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The θ_1-eigenpolytopes of the Hamming graphs

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Abstract

We can introduce geometric methods into the study of the Hamming graphs as follows. Let X be the Hamming graph $H(n,q)$ and identify the i-th vertex of X with the i-th standard basis vector e_i in $\boldsymbol{R}^{q^n}$. Let θ_1 be the second largest eigenvalue of the Hamming graph and let U be the corresponding eigenspace. We then associate to the i-th point of X the image of e_i under orthogonal projection onto U. U has dimmension $n(q-1)$ and we have a mapping from X into $\boldsymbol{R}^{q^n}$. Then it can be shown that the image of X lies in a sphere centred at the origin, and that the cosine of the angle between the vectors representing two vertices u and v is determined by the distance between them in X. In this paper we describe about convex polytopes of the image of X (it is called θ_1-eigenpolytopes of the Hamming graphs) and their faces.

1 Intoroduction

A convex polytope is the convex hull of finite poinits. An affine hyperplane $\mathcal{H}$ is a suppotting hyperplane for a convex polytope $\mathcal{P}$, if it contains at least one point of $\mathcal{P}$, and if all points of $\mathcal{P}$ not on $\mathcal{H}$ lie on the same side of $\mathcal{H}$. A face of $\mathcal{P}$ is any set of points $\mathcal{P}\bigcap\mathcal{H}$, where $\mathcal{H}$ is a suppporting hyperplane. We will determine the faces of the θ_1-eigenpolytopes of the Hamming graphs.

2 Representations of graphs

We define a representation of a graph .

Definition 2. 1 *A representation of a graph Γ in $\boldsymbol{R}^m$ is a map, ρ, from $V(\Gamma)$ into $\boldsymbol{R}^m$, such that for any two vertices u and v, the inner product $\langle\rho(u),\rho(v)\rangle$ depends only on the distance $\partial(u,v)$ between u and v in Γ.*

Because any vertex is at distance 0 from itself, this implies that the image of $V(\Gamma)$ under ρ lies on a sphere centred at the origin. We introduce one concept from linear algebra .

Definition 2. 2 *The Gram matrix of a set of vectors $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_n$ is the $n\times n$ matrix $\boldsymbol{M}$ with $\boldsymbol{M}_{i,j}$ equal to $\langle\boldsymbol{v}_i,\boldsymbol{v}_j\rangle$.*

It is not hard to see that a Gram matrix is a positive semi-definite matrix and symmetric.The converse is also true, any symmetric positive semi-definite matrix is a Gram matrix. If Γ is a graph with diameter d, let Γ_i denote the graph with the same vertex set as Γ, with two vertices adjacent in Γ_i if and only if they are at distance i in Γ. Let A_i denote the adjacency matrix of Γ_i, with the understanding that A_0=I. We call $A_0,\cdots,A_d$ the distance matrices of Γ. If ρ is a representation of Γ in $\boldsymbol{R}^m$, then the matrix $\boldsymbol{M}(\rho)$ has rows and colunms indexed by the vertices of Γ and, if $u,v\in V(\Gamma)$, then

$$(\boldsymbol{M}(\rho))_{u,v}=\langle\rho(u),\rho(v)\rangle \qquad \cdots(2,1).$$

The matrix $\boldsymbol{M}(\rho)$ is a linear combination of the matrices A_i. It is a Gram matrix, of the vectors ρ for u in $V(\Gamma)$, and so it is positive semi-definite. As every symmetric positive semi-definite matrix is a Gram matrix, we see conversely that each positive semi-definite matrix in the span of $A_0,\cdots,A_d$ the distance matrices determines a representation of Γ.

3 Distance regular graphs

Definition 3. 1 *A graph is distance regular if it is connected and,given any two vertices u and v at distace i, the number of vertices x at distance j from u and k from v is determined by the triple (i,j,k).*

Let Γ be a distance regular graph with the vertex set X=$\{x_1, \cdots, x_n\}$, and the distance matrices $A_0, \cdots, A_d$. We identify the vertex x_i with the standard basis vector in $\boldsymbol{R}^n$

. The distance matrices $A_0, \cdots, A_d$ form a basis of a commutative algebra of dimension $d+1$ over the reals, called the Bose-Mesner algebra $\Im$. Thus, if ρ is a representation of Γ, then $\boldsymbol{M}(\rho)$ is a positive semi-definite matrix in $\Im$; conversely every such matrix gives rise to a representation of Γ.

One class of positive semi-definite matrices can be obtained using projections. Let θ be an eigenvalue of A=A_1 with multiplicity m, let U be the corresponding eigenspace and let E_θ be the matrix representating orthogonal projection on U. Then E_θ can be shown to be a polynomial in A, and therefore it lies in the Bose-Mesner algebra of Γ(see [4]). Hence we obtain a representation of Γ in $\boldsymbol{R}^n$, which we call an eigenspace representation of Γ. As E_θ lies in the span of the matrices $A_0, \cdots, A_d$, its diagonal entries are all equal. Because E_θ^2=E_θ, all eigenvalues of E_θ are equal to 0 or 1; therefore $traceE_\theta$=$rankE_\theta$ and each diagonal entry of $\boldsymbol{E}_\theta$ is equal to $\frac{m}{V(\Gamma)}$.

So there are constants $\omega_0, \cdots, \omega_d$, with ω_0=1, such that

$$\frac{|V(\Gamma)|}{m} E_\theta = \sum_{i=0}^{d} \omega_i A_i.$$

As this matrix is positive semi-definite, each principal 2×2 submatrix has non-negative determinant, which implies that

$$|\omega_i| \leq 1.$$

We can view ω_i as the cosine of the angle between $\rho(u)$ and $\rho(v)$, for any two vertices u and v at distance i in Γ.

We call $\omega_0, \cdots, \omega_d$ the sequence of cosines of Γ; this sequence depends on the eigenvalue θ. We can summarise our conclusions as follows.

Lemma 3. 1 *Let Γ be a distance regular graph with diameter d, let θ be an eigenvalue of Γ, with cosine sequence $\omega_0, \cdots, \omega_d$ and let ρ be the corresponding representation of Γ. Then*

$$M(\rho) = \sum_{i=0}^{d} \omega_i A_i.$$

4 Convex polytope

In this section, we describe about convex polytopes.

Definition 4. 1 *Let X be a non-empty subset of $\boldsymbol{R}^N$. Then the covex hull of X, denoted by $CONV(X)$, is the smallest convex set containing X.*

In perticular, if X is finite, then we obtain the following proposition.

Proposition 4. 1 *Let X be a finite set $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_v\}$ in $\boldsymbol{R}^N$. Then the convex hull of X is the following;*

$$CONV(X) = \{\sum_{i=1}^{v} t_i \boldsymbol{x}_i | 0 \le t_i \in \boldsymbol{R}, \sum_{i=1}^{v} t_i = 1\}.$$

proof: See theoreom 2.2 [1].□

Definition 4. 2 *A subset $\mathcal{P}$ in $\boldsymbol{R}^N$ is a convex polytope, for short 'polytope', if there exists a finite set in $\boldsymbol{R}^N$ such that $\mathcal{P} = CONV(X)$.*

The dimension of a polytope is the dimension of the smallest affine space which contains all its points; we will often refer to a polytope of dimension m as an m-polytope.

Definition 4. 3 *A subset $\mathcal{H}$ in $\boldsymbol{R}^N$ is an affine hyperplane, if there exist the vector $\boldsymbol{a}$ and the real b such that the following equation are satisfied;*

$$\mathcal{H} = \{\boldsymbol{x} \in \boldsymbol{R}^N | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b\}.$$

Let $\mathcal{H} = \{\boldsymbol{x} \in \boldsymbol{R}^N | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b\}$ be an affine hyperplane in $\boldsymbol{R}^N$. Then we define $\mathcal{H}^{(+)}$ as the set $\mathcal{H} = \{\boldsymbol{x} \in \boldsymbol{R}^N | \langle \boldsymbol{a}, \boldsymbol{x} \rangle \le b\}$, and $\mathcal{H}^{(-)}$ as the set $H = \{\boldsymbol{x} \in \boldsymbol{R}^N | \langle \boldsymbol{a}, \boldsymbol{x} \rangle \ge b\}$, where $\boldsymbol{a}$ is the vector in $\boldsymbol{R}^N$, and b is real.

Definition 4. 4 *Let $\mathcal{P}$ be a polytope in $\boldsymbol{R}^N$, and let $\mathcal{H}$ be an affine hyperplane in $\boldsymbol{R}^N$. Then $\mathcal{H}$ is the supporting hyperplane for $\mathcal{P}$, if the following conditions (1) and (2) are satisfied;*

$$(1)\ \ \mathcal{P} \subset \mathcal{H}^{(+)},\ \ or\ \mathcal{P} \subset \mathcal{H}^{(-)};$$

$$(2)\quad \emptyset \neq P \underset{\neq}{\subset} H \subset P.$$

Definition 4.5 *A face of a polytope $\mathcal{P}$ is any set of points $\mathcal{P} \bigcap \mathcal{H}$, where $\mathcal{H}$ is a supporting hyperplane for $\mathcal{P}$.*

Since any face is itself a convex polytope, a face with dimension i with respect to polytope is called an i-face. A 0-face is usually called a vertex and a 1-face is called an edge. An $(m-1)$-face of an m-polytope is a facet. The vertices and edges of a polytope form a graph, which is called the 1-skelton of the polytope.

Proposition 4.2 *Let $\mathcal{P}$ be a polytope which is the convex hull of a finite set X in $\boldsymbol{R}^N$, and let $\mathcal{H}$ be the supporting hyperplane for $\mathcal{P}$. Then*

$$\mathcal{P} \bigcap \mathcal{H} = CONV(X \bigcap \mathcal{H}).$$

proof: See pp.21-22 [5].□

Proposition 4.3 *Let V be the vertex set of a polytope $\mathcal{P}$. Then*

$$\mathcal{P} = CONV(V).$$

proof: See theorem 7.2 [1].□

Proposition 4.4 *Let V be the vertex set of a polytope $\mathcal{P}$, and let $\mathcal{F}$ be a face of $\mathcal{P}$. Then the vertex set of $\mathcal{F}$ is equal to $V \bigcap \mathcal{P}$.*

proof: See Theorem 7.3 [2].□

Proposition 4.5 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be faces of a polytope $\mathcal{P}$. If $\mathcal{F}_1$ contains $\mathcal{F}_2$, then $\mathcal{F}_2$ is also a face of $\mathcal{F}_1$. In perticular, a face of a face of $\mathcal{P}$ is a face of $\mathcal{P}$.*

proof: See Theorem 5.2 [1].□

Proposition 4. 6 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be faces of a polytope $\mathcal{P}$ in $\boldsymbol{R}^{q^n}$. Then*

(a) If $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_1 \neq \mathcal{F}_2$, then dim $\mathcal{F}_1 \leq$ dim $\mathcal{F}_2$;

(b) $\mathcal{F}_1 \bigcap \mathcal{F}_2$ is a face of $\mathcal{P}$.

proof: See Corollary 5.4 and Theorem 5.9 [1].□

5 The Hamming graphs $H(n, q)$

In this section, we introduce the Hamming graphs, which are very famous examples of distance regular graphs, and its θ_1-eigenpolytopes.
Let Q be a fixed set of size q. The Hamming scheme $H(n, q)$ has vertex set Q^n, and two elements of Q^n are adjacent if they differ in exactly one coordinate. (We can identify Q with the set $\{0, 1, \cdots, q-1\}$.) We denote the vertex set of $H(n, q)$ by $X_{H(n,q)}$ and the vertex $(0, \cdots, 0)$ by x_0.

The cosine sequence of the Hamming graph $H(n, q)$ is also known (See [4]).

Proposition 5. 1 *Let $\theta_0 < \theta_1 < \cdots < \theta_n$ be the eigenvalues of the Hamming graph $H(n, q)$, and let $\omega_0, \omega_1, \cdot, \omega_n$ be the cosine sequence of $H(n, q)$ corresponding to θ_1. Then*

$$\omega_j = 1 - \frac{qj}{(q-1)n} \quad (0 \leq j \leq n).$$

proof See [4].□

6 The θ_1-eigenpolytopes of the Hamming schemes

Let Γ be a distance regular graph with diameter d and with eigenvalues $\theta_0, \cdots, \theta_d$ in decresing order and let ρ be an eigenspace representation of Γ into $\boldsymbol{R}^m$ corresponding to the eigenvalue θ_1. We call the convex hull of the points in the image of $\rho(\Gamma)$ a θ_1-*eigenpolytope* of Γ.

In this section we describe about the θ_1-eigenpolytopes of the Hamming graphs $H(n,q)$ and properties of them.
Let $\theta_0 < \theta_1 < \cdots < \theta_n$ be the eigenvales of the Hamming graph $H(n,q)$ and let ρ be the eigenspace representation of $H(n,q)$ corresponding to θ_1. Since the vertex set $X_{H(n,q)}$ of the Hamming graph $H(n,q)$ has q^n elements, ρ is the representation from $X_{H(n,q)}$ into $\boldsymbol{R}^{q^n}$. We denote the θ_1-eigenpolytope of the Hamming graph $H(n,q)$ by $P_{H(n,q)}$, its 1-skelton by $\Gamma_{H(n,q)}$, and the eigenspace representation of $H(n,q)$ corresponding to θ_1 as ρ.

Theorem 6. 1 *Let $P_{H(n,q)}$ be the θ_1-eigenpolytope of the Hamming graph $H(n,q)$ and let $\Gamma_{H(n,q)}$ be the 1-skelton of $P_{H(n,q)}$. Then $\Gamma_{H(n,q)}$ is isomorphic to $H(n,q)$.*

proof: See [3].□
From theorem 6.1 we see that $\rho(X_{H(n,q)})$ is the vertex set of the θ_1- eigenpolytope $P_{H(n,q)}$, and we denote it by $V_{H(n,q)}$. Moreover, we obtain the following lemma with respect to the edge of $P_{H(n,q)}$.

Lemma 6. 1 *Let x,$y \in X_{H(n,q)}$. Then x and y in the vertex set $X_{H(n,q)}$ of $H(n,q)$ are adjacent in $X_{H(n,q)}$ if and only if $\rho(x)$ and $\rho(y)$ in the vertex set $V_{H(n,q)}$ of $X_{H(n,q)}$ are adjacent in $\Gamma_{H(n,q)}$.*

proof: Let x,$y \in X_{H(n,q)}$ and let $\omega_0, \cdots, \omega_n$ be the cosine sequence of $H(n,q)$ corresponding to θ_1. Suppose that x and y are adjacent. Then from (2,1) and lemma 3.1 we have $\langle\rho(x), \rho(y)\rangle = \omega_1$. Since $\omega_0 < \cdots < \omega_n$ by proposition 5.5, for any z$\in X_{H(n,q)} - \{x, y\}$, we have $\langle\rho(x), \rho(z)\rangle \leq \omega_1$ *and* $\langle\rho(y), \rho(z)\rangle \leq \omega_1$. Then an affine hyperplane $H = \{v \in R^{q^n} | \langle v, \rho(x) + \rho(y)\rangle = \omega_0 + \omega_1\}$ is a supporting hyperplane for $P_{H(n,q)}$, and we have $P_{H(n,q)} \bigcap H = CONV\{\rho(x), \rho(y)\}$ Since $CONV\{\rho(x), \rho(y)\}$ is an 1-face of $P_{H(n,q)}$, $\rho(x)$ and $\rho(y)$ are adjacent in $\Gamma_{H(n,q)}$.□
From lemma 6.1 and theorem 6.1 we can identify $H(n,q)$ with $\Gamma_{H(n,q)}$.

Definition 6. 1 *An m-polytope is simple if every k-face lies in exactly $m-k$ facets.*

There is a more intuitive characterization, given as Theorem 12.12 [1]

Theorem 6. 2 *An m-polytope is simple if and only if its 1-skelton is regular of valency m.*□

Theorem 6. 3 *If a polytope $\mathcal{P}$ is simple, then every face of $\mathcal{P}$ is simple.*

proof: See Theorem 12.15 [1].□
Since $m_1 = (q-1)n$ by proposition 5.1 and $\Gamma_{H(n,q)}$ has the valency $(q-1)n$, $(q-1)n$-polytope $\mathcal{P}_{H(n,q)}$ is simple.

Lemma 6. 2 *Let $\mathcal{P}$ be a simple polytope, let u and $v_1, \cdots, v_k$ be vertices of $\mathcal{P}$ such that uv_i is an edge of $\mathcal{P}$ for $i = 1, \cdots, k$ and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be faces of $\mathcal{P}$ which contain u and v_1. Then if $dim\mathcal{F}_1 = dim\mathcal{F}_2$, then we have $\mathcal{F}_1 = \mathcal{F}_2$.*

proof: Since $\{u, v_1, \cdots, v_k\} \subset \mathcal{F}_1 \bigcap \mathcal{F}_2$, $\mathcal{F}_1 \bigcap \mathcal{F}_2$ is not empty and a face of $\mathcal{P}$ by proposition 4.6 (a). It follows by theorem 6.3 that the face $\mathcal{F}_1 \bigcap \mathcal{F}_2$ is simple and $dim(\mathcal{F}_1 \bigcap \mathcal{F}_2) \geq k$. On the other hand $\mathcal{F}_1 \bigcap \mathcal{F}_2 \subset \mathcal{F}_1$ and $\mathcal{F}_1 \bigcap \mathcal{F}_2 \neq \mathcal{F}_1$ by assumption. Then $dim(\mathcal{F}_1 \bigcap \mathcal{F}_2) \leq k$ and it is contradiction. Therefore we have $\mathcal{F}_1 = \mathcal{F}_2$.□

7 The faces of the θ_1-eigenpolytopes of the Hamming graphs

In this section we shall determine the faces of the θ_1-eigenpolytopes of the Hamming graphs and calculate some numbers with respect to the faces.

Let Y_l $(1 \leq l \leq n)$ be non-empty subsets of $Q = \{0, 1, \cdots, q-1\}$. We define the subset $Y_1 \times \cdots \times Y_n$ of $X_{H(n,q)}$ as the following;

$$Y_1 \times \cdots \times Y_n = \{(y_1, \cdots, y_n) \in X_{H(n,q)} |\ y_l \in Y_l\ (1 \leq l \leq n)\}$$

We define the subset $\Gamma_i(x)$ $(1 \leq i \leq n,\ x \in X_{H(n,q)})$ of $X_{H(n,q)}$ as the following;

$$\Gamma_i(x) = \{y \in X_{H(n,q)} |\ \partial(x, y) = i\}$$

We can determine the faces of $\mathcal{P}_{H(n,q)}$ by the following theorem.

Theorem 7. 1 *Let Y_l $(1 \le l \le n)$ be non-empty subsets of $Q = \{0, 1, \cdots, q-1\}$. Then the following (1) (2) and (3) hold.*
(1) $CONV(\rho(Y_1 \times \cdots \times Y_n))$, denoted by $\mathcal{F}(Y_1 \times \cdots \times Y_n)$, is a face of $\mathcal{P}_{H(n,q)}$ with vertex set $\rho(Y_1 \times \cdots \times Y_n)$.
(2) If $\sum_{l=1}^{n} |Y_l| = n + i$ then $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ is an i-face of $\mathcal{P}_{H(n,q)}$.
(3) Any face $\mathcal{F}$ of $\mathcal{P}_{H(n,q)}$ is equal to $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ for some non-empty subsets Y_l $(1 \le l \le n)$ of Q.

proof: (1) Suppose $Y_l = \{p_{l1}, \cdots, p_{lm_l}\}$ $(1 \le l \le n,\ 1 \le m_l \le q-1)$. Let us fix $u \in Y_1 \times \cdots \times Y_n$ arbitrarily. Then we may assume that $u = (p_{11}, p_{21}, \cdots, p_{n1})$ without loss of generality. By definition of distance in the Hamming graph it is not hard to show that

$$| \Gamma(u) \bigcap (Y_1 \times \cdots \times Y_n) | = \sum_{l_1 < \cdots < l_i} (m_{l_1} - 1) \cdots (m_{l_i} - 1) \quad \cdots (7.1)$$

We see that $| \Gamma(u) \bigcap (Y_1 \times \cdots \times Y_n) |$ is determined independent of u. Then we denote $| \Gamma(u) \bigcap (Y_1 \times \cdots \times Y_n) |$ by C_i, i.e. $| \Gamma(u) \bigcap (Y_1 \times \cdots \times Y_n) | = C_i$ for any $v \in Y_1 \times \cdots \times Y_n$. Let $w = (w_1, \cdots, w_n) \in X_{H(n,q)} - (Y_1 \times \cdots \times Y_n)$. Then there exist at least one w_j such that $w_j \notin Y_j$. Then it is not hard to show that

$$| \Gamma_i(w) \bigcap (Y_1 \times \cdots \times Y_n) | \le C_i,$$

and in particular

$$| \Gamma_0(w) \bigcap (Y_1 \times \cdots \times Y_n) | = 0 < 1 = C_0.$$

Let $\omega_0, \cdots, \omega_n$ be cosine sequence of $H(n,q)$ corresponding to the second largest eigenvalue θ_1. Then from proposition 5.1 we see that $\omega_0 < \cdots < \omega_n$. Set $\boldsymbol{a} = \sum_{x \in Y_1 \times \cdots \times Y_n} \rho(x) \in \boldsymbol{R}^{q^n}$ and $b = \sum_{i=0}^{n} C_i \omega_i$. Since $| \Gamma(u) \bigcap (Y_1 \times \cdots \times Y_n) | = C_i$ for any $v \in Y_1 \times \cdots \times Y_n$, we obtain

$$\langle \boldsymbol{a}, \rho(b) \rangle = b \quad (u \in Y_1 \times \cdots \times Y_n) \cdots (7.2).$$

Since $| \Gamma_i(w) \bigcap (Y_1 \times \cdots \times Y_n) | \le C_i$ and $| \Gamma_0(w) \bigcap (Y_1 \times \cdots \times Y_n) | < C_0$ we obtain

$$\langle \boldsymbol{a}, \rho(w) \rangle < b \quad (w \in X_{H(n,q)} - (Y_1 \times \cdots \times Y_n)) \cdots (7.3).$$

We define affine hyperplane $\mathcal{H} \subset \boldsymbol{R}_{q^n}$ as the following;

$$\mathcal{H} = \{z \in \boldsymbol{R}^{q^n} \mid \langle z, a\rangle = b\}.$$

From (7.2) and proposition 4.1 we obtain

$$\emptyset \neq \mathcal{P}_{H(n,q)} \bigcap \mathcal{H} = CONV(\rho(Y_1 \times \cdots \times Y_n)) \subset \mathcal{P}_{H(n,q)} \cdots (7.4).$$

From (7.2) and (7.3) we obtain

$$\mathcal{P}_{H(n,q)} \subset \mathcal{H}^{(+)} \cdots (7.5).$$

From (7.4) and (7.5) we see that an affine hyperplane $\mathcal{H}$ is a supporting hyperplane for $\mathcal{P}_{H(n,q)}$ and $CONV(\rho(Y_1 \times \cdots \times Y_n))$ is a face of $\mathcal{P}_{H(n,q)}$.

Furthermore, from proposition 4.4 we see that $\rho(Y_1 \times \cdots \times Y_n)$ is vertex set of a face $CONV(\rho(Y_1 \times \cdots \times Y_n))$, that is $\mathcal{F}(Y_1 \times \cdots \times Y_n)$.
(2) Suppose $Y_l = \{p_{l1}, \cdots, p_{lm_l}\}$ $(1 \leq l \leq n,\ 1 \leq m_l \leq q-1)$ and $\sum_{l=1}^{n} |Y_l| = n+i$. From (1) we see that $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ is a face of $\mathcal{P}_{H(n,q)}$ with vertex set $\rho(Y_1 \times \cdots \times Y_n)$. It follows from proposition 4.5 and 6.1 that 1-skelton of $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ is an induced subgraph of that of $\mathcal{P}_{H(n,q)}$, $\Gamma_{H(n,q)}$. Further more, it follows from theorem 6.2 and theorem 6.3 that the $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ is simple and hence its 1-skelton is regular graph. Then we see that (2) holds if and only if 1-skelton of $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ is regular of valency i.

Set $u = (p_{11}, p_{21}, \cdots, p_{lm_l}) \in Y_1 \times \cdots \times Y_n$. From (7.1) we obtain

$$|\, \Gamma_1 \bigcap (Y_1 \times \cdots \times Y_n)| = \sum_{l=1}^{n} (m_l - 1) = \sum_{l=1}^{n} |Y_l| - n = i,$$

and hence $\rho(u)$ is adjacent to i vertices in 1-skelton of $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ by lemma 6.1, i.e. 1-skelton of $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ is regular of valency i.
(3) Let $\mathcal{F}$ be a face of $\mathcal{P}_{H(n,q)}$ with vertex set $V_{\mathcal{F}}$. Then from proposition 4.4 we see that $V_{\mathcal{F}} \subset V_{H(n,q)}$. Suppose that $\rho(u) \in V_{\mathcal{F}}$ $(u \in X_{H(n,q)})$, and that $\{\rho(v_1), \cdots, \rho(v_k)\} \subset V_{\mathcal{F}}$ is all vertices adjacent to $\rho(u)$ in 1-skelton of $\mathcal{F}$. Since $\mathcal{F}$ is simple by theorem 6.3, from theorem6.2 we see that 1-skelton os $\mathcal{F}$ is regular of valency k and hence $\mathcal{F}$ is a k-face of $\mathcal{P}_{H(n,q)}$. Since $\rho(u)$ and $\rho(v_l)$ $(1 \leq l \leq k)$ are adjacent in 1-skelton of $\mathcal{F}$, we see that u and v_l $(1 \leq l \leq k)$ are adjacent in $H(n,q)$, i.e.

$$\partial(u, v_l) = 1 \ (1 \leq l \leq k) \cdots (7.6).$$

Set $u = (u_1, \cdots, u_n) \in X_{H(n,q)}$ and $v_l = (v_{l1}, \cdots, v_{ln}) \in X_{H(n,q)}$ $(1 \le l \le n)$. We define non-empty subsets $\acute{Y}_j$ $(1 \le j \le n)$ of $Q = 0, 1, \cdots, q-1$ as the following;

$$\acute{y}_j = \{u_j\} \bigcup \{v_{lj} | \ 1 \le l \le k, \ v_{lj} \ne u_j\}.$$

Then from (7.6) it is not hard to show that

$$u, v_1, \cdots, v_k \in \acute{Y}_1 \times \cdots \times \acute{Y}_n \ \cdots (7.7),$$

$$\text{and} \ \sum_{j=1}^{n} |\acute{Y}_j| = n + k \ \cdots (7.8).$$

From (1), (2), and (7.8) we see that $\mathcal{F}(\acute{Y}_1 \times \cdots \times \acute{Y}_n)$ is a k-face of $\mathcal{P}_{H(n,q)}$ with vertex set $\rho(\acute{Y}_1 \times \cdots \times \acute{Y}_n)$. From (7.7) we see that $\rho(u), \rho(v_1), \cdots, \rho(v_k) \in \rho(\acute{Y}_1 \times \cdots \times \acute{Y}_n)$. Therefore, since $\mathcal{F}$ and $\mathcal{F}(\acute{Y}_1 \times \cdots \times \acute{Y}_n)$ are simple, we obtain $\mathcal{F} = \mathcal{F}(\acute{Y}_1 \times \cdots \times \acute{Y}_n)$ from lemma 6.2. □

We denote the set of all faces of $\mathcal{P}_{H(n,q)}$ by $AF_{H(n,q)}$ and the set $\{1, \cdots, q\} \subset \boldsymbol{N}$ by K_q. We define the mapping $\psi : \ AF_{H(n,q)} \longrightarrow K_q^n$ as the following;

$$\psi(\mathcal{F}(Y_1 \times \cdots \times Y_n)) = (|Y_1|, \cdots, |Y_n|) \in K_q^n \ \text{ for } \mathcal{F}(Y_1 \times \cdots \times Y_n) \in AF_{H(n,q)}.$$

Then from theorem 7.1 it is not hard to show that the mapping ψ is well defined and surjective. For non negative integers $s_1, \cdots, s_q$, we define the subset $L[s_1, \cdots, s_q]$ of K_q^n as the following;

$$L[s_1, \cdots, s_q] = \{ \ (k_1, \cdots, k_n) | \ {}^{\sharp}\{k_j | \ k_j = l\} = s_l, \ 1 \le l \le q\}.$$

Then it is not hard to show that

$$\sum_{l=1}^{q} s_l = n \ \text{ if and only if } \ L[s_1, \cdots, s_q] \ne \emptyset \cdots (7,9),$$

$$\text{and} \ \bigcup_{s_1, \cdots, s_q \ge 0} L[s_1, \cdots, s_q] = K_q^n.$$

We call a face $\mathcal{F}$ of $\mathcal{P}_{H(n,q)}$ a $[s_1, \cdots, s_q]$-type face if $\psi(\mathcal{F}) \in L[s_1, \cdots, s_q]$. Now we want to caluculate the three of numbers. The first is the number of all i-face of $\mathcal{P}_{H(n,q)}$, denoted by $NF(i)$. We denote all $[s_1, \cdots, s_q]$-type faces of $\mathcal{P}_{H(n,q)}$ by $F[s_1,\cdots,s_q]$. From (7.9) we may assume $\sum_{l=1}^{n} s_l = n$. It is not hard to show that

$$| F[s_1,\cdots,s_q] | = |\psi^{-1}(L[s_1,\cdots,s_q])| \cdots (7.10),$$

$$\text{and} \quad | L[s_1,\cdots,s_q] | = \frac{n!}{s_1!\cdots s_q!} \cdots (7.11).$$

Let $(k_1,\cdots,k_n) \in L[s_1,\cdots,s_q]$. Since $\psi(\mathcal{F}(Y_1 \times \cdots \times Y_n)) = (k_1,\cdots,k_n)$ if and only if $|Y_l| = k_l$ $(1 \le l \le n)$, we obtain

$$| \psi^{-1}(k_1,\cdots,k_n) | = \binom{q}{k_1}\cdots\binom{q}{k_n} = \prod_{j=1}^{q}\binom{q}{s_j}^{s_j} \cdots (7.12).$$

From (7.10) (7.11) and (7.12) we obtain

$$| F[s_1,\cdots,s_q] | = n! \prod_{j=1}^{q} \frac{\binom{q}{j}^{s_j}}{s_j!} \cdots (7.13).$$

Let $\mathcal{F}$ be a $[s_1,\cdots,s_q]$-type face of $\mathcal{P}_{H(n,q)}$. From theorem 7.1 it is not hard to show that $\mathcal{F}$ is i-face if and only if $\sum_{l=i}^{q}(l \cdot s_l) = n + i$. We define the subset $D(i,n,q)$ of Z^q as the following;

$$D(i,n,q) = \{(s_1,\cdots,s_q) \in \boldsymbol{Z}^q \mid s_l \ge 0, \ \sum_{l=1}^{q} s_l = n, \ \textit{and} \ \sum_{l=1}^{q}(l \cdot s_l) = n + i \ \}.$$

Then it is not hard to show that

$$NF(i) = \sum_{(s_1,\cdots,s_q)\in D(i,n,q)} | F[s_1,\cdots,s_q] | \cdots (7,14).$$

From (7.13) and (7.14) we obtain the following proposition.

Proposition 7.1 *The number of all i-faces of $\mathcal{P}_{H(n,q)}$ is equal to*

$$NF(i) = n! \sum_{(s_1,\cdots,s_q)\in D(i,n,q)} \prod_{j=1}^{q} \frac{\binom{q}{j}^{s_j}}{s_j!}. \square$$

The second is the number of all $(i-1)$-faces of $\mathcal{P}_{H(n,q)}$ contained in a fixed i-face of $\mathcal{P}_{H(n,q)}$. Let $\mathcal{F} = \mathcal{F}(Y_1 \times \cdots \times Y_n)$ be $[s_1, \cdots, s_q]$-type i-face of $\mathcal{P}_{H(n,q)}$. Then we see that $\sum_{l=1}^{q} s_l = n$ and $\sum_{l=1}^{q}(l \cdot s_l) = i + n$. Let $\mathcal{F}(Z_1 \times \cdots \times Z_n)$ be a $(i-1)$-face of $\mathcal{P}_{H(n,q)}$ contained in $\mathcal{F}$. Then it is not hard to show that

$$Z_l \subset Y_l \ (1 \leq l \leq n), \quad \text{and} \quad \sum_{l=1}^{n}(\ |Y_l| - |Z_l|\ ) = 1 \cdots (7.15).$$

From (7.15) there exists $\delta \in \{1, \cdots, n\}$ such that $|Z_\delta| = |Y_\delta| - 1$, $|Y_\delta| \geq 2$, and $Z_l = Y_l$ for $l \in \{1, \cdots, \delta-1, \delta+1, \cdots, n\}$. Conversely, if $|Y_l| \geq 2$, then for any $y \in Y_l$, $\mathcal{F}(Y_1 \times \cdots Y_{l-1} \times (\ Y_l - \{y\}\ ) \times Y_{l+1} \times \cdots \times Y_n)$ is $(i-1)$-face of $\mathcal{P}_{H(n,q)}$, contained in $\mathcal{F}$. Then the number of all $(i-1)$-faces of $\mathcal{P}_{H(n,q)}$ contained in $\mathcal{F}$ is equal to

$$\sum_{|Y_l| \neq 1} |Y_l|.$$

It is not hard to show that

$$\sum_{|Y_l| \neq 1} |Y_l| = \sum_{l=2}^{q}(l \cdot s_l).$$

Therefore we obtain the following proposition.

Proposition 7. 2 *The number of all $(i-1)$-faces of $\mathcal{P}_{H(n,q)}$ contained in a fixed $[s_1, \cdots, s_q]$-type i-face of $\mathcal{P}_{H(n,q)}$ is equal to*

$$\sum_{l=2}^{q}(l \cdot s_l).\square$$

The third is the number of $(i+1)$-faces of $\mathcal{P}_{H(n,q)}$ which contain a fixed i-face of $\mathcal{P}_{H(n,q)}$. Let us fix an i-face $\mathcal{F} = \mathcal{F}(Y_1 \times \cdots \times Y_n)$ of $\mathcal{P}_{H(n,q)}$ and let $\mathcal{F}(Z_1 \times \cdots \times Z_n)$ be $(i+1)$-face of $\mathcal{P}_{H(n,q)}$ which contains $\mathcal{F}$. Then it is not hard to show that

$$Y_l \subset Z_l \ (1 \leq l \leq n), \quad \text{and} \quad \sum_{l=1}^{n}(\ |Z_l| - |Y_l|\ ) = 1 \cdots (7.16).$$

From (7.16) there exists $\gamma \in \{1, \cdots, n\}$ such that $|Z_\delta| = |Y_\delta| + 1$, and $Z_l = Y_l$ for $l \in \{1, \cdots, \gamma-1, \gamma+1, \cdots, n\}$. Conversely, if $|Y_l| < q$, then for any

$z \in (Q - Y_l)$, $\mathcal{F}(Y_1 \times \cdots Y_{l-1} \times (Y_l \bigcap \{z\}) \times Y_{l+1} \times \cdots \times Y_n)$ is $(i-1)$-face of $\mathcal{P}_{H(n,q)}$, which contains $\mathcal{F}$. Then the number of all $(i+1)$-faces of $\mathcal{P}_{H(n,q)}$ which contains $\mathcal{F}$ is equal to

$$\sum_{l=1}^{n}(q - |Y_l|) = n(q-1) - i$$

since $\sum_{l=1}^{n} |Y_l| = n + i$.
Therefore we obtain the following proposition.

Proposition 7. 3 *The number of all $(i+1)$-faces of $\mathcal{P}_{H(n,q)}$ which contain a fixed i-face of $\mathcal{P}_{H(n,q)}$ is equal to*

$$n(q-1) - i.\square$$

8 The 1-skeston of face

In this section we describe about the 1-skelton of face of $\mathcal{P}$. We denote the 1-skelton of a face $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ of $\mathcal{P}_{H(n,q)}$ by $G(Y_1 \times \cdots \times Y_n)$. Then we see that $G(Y_1 \times \cdots \times Y_n)$ is induced subgraph of $\Gamma_{H(n,q)}$. We identify $Y_1 \times \cdots \times Y_n \subset X_{H(n,q)}$ with induced subgraph of $H(n,q)$. Then from lemma 6.1 it is not hard to show that $Y_1 \times \cdots \times Y_n$ is isomorphic to $G(Y_1 \times \cdots \times Y_n)$ as a graph and we can identify $G(Y_1 \times \cdots \times Y_n)$ with $Y_1 \times \cdots \times Y_n$. Let $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ be a $[s_1, \cdots, s_q]$-type of $\mathcal{P}_{H(n,q)}$, let $Y_l = \{p_{l1}, \cdots, p_{lm_l}\}$ $(1 \le l \le n,\ 1 \le m_l \le q)$, and let $u \in Y_1 \times \cdots \times Y_n$. Then we may assume $u = (p_{11}, p_{21}, \cdots, p_{n1})$ without loss of genarality. We see that for any $l \in \{1, \cdots, n\}$ the M_l vertices $u, (p_{11}, \cdots, p_{(l-1)1}, p_{lj}, p_{(l+1)1}, \cdots, p_{n1})$ $(2 \le j \le m_l)$ are adjacent to each other in $Y_1 \times \cdots \times Y_n$, i.e. they form complete subgraph of $Y_1 \times \cdots \times Y_n$ with m_l vertices, denoted by C_{m_l}. From definition of $[s_1, \cdots, s_q]$-type face of $\mathcal{P}_{H(n,q)}$ we obtain following proposition.

Proposition 8. 1 *Let $\mathcal{F}(Y_1 \times \cdots \times Y_n)$ be a $[s_1, \cdots, s_q]$-type face of $\mathcal{P}_{H(n,q)}$ and let u be a vertex of $\mathcal{F}(Y_1 \times \cdots \times Y_n)$, i.e. $G(Y_1 \times \cdots \times Y_n)$. Then there exist exact s_j $(1 \le j \le q)$ complete subgraph C_j of $G(Y_1 \times \cdots \times Y_n)$, which contain a fixed vertex of $\mathcal{P}_{H(n,q)}$.*$\square$

9 Conclution

We have determined the faces of the θ_1-eigenpolytopes of the Hamming graphs $H(n,q)$ as the subsets of the vertex set of $H(n,q)$. Since we know the cosine sequence of $H(n,q)$ corresponding to θ_1, we can determine the θ_1-eigenpolytopes of $H(n,q)$ geometrically for small n and q. For example there appear regular triangles, squares, and cubes as 2-faces and 3-faces in the θ_1-eigenpolytope of $H(3,3)$. Some unsolved problems are (1) Can we express the number in proposition 7.1 and proposition 7.2 more simply? (2) Are faces of some type equal to equvalent other as polytopes? (3) Can we determine the θ_i-eigenpolytopes of $H(n,q)$ for $i \geq 2$?

Acknowledgement The notation of chapter 2 and chapter 3 is based on C.D.Godsil [2]

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