)}\{z^{m+1}\nabla(z^{-m}\rho) + \nabla(z^m\theta)\} + \mathbf{g}\,, \tag{19}$$

$$\lambda z^m\theta - \Delta\theta = -\mathcal{R}z^m u_n - \mathcal{R}(\gamma-1)z^{m+1}\nabla\cdot\mathbf{u} + h\,, \tag{20}$$

in the strip $\Omega = \{z_0 \le z = x_n \le z_0+1\}$, with the boundary condition (17).

Theorem 2 *The linearized system (18-20) forms a sectorial operator for any Rayleigh number for* $f\in H^1$, $\mathbf{g}\in\mathbf{L}^2$, $h\in L^2$.

See Pyi Aye [6].

Hereafter we consider the two-dimensional problem and since we assume the periodicity with respect to the horizontal direction x, we may consider the eigenfunctions in the form :

$$\rho = \rho(\lambda,z)\cos(nx)\,,\quad u_1 = u(\lambda,z)\sin(nx)\,,$$

$$u_2 = w(\lambda,z)\cos(nx)\,,\quad \theta = \theta(\lambda,z)\cos(nx)\quad \text{for } z_0 \le z \le z_0+1\,.$$

Then the system for the eigenvalue problem becomes the following system of ordinary differential equations.

$$\left(\frac{\lambda}{\mathcal{R}z^m} + \frac{3\mathcal{R}z}{4b\gamma(m+1)}\right)\frac{d\rho}{dz} = \left(\frac{2\lambda m}{\mathcal{R}z^{m+1}} + \frac{3\mathcal{R}m}{4b\gamma(m+1)}\right)\rho + \frac{mn}{z}u - \frac{3}{4}n\frac{du}{dz}$$

$$
\begin{aligned}
& - \Big(\frac{3\lambda}{4\mathcal{P}_r} z^m + \frac{3}{4} n^2 - \frac{m(m+1)}{z^2}\Big) w \\
& - \frac{3\mathcal{R}}{4b\gamma(m+1)} \Big(m z^{m-1}\theta + z^m \frac{d\theta}{dz} \Big) \qquad (21)
\end{aligned}
$$

$$
\frac{dw}{dz} = -nu - \frac{m}{z} w - \frac{\lambda}{\mathcal{R} z^m} \rho \qquad (22)
$$

$$
\begin{aligned}
\frac{d^2u}{dz^2} &= -\frac{\mathcal{R} n z}{b\gamma(m+1)} \rho + \Big(\frac{\lambda}{\mathcal{P}_r} z^m + \frac{4}{3} n^2\Big) u \\
& \quad + \frac{1}{3} n \frac{dw}{dz} - \frac{\mathcal{R} n z^m}{b\gamma(m+1)} \theta , \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2\theta}{d^2z} &= \mathcal{R}(\gamma-1) n z^{m+1} u + \mathcal{R}(\gamma-1) z^{m+1} \frac{dw}{dz} \\
& \quad + \mathcal{R} z^m w + (\lambda z^m + n^2)\theta , \qquad (24)
\end{aligned}
$$

in the interval $z_0 \le z = x_n \le z_0 + 1$, with the boundary condition

$$
\frac{du}{dz} = 0 , \quad w = \theta = 0 \quad \text{at} \quad \mathrm{z} = \mathrm{z}_0 \quad \text{and} \quad \mathrm{z} = \mathrm{z}_0 + 1 . \qquad (25)
$$

By this formulation, the original problem of instability is reduced to investigate the behavior of the real part of the eigenvalue λ when the parameters $\mathcal{R}$, z_0 and n vary. In order to see the instability we use the method given in [5] to prove the existence of the purely imaginary eigenvalue and the critical Rayleigh number in a small neighbourhood of the computed purely imaginary eigenvalue and critical Rayleigh number based on the Newton method. To obtain the eigenvalue and the eigenfunction for (21-25), we use the shooting method, i.e., we consider the fundamental solutions of the initial value problem for (21-24) in $z \ge z_0$ and express the eigenfunction by them as

$$
\begin{aligned}
&\rho = a\rho_1(z) + b\rho_2(z) + c\rho_3(z) , \quad u = a u_1(z) + b u_2(z) + c u_3(z) , \\
&w = a w_1(z) + b w_2(z) + c w_3(z) , \quad \theta = a\theta_1(z) + b\theta_2(z) + c\theta_3(z) \qquad (26)
\end{aligned}
$$

where $\rho_j(z)$, $u_j(z)$, $w_j(z)$, $\theta_j(z)$, $j=1,2,3$ satisfy (21-24) in $z>z_0$ and the initial conditions at $z=z_0$

$$\left\{\begin{array}{llllll} u_j'(z_0) = 0\,, & w_j(z_0) = 0\,, & \theta_j(z_0) = 0\,, & j = 1,2,3, \\ \rho_1(z_0) = 1\,, & u_1(z_0) = 0\,, & \theta_1'(z_0) = 0\,, & \\ \rho_2(z_0) = 0\,, & u_2(z_0) = 1\,, & \theta_2'(z_0) = 0\,, & \\ \rho_3(z_0) = 0\,, & u_3(z_0) = 0\,, & \theta_3'(z_0) = 1\,, & \end{array}\right. \tag{27}$$

a , b and c are constants to be determined. In order that the function (26) is the eigenfunction, it must satisfy the boundary condition (25) . This condition is written as follows

$$\begin{pmatrix} \frac{du_1}{dz}(z_0+1) & \frac{du_2}{dz}(z_0+1) & \frac{du_3}{dz}(z_0+1) \\ w_1(z_0+1) & w_2(z_0+1) & w_3(z_0+1) \\ \theta_1(z_0+1) & \theta_2(z_0+1) & \theta_3(z_0+1) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\,. \tag{28}$$

Then for the eigenfunction, we have to have

$$\mathcal{F}(\mathcal{R},\lambda\,;\,\mathcal{P}_r,z_0,n) \equiv \det\, A = 0\,, \tag{29}$$

where the matrix in (28) is denoted by $A=(a_{ij})$. Thus, we come to the position to search the values of $\mathcal{R}=\mathcal{R}_c$, $\lambda=\mathrm{i}\omega_c$ satisfying (29) for the fixed parameters $\mathcal{P}_r$, z_0 and n . Noting that (29) can be rewritten as

$$\mathcal{F}(\mathcal{R},\lambda) = \mathcal{F}(\mathcal{R}_0,\lambda_0) + \frac{\partial\mathcal{F}}{\partial\mathcal{R}}(\mathcal{R}-\mathcal{R}_0) + \frac{\partial\mathcal{F}}{\partial\lambda}(\lambda-\lambda_0) = 0\,,$$

we can state our criterion for existence of the critical eigenvalue based on the simplified Newton method as follows :

Theorem 3 *Suppose, for a small $\varepsilon > 0$, there exist $\mathcal{R}_0$ and λ_0 such that*

$$\|\mathcal{F}(\mathcal{R}_0,\ \lambda_0)\| \ < \ \varepsilon \ . \tag{30}$$

Put

$$L_0 \ \equiv \ \left(\overline{\frac{\partial \mathcal{F}}{\partial \mathcal{R}}}(\mathcal{R}_0, \lambda_0) \ , \ \overline{\frac{\partial \mathcal{F}}{\partial \lambda}}(\mathcal{R}_0, \lambda_0) \right) \ , \tag{31}$$

where the bar means an appropriate approximation of the quantity. Suppose further that, for a small δ, there is a ρ_1 such that the estimate

$$\| \ D\mathcal{F}(\mathcal{R} \ , \ \lambda) \ - \ L_0 \ \| \ < \ \delta \tag{32}$$

holds for any $(\mathcal{R} \ , \ \lambda)$ such that

$$(\mathcal{R} \ - \ \mathcal{R}_0)^2 \ + \ |\lambda \ - \ \lambda_0|^2 \ < \ \rho_1^2 \ .$$

For ε, ρ_1, δ and L_0 as above, if it holds that

$$\| \ L_0^{-1} \ \| \left(\frac{\varepsilon}{\rho_1} \ + \ \delta \right) \ \leq \ 1 \ , \tag{33}$$

then there exist some $\mathcal{R}_c$ and λ_c in the ρ_1-neighborhood of $\mathcal{R}_0$ and λ_0 satisfying

$$\mathcal{F}(\ \mathcal{R}_c \ , \ \lambda_c \ ) \ = \ 0 \ . \tag{34}$$

To utilize this criterion to our problem, we need to justify the following steps:

(*i*) To find appropriate values $\mathcal{R}_0$ and λ_0, we use the numerical computation by the shooting method and Newton method. The fundamental solutions are obtained by the fourth order Taylor finite difference scheme .

(*ii*) To estimate ε we need the interval analysis by a computer software for the bound of round-off errors in the computation of the fundamental solutions and the theory of pseudo trajectory to estimate the difference between the genuine fundamental solutions and the numerically computed ones.

(*iii*) At this pair of $\mathcal{R}_0$, λ_0, find an approximate derivative L_0 and estimate the norm $\| L_0^{-1} \|$;

(*iv*) Estimate δ for which the estimate (32) holds in the ρ_1-neighborhood of $\mathcal{R}_0$ and λ_0;

(*v*) For these values in (*i, ii, iii, iv*), prove that the criterion (33) holds.

Following these steps we see that there exist the exact eigenvalue $\lambda = i\omega_c$ and the critical Rayleigh number $\mathcal{R} = \mathcal{R}_c$ for (21-25) in the ρ_1 -neighborhood of numerically computed values ($\mathcal{R}_0$, λ_0) in (*i*).

In order to see the motion of the eigenvalue crossing the imaginary axis when $\mathcal{R}$ increases, we can apply such arguments as in [5] which uses the adojoint system of the equations to (21-24). For notational convenience we write the eigenvalue λ_c and the eigenfunction $\Phi = (\rho, u, w, \theta)$ with the critical Rayleigh number $\mathcal{R}_c$ for the system of equations (21-24) and the boundary conditions (25) as

$$L\,\Phi = 0 \quad \text{and} \quad B\,\Phi = 0\,. \tag{35}$$

Let us denote the eigenvalue $\overline{\lambda_c}$ and the eigenfunction $\Psi = (\rho^*, u^*, w^*, \theta^*)$ which satisfy the the adjoint problem

$$L^*\Psi = 0 \quad \text{and} \quad B^*\Psi = 0\,.$$

Taking the derivative of (35) with respect to the Rayleigh number and the

$L^2(0,1)$-inner product with Ψ, we obtain

$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c} = -\frac{\left(\frac{\partial L}{\partial \mathcal{R}}\Phi\,,\,\Psi\right)_{L^2}}{\left(\frac{\partial L}{\partial \lambda}\Phi\,,\,\Psi\right)_{L^2}}\,.$$

Example 1. We take $\gamma = 5/3$, $c_p = 1$, $c_v = 0.6$, $m = 1.4$, $\mathcal{P}_r = 1$, and $b = 0.04$.

z_0	n	λ	$\mathcal{R}_0$	$\mathcal{R}_m$
0.125	2.4797	0.0	54.8836	1292.61
0.25	2.3925	0.0	42.6609	1084.35
0.5	2.3136	0.0	29.7561	885.428
1.0	2.2614	0.0	19.0653	754.139
2.0	2.2357	0.0	11.5268	691.377
4.0	2.2258	0.0	6.6749	667.851
8.0	2.2226	0.0	3.7447	660.399
16.0	2.2217	0.0	2.0581	658.276
32.0	2.2215	0.0	1.1176	657.708
64.0	2.22146	0.0	0.60306	657.561
128.0	2.22145	0.0	0.32429	657.523
Bouss.	2.22144	0.0		657.511

Here $\mathcal{R}_m$ is the Rayleigh number on the middle plane $z = z_0 + 0.5$

$$\mathcal{R}_m = \mathcal{R}_a(z_0 + 0.5)\,.$$

It approaches to that of Boussinesq approximation for heat convection as z_0 gets large.

This example suggests the occurrence of the stationary bifurcation at the critical Rayleigh number. However the usual bifurcation theory does not apply to the original system (8-13) , because the mass conservation law has a high nonlinearity and the sectorial properties of the theorem 2 is not sufficient to guarantee the bifurcation. Further investigations are required.

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