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Coinvariant Algebras of Some Finite Groups

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0. Recently Y.Ito and I.Nakamura [IN2], [N2] studied the Hilbert scheme of G-orbits $Hilb^G(\mathbf{C}^2)$ for a finite group $G \subset SL(2,\mathbf{C})$ and showed a direct correspondence between the representation graph of G(McKay observation) and the singular fiber of the minimal resolution of $\mathbf{C}^2/G$(Dynkin curve). In this article we report some attempts to extend the results to finite subgroups of $SL(3,\mathbf{C})$, which is being studied jointly with Iku Nakamura(Hokkaido Univ.) and Yasushi Gomi(Sophia Univ.). For simplicity we take the complex number field $\mathbf{C}$ as a ground field and representations considered are complex representations.

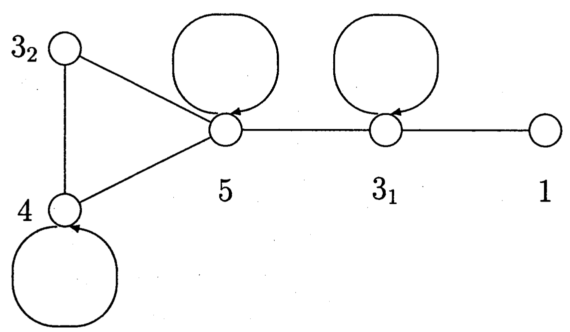
1. Let G be a finite group, $Irr(G) = \{\chi_1, ..., \chi_s\}$ be the set of all irreducible characters of G and $Irr(G)^\sharp = Irr(G) - \{1_G\}$. Given a character χ of G, we can form the representation graph $\Gamma(G) = \Gamma_\chi(G)$ as follows: the set of vertices is $Irr(G)$ and the directed edge of weight m_{ij} from χ_i to χ_j is determined by the relation

$$\chi \cdot \chi_i = \sum_{j=1}^{s} m_{ij}\chi_j, \qquad i = 1, ..., s.$$

We use the convention that a pair of opposing directed edges of weight 1 is represented by a single edge and the weight m_{ij} is omitted if $m_{ij} = 1$.

Example 1. Let G be the quaternion group of order 8. Then $Irr(G)$ consists of 4 linear charcters and the character χ of 2-dimensional representaion. Then $\Gamma_\chi(G)$ is eactly the extended Dynkin diagram of type D_4 centered at χ.

Example 2. Let G be the alternating group of degree 5, A_5. Then $Irr(G) = \{1, \chi = 3_1, 3_2, 4, 5\}$, (where the characters are expressed by the degrees of the corresponding representations), and $\Gamma_\chi(G)$ becomes as follows:



2. In [M] J. McKay stated the following which is now famous as McKay observation.

Proposition. Let G be a finite subgroup of $SL(2,\mathbf{C})$ and χ be the character of the inclusion representation. Then $\Gamma_\chi(G)$ is an extended Dynkin diagram of type A, D or E.

Conversely every such extended Dynkin diagram is obtained as a representation graph of a subgroup of $SL(2, \mathbf{C})$.

Thus McKay observation establishes a bijective correspondence between subgroups G of $SL(2, \mathbf{C})$ and the extended Dynkin diagram $\overline{X}_G$ of type A, D and E.

3. There is another famous correspondence between subgroups G of $SL(2, \mathbf{C})$ and the Dynkin diagram X_G of type A, D and E.(The extended Dynkin diagram of X_G is $\overline{X}_G$.) Let $S = \mathbf{C}^2/G$ and $p : \widetilde{S} \to S$ be the minimal resolution of sigularity. Then the singular fiber, $p^{-1}(0)$, is a union of projective lines, Dynkin curve of type X_G, having intersection matrix $-C$, where C is the Cartan matrix of type X_G. In particular the graph obtained by Dynkin curve as follows is the Dynkin diagram X_G: the set of vertices is that of projective lines appearing in Dynkin curve and two lines are joined iff they meet. For details, please see a survey article of R.Steinberg[St] or P.Slodowy[Sl].

These two correspondences were famous, but relations between them had not been clear. Recently an explanation of these correspondences was given by Y.Ito and I.Nakamura[IN1], [IN2] and I.Nakamura[N1], [N2], using Hilbert schemes.

4. Let $Hilb^n(\mathbf{C}^m)$ be the Hilbert scheme of $\mathbf{C}^m$ parametrizing all the 0-dimensional subschemes of length n and let $Symm^n(\mathbf{C}^m)$ be the n-th symmetric product of $\mathbf{C}^m$, that is, the quotient of n-copies of $\mathbf{C}^m$ by the natural action of the symmetric group of degree n. There is a canonical morphism π from $Hilb^n(\mathbf{C}^m)$ to $Symm^n(\mathbf{C}^m)$ associating to each 0-dimensional subscheme of $\mathbf{C}^m$ its support. Let G be a finite subgroup of $SL(m, \mathbf{C})$. The group G acts on $\mathbf{C}^m$ so that it acts naturally on both $Hilb^n(\mathbf{C}^m)$ and $Symm^n(\mathbf{C}^m)$. Since π is G-equivariant, π induces a morphism from the G-fixed point set $Hilb^n(\mathbf{C}^m)^G$ to the G-fixed point set $Symm^n(\mathbf{C}^m)^G$.

Now consider the special situation that n is the order of the group G and $m = 2$. Then $Symm^n(\mathbf{C}^2)^G$ is isomorphic to the quotient space $\mathbf{C}^2/G$ and there is a unique irreducible component of $Hilb^n(\mathbf{C}^2)^G$ dominating $Symm^n(\mathbf{C}^2)^G$, which we denote by $Hilb^G(\mathbf{C}^2)$ and call it the Hilbert scheme of G-orbits, following the notation and the definition by I.Nakamura. Notice that we have a morphism $p : Hilb^G(\mathbf{C}^2) \to \mathbf{C}^2/G$ induced by π. The following theorem is proved in a unified way.

Theorem. [IN2]. $Hilb^G(\mathbf{C}^2)$ is nonsingular and $p : Hilb^G(\mathbf{C}^2) \to \mathbf{C}^2/G$ is a minimal resolution of singularity.

5. Let $R = \mathbf{C}[x, y]$ be the ring of regular functions on $\mathbf{C}^2$ and M be the maximal ideal corresponding to the origin, that is $M = (x, y)$. For a finite group $G \subset SL(2, \mathbf{C})$ of order n, let R^G be the invarint algebra of G and N be the ideal of R generated by invariant homogeneous polynomials of positive degree which generate R^G. The ring $R_G = R/N$ is called the coinvariant algebra of G.

We identify a G-invariant 0-dimensional subscheme with its defining ideal of R. For $I \in Hilb^G(\mathbf{C}^2)$ with support origin, put $V(I) = I/(MI + N)$. Then $V(I)$ is a G-module and we denote its character by $\chi_{V(I)}$. Let E be the exceptional set of p and $Irr(E)$ be the set of irreducible components of E. For $\chi \in Irr(G)^\sharp$, define

$$E(\chi) = \{I \in E | (\chi, \chi_{V(I)})_G \neq 0\}$$

where $(,)_G$ is the usual inner product on functions on G. Then by verifying every case the following theorem is obtained.

Theorem. [IN2],[N2].

$$E = \{\ I \ \mid \ G\text{-invarinant ideal of } R, N \subset I \subset M, R/I \simeq \mathbf{C}G\}$$

and the map $\chi \mapsto E(\chi)$ gives a bijective correspondence between $Irr(G)^\sharp$ and $Irr(E)$.

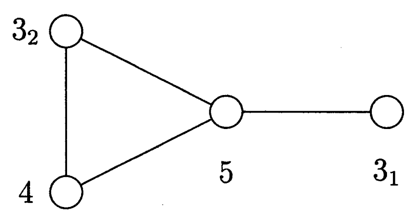
6. Let G be a subgroup of $SL(3, \mathbf{C})$. R, R^G, R_G, M and N are defined similarly for $\mathbf{C}^3$ and G as in **5**. Now theorem 5 suggests the necessity to study

$$F_G := \{\ I \ \mid \ G\text{-invarinant ideal of } R, N \subset I \subset M, R/I \simeq \mathbf{C}G\},$$

which would be a fiber of the origin of the quotient space $\mathbf{C}^3/G$ in the Hilbert scheme of G-orbits. For that purpose we need detailed structures of the coinvariant algebras R_G. What we have mainly obtained so far are

- decomposition of R_G (or its overalgebra) into irreducible components, particularly for groups of orders $60(A_5)$, $168(PSL(2,7))$, 108, 180, 216, 504, 648, and 1080,
- explicit determination of basis for each irreducible component above for A_5 and $PSL(2, 7)$.

As an outcome of these calculations we can show that F_{A_5} is a union of projective lines whose graph is given by



and a graph for $PSL(2, 7)$ also can be given. Details will appear in [GNS].

References

[GNS] Y.Gomi, I.Nakamura and K.Shinoda, Coinvariant algebras of some finite groups, (in preparation).

[IN1] Y.Ito and I.Nakamura, Hilbert schemes and simple singularities, to appear in Proc. Japan Academy.

[IN2] ———, Hilbert schemes and simple singularities A_n and D_n, (preprint).

[M] McKay, Graphs, singularities, and finite groups, Proc. Symp. Pure Math., AMS 37(1980),183-186.

[N1] I.Nakamura, Simple singularities, McKay correspondence and Hilbert schemes of G-orbits, (preprint).

[N2] ———, Hilbert schemes and simple singulariries E_6, E_7 and E_8,(preprint).

[Sl] P.Slodowy, Simple singularities, Springer Lecture Note 815(1980).

[St] R.Steinberg, Kleinian singularities and unipotent elements, Proc. Symp. Pure Math., AMS 37(1980),265-270.